ON TRIPLY-GENERATED TELESCOPIC SEMIGROUPS AND CHAINS OF SEMIGROUPS

GRETCHEN L. MATTHEWS

Abstract. Given a numerical semigroup $S = \langle a_1, a_2, \dots, a_\nu \rangle$ in canonical form, let $M(S) := S \setminus \{0\}$. Define associated numerical semigroups $B(S) := \{x \in \mathbb{N}_0 : x + M(S) \subseteq M(S)\}$ and $L(S) := \langle a_1, a_2 - a_1, \dots, a_\nu - a_1 \rangle$. Set $B_0(S) = S$, and for $i \geq 1$, define $B_i(S) := B(B_{i-1}(S))$. Similarly, set $L_0(S) = S$, and for $i \geq 1$, define $L_i(S) := L(L_{i-1}(S))$. These constructions define finite ascending chains of semigroups $S = B_0(S) \subseteq B_1(S) \subseteq \cdots \subseteq B_{\beta(S)}(S) = \mathbb{N}_0$ and $S = L_0(S) \subseteq L_1(S) \subseteq \cdots \subseteq L_{\lambda(S)}(S) = \mathbb{N}_0$. It is shown that if S is a triply-generated telescopic semigroup, then $B_j(S) = L_1(S)$ for some j, $1 \leq j \leq \beta(S)$. From this, it follows that certain triply-generated telescopic semigroups S satisfy $B_i(S) \subseteq L_i(S)$ for all $0 \leq i \leq \beta(S)$.

1. Introduction

A *numerical semigroup* is a submonoid of the moniod $\mathbb{N}_0$ of non-negative integers under addition. It is well known that each numerical semigroup is finitely generated. More precisely, given a numerical semigroup S, there exist $a_1, a_2, \dots, a_\nu \in \mathbb{N}$ such that $S = \langle a_1, a_2, \dots, a_\nu \rangle$; that is $S = \{\sum_{i=1}^{\nu} c_i a_i : c_i \in \mathbb{N}_0\}$. We adopt the conventions of [1] and [2]. In particular, we will consider those numerical semigroups S with the property that the set of elements of S has greatest common divisor 1. (Note that while not every numerical semigroup satisfies this property, every numerical semigroup is isomorphic to one that does.) Then each numerical semigroup S has a *canonical form* description so that $S = \langle a_1, a_2, \dots, a_\nu \rangle$ where $a_1 < a_2 < \cdots < a_\nu$, $a_j \notin \langle \{a_i : 1 \leq i \leq \nu, i \neq j\} \rangle$ for all $1 \leq j \leq \nu$, and $\gcd\{a_1, a_2, \dots, a_\nu\} = 1$. The *embedding dimension* of S, denoted $e(S)$, is the number of generators of S in its canonical form description; that is, $e(S) = \nu$. It can be shown that $e(S) \leq a_1$. Thus, a numerical semigroup is said to be of *maximal embedding dimension* if $e(S) = a_1$, the least positive element of S. The assumption that $\gcd\{a_1, a_2, \dots, a_\nu\} = 1$ ensures that $\mathbb{N}_0 \setminus S$ is finite. The *Frobenius number* of S, denoted $g(S)$, is the largest integer in $\mathbb{N}_0 \setminus S$.

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Suppose $S = \langle a_1, a_2, \dots, a_\nu \rangle$ is a canonical form description of a numerical semigroup S. Let $M := S \setminus \{0\}$. One may consider associated numerical semigroups

$$B(S) := \{x \in \mathbb{N}_0 : x + M \subseteq M\}$$

and

$$L(S) := \langle a_1, a_2 - a_1, \dots, a_\nu - a_1 \rangle .$$

Clearly, $S \subseteq B(S)$ and $S \subseteq L(S)$. It is also not hard to see that $B(S) \subseteq L(S)$ since $x \in B(S)$ implies $x + a_1 = \sum_{i=1}^{\nu} c_i a_i$ for some $c_i \in \mathbb{N}_0$. As in [1], one may iterate the B and L constructions to obtain two ascending chains of numerical semigroups

$$B_0(S) := S \subseteq B_1(S) := B(B_0(S)) \subseteq \cdots \subseteq B_{h+1}(S) := B(B_h(S)) \subseteq \dots$$

and

$$L_0(S) := S \subseteq L_1(S) := L(L_0(S)) \subseteq \cdots \subseteq L_{h+1}(S) := L(L_h(S)) \subseteq \dots .$$

Note that $S \subsetneqq B(S)$ and $S \subsetneqq L(S)$ for all numerical semigroups $S \neq \mathbb{N}_0$. This, together with the fact that $\mathbb{N}_0 \setminus S$ is finite, implies that there exist smallest non-negative integers $\beta(S)$ and $\lambda(S)$ such that $B_{\beta(S)}(S) = \mathbb{N}_0 = L_{\lambda(S)}(S)$. Thus, the B and L constructions give rise to finite strictly increasing chains of numerical semigroups. Since $B_0(S) = S = L_0(S)$, $B_1(S) \subseteq L_1(S)$, and $B_{\beta(S)}(S) = \mathbb{N}_0 = L_{\lambda(S)}(S)$, it is natural to compare the two chains. In particular, it is natural to ask, as in [1], if $B_i(S) \subseteq L_i(S)$ for all $0 \leq i \leq \beta(S)$. In [2], we show that while this containment does not hold for all numerical semigroups S, it does hold if S is a doubly-generated semigroup $\langle a_1, a_2 \rangle$.

This brings us to the focus of this work. Doubly-generated semigroups $S = \langle a_1, a_2 \rangle$ are examples of so-called telescopic semigroups. A numerical semigroup $S = \langle a_1, a_2, \dots, a_\nu \rangle$ in canonical form is *telescopic* if $\frac{a_i}{d_i} \in S_{i-1}$ for all $2 \leq i \leq \nu$, where $d_i := \gcd\{a_1, a_2, \dots, a_i\}$ and $S_i := \left\langle \frac{a_1}{d_i}, \frac{a_2}{d_i}, \dots, \frac{a_i}{d_i} \right\rangle$. In this paper, we consider triply-generated telescopic semigroups; that is, numerical semigroups $S = \langle a_1, a_2, a_3 \rangle$ such that $a_3 \in \left\langle \frac{a_1}{d}, \frac{a_2}{d} \right\rangle$ where $d := \gcd\{a_1, a_2\}$. The main result, Theorem 2.3, states that for such S, $B_{\frac{a_1}{d}+d-2}(S) = L_1(S)$. It follows immediately that $B_i(S) \subseteq L_i(S)$ for $0 \leq i \leq \frac{a_1}{d} + d - 2$. As a consequence, we see in Corollary 2.4 that for certain triply-generated telescopic semigroups S, $B_i(S) \subseteq L_i(S)$ for all $0 \leq i \leq \beta(S)$.

For background on numerical semigroups, see [4], [1]. For background on telescopic semigroups, see [3].

2. Results

We collect some results on numerical semigroups from [1] that will be used in the proof of the main result. Recall that a numerical semigroup S

is *symmetric* if the map $S \cap \{0, 1, \ldots, g(S)\} \to (\mathbb{N}_0 \setminus S) \cap \{0, 1, \ldots, g(S)\}$ defined by $s \mapsto g(S) - s$ is a bijection.

Proposition 2.1. *Let $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ be a numerical semigroup in canonical form. Then:*
(a) $g(B_1(S)) = g(S) - a_1$.
(b) If S is symmetric, then $B_1(S) = \langle a_1, a_2, \ldots, a_\nu, g(S) \rangle$.
(c) S is of maximal embedding dimension if and only if $B_1(S) = L_1(S)$.

The next proposition contains some useful results on telescopic semigroups from [3].

Proposition 2.2. *(a) Suppose $S = \langle a_1, a_2, a_3 \rangle$ is a canonical form description of a numerical semigroup S. If S is telescopic, then $g(S) = \frac{a_1 a_2}{d} + (d-1)a_3 - a_1 - a_2$ where $d := \gcd\{a_1, a_2\}$.*
(b) If a numerical semigroup S is telescopic, then S is symmetric.

Theorem 2.3. *If $S = \langle a_1, a_2, a_3 \rangle$ is a telescopic semigroup and $d := \gcd\{a_1, a_2\}$, then $B_i(S) \subseteq L_i(S)$ for $0 \le i \le \frac{a_1}{d} + d - 2$. Moreover, $B_{\frac{a_1}{d}+d-2}(S) = L_1(S)$.*

Proof. It will be convenient to write B_i and L_i instead of $B_i(S)$ and $L_i(S)$, respectively. Let $g := g(S)$ and $d := \gcd\{a_1, a_2\}$. According to Proposition 2.2(a),

$$g(S) = \frac{a_1 a_2}{d} + (d-1)a_3 - a_1 - a_2.$$

Much of the proof is devoted to proving the claim that if $1 \le i \le \frac{a_1}{d} + d - 3$, then

$$B_i = \langle \{a_1, a_2, a_3\} \cup T_i \rangle$$

where

$$T_i := \left\{ \begin{array}{ll} g - r_1 a_1 - r_2 a_2 - r_3 a_3 : & r_1 + r_2 + r_3 = i - 1, \\ & 0 \le r_1 \le i - 1, \\ & 0 \le r_2 \le \frac{a_1}{d} - 1, \\ & 0 \le r_3 \le d - 1 \end{array} \right\}.$$

First, we establish the claim in the case $i = 1$. According to Proposition 2.2(b), S is symmetric since S is telescopic. This implies that $B_1 = \langle a_{1,2}, a_3, g \rangle$ by Proposition 2.1(b). It follows that $B_1 = \langle a_1, a_2, a_3\} \cup T_1 \rangle$ as $T_1 = \{g\}$, and the claim holds in the case $i = 1$.

We now proceed by induction on $i \ge 1$. Suppose the claim holds for all j, $1 \le j \le i - 1$; that is, assume that $B_j = \langle \{a_1, a_2, a_3\} \cup T_j \rangle$ for all j, $1 \le j \le i - 1$. By definition of B_i, to show that $\langle \{a_1, a_2, a_3\} \cup T_i \rangle \subseteq B_i$, it suffices to show that $T_i \subseteq B_i$. Let $x \in T_i$. Then

$$x := g - r_1 a_1 - r_2 a_2 - r_3 a_3,$$

where $r_1 + r_2 + r_3 = i - 1$, $0 \le r_1 \le i - 1$, $0 \le r_2 \le \frac{a_1}{d} - 1$, and $0 \le r_3 \le d - 1$. We must show that $x + B_{i-1} \subseteq B_{i-1}$. Since $B_{i-1} = \langle \{a_1, a_2, a_3\} \cup T_{i-1} \rangle$

by the induction hypothesis, it suffices to show that $x + S \subseteq B_{i-1}$ and $x + T_{i-1} \subseteq B_{i-1}$. Note that $x + a_1 = g - (r_1 - 1)a_1 - r_2a_2 - r_3a_3$. If $0 < r_1 \leq i-1$, then $x + a_1 \in T_{i-1}$. If $r_1 = 0$, then $x + a_1 = g + a_1 - r_2a_2 - r_3a_3 = (\frac{a_1}{d} - 1 - r_2)a_2 + (d - 1 - r_3)a_3 \in S$. Thus, $x + a_1 \in B_{i-1}$. Similarly, one may check that $x + a_2, x + a_3 \in B_{i-1}$. Then $x + S \subseteq B_{i-1}$.

It remains to show that $x + T_{i-1} \subseteq B_{i-1}$. Let $y \in T_{i-1}$. Then

$$y := g - l_1a_1 - l_2a_2 - l_3a_3,$$

where $l_1 + l_2 + l_3 = i - 2$, $0 \leq l_1 \leq i-2$, $0 \leq l_2 \leq \frac{a_1}{d} - 1$, and $0 \leq l_3 \leq d - 1$. We will show that $x + y \in B_{i-1}$. Notice that $x + y$

$$\begin{aligned} &= g - r_1a_1 - r_2a_2 - r_3a_3 + g - l_1a_1 - l_2a_2 - l_3a_3 \\ &= g - (r_1 + l_1 + 1)a_1 - (r_2 + l_2 - (\tfrac{a_1}{d} - 1))a_2 - (r_3 + l_3 - (d-1))a_3. \end{aligned}$$

If $r_1 + l_1 \leq i - 3$, then $(r_1 + l_1 + 1) + (r_2 + l_2 - (\frac{a_1}{d} - 1)) + (r_3 + l_3 - (d-1)) = (r_1 + r_2 + r_3) + (l_1 + l_2 + l_3) - (\frac{a_1}{d} + d - 3) = (i-1) + (i-2) - (\frac{a_1}{d} + d - 3) \leq i - 1 + i - 2 - i = i - 3$ and $0 \leq r_1 + l_1 + 1 \leq i - 2$. Since $0 \leq r_2, l_2 \leq \frac{a_1}{d} - 1$, $0 \leq r_2 + l_2 - (\frac{a_1}{d} - 1) \leq \frac{a_1}{d} - 1$. Since $0 \leq r_3, l_3 \leq d-1$, $0 \leq r_3 + l_3 - (d-1) \leq d - 1$. By definition of T_{i-1}, this implies that $x + y \in T_{i-1} \subseteq B_{i-1}$ in the case $r_1 + l_1 \leq i - 3$.

We now assume that $r_1 + l_1 \geq i - 2$. Notice that $x + y > g(B_{i-1})$ implies $x + y \in B_{i-1}$. Thus, we only need to consider the case in which $x + y \leq g(B_{i-1})$. Repeated applications of Proposition 2.1(a) lead to $g(B_{i-1}) = g - (i-1)a_1$ since $a_1 < g - (\frac{a_1}{d} - 3)a_2 - (d-1)a_3 \leq z$ for all $z \in T_{i-1}$. Thus, $x + y \leq g - (i-1)a_1$. This implies

$$A_2a_2 + A_3a_3 \leq A_1a_1, \tag{1}$$

where $A_1 := r_1 + l_1 - (i-2)$, $A_2 := \frac{a_1}{d} - 1 - (r_2 + l_2)$, and $A_3 := d - 1 - (r_3 + l_3)$. Note that

$$A_2 + A_3 > A_1.$$

Otherwise, $A_2 + A_3 \leq A_1$, which implies that

$$\frac{a_1}{d} - 1 - (r_2 + l_2) + d - 1 - (r_3 + l_3) \leq r_1 + l_1 - (i-2).$$

As a result, $\frac{a_1}{d} + d - 3 + i - 2 \leq (r_1 + r_2 + r_3) + (l_1 + l_2 + l_3) = i - 1 + i - 1$, and so $\frac{a_1}{d} + d - 3 \leq i - 2$. This contradicts the fact that $1 \leq i \leq \frac{a_1}{d} + d - 3$. If $A_2 \leq 0$, then $a_2 < a_3$ implies that $A_2a_2 \geq A_2a_3$. Then (1) gives

$$(A_2 + A_3)a_3 \leq A_2a_2 + A_3a_3 \leq A_1a_1 \leq A_1a_3.$$

It follows that $A_2 + A_3 \leq A_1$, which is a contradiction. Thus, $A_2 > 0$. If $A_3 \geq 0$, then $(A_2 + A_3)a_1 \leq A_2a_2 + A_3a_3 \leq A_1a_1$. This implies $A_2 + A_3 \leq A_1$, which is a contradiction.

Thus, we have reduced to the case where $A_1 \geq 0$, $A_2 > 0$, and $A_3 < 0$; that is, $r_1 + l_1 \geq i - 2$, $\frac{a_1}{d} - 1 > r_2 + l_2$, and $r_3 + l_3 > d - 1$. Then $x + y$

$$\begin{aligned} &= g - r_1a_1 - r_2a_2 - r_3a_3 + g - l_1a_1 - l_2a_2 - l_3a_3 \\ &= g - (r_1 + l_1 - \tfrac{a_2}{d} + 1)a_1 - (r_2 + l_2 + 1)a_2 - (r_3 + l_3 - (d-1))a_3. \end{aligned}$$

If $r_1+l_1-\frac{a_2}{d}+1<0$, then substituting for g and simplifying gives $x+y=(\frac{a_1}{d}-1-(r_2+l_2+1))a_2+((d-1)-(r_3+l_3-(d-1)))a_3-(r_1+l_1-\frac{a_2}{d}+1+1)a_1\in S$. Thus, the only case left to consider is $r_1+l_1-\frac{a_2}{d}+1\geq 0$. Here, $r_1+l_1-\frac{a_2}{d}+1=(i-1-(r_2+r_3))+(i-2-(l_2+l_3))-\frac{a_2}{d}+1<i-1+i-2-(d-1)-\frac{a_2}{d}+1<i-1+i-1-(\frac{a_1}{d}+d-3)<i-2$. We also have that $(r_1+l_1-\frac{a_2}{d}+1)+(r_2+l_2+1)+(r_3+l_3-(d-1))\leq i-3$. Therefore, $x+y\in S\cup T_{i-1}\subseteq B_{i-1}$. This shows that $x+T_{i-1}\subseteq B_{i-1}$. It follows that $x+B_{i-1}\subseteq B_{i-1}$, and so $x\in B_i$. Therefore, $\langle\{a_1,a_2,a_3\}\cup T_i\rangle\subseteq B_i$.

In order to complete the proof of the claim, we must show that $B_i\subseteq\langle\{a_1,a_2,a_3\}\cup T_i\rangle$. Clearly, by the induction hypothesis,

$$B_{i-1}=\langle\{a_1,a_2,a_3\}\cup T_{i-1}\rangle\subseteq\langle\{a_1,a_2,a_3\}\cup T_i\rangle\,.$$

Next, we show that $B_i\setminus B_{i-1}\subseteq\langle\{a_1,a_2,a_3\}\cup T_i\rangle$. Let $z\in B_i\setminus B_{i-1}$. Then $z+B_{i-1}\subseteq B_{i-1}$. Since $z\notin B_1$, there exists $j\in\{1,2,3\}$ such that $z+a_j\in B_{i-1}\setminus S=\langle\{a_1,a_2,a_3\}\cup T_{i-1}\rangle\setminus S$. This leads to $z+a_j=g-r_1a_1-r_2a_2-r_3a_3+m$ for some $g-r_1a_1-r_2a_2-r_3a_3\in T_{i-1}$ and $m\in B_{i-1}$. Note that $r_1+r_2+r_3=i-2$. Thus, $z=g-r_1a_1-r_2a_2-r_3a_3-a_j+m$. Using the definition of T_i together with the fact that $T_i+B_{i-1}\subseteq B_{i-1}$, we see that $z\in T_i\cup(T_i+B_{i-1})\subseteq T_i\cup B_{i-1}\subseteq\langle\{a_1,a_2,a_3\}\cup T_i\rangle$ except in the cases $j=2$ with $r_2=\frac{a_1}{d}-1$ and $j=3$ with $r_3=d-1$. If $z=g-r_1a_1-\frac{a_1}{d}a_2-r_3a_3$, then $g=z+r_1a_1+\frac{a_1}{d}a_2+r_3a_3=z+r_1a_1+\frac{a_2}{d}a_1+r_3a_3\in z+(r_1+\frac{a_2}{d}+r_3)M\in z+(i-2-(\frac{a_1}{d}-1)+\frac{a_2}{d})M\in z+iM\in B_i+iM\in S$, where $M:=S\setminus\{0\}$. This is a contradiction, since by definition $g\notin S$. Thus, $z\neq g-r_1a_1-\frac{a_1}{d}a_2-r_3a_3$. Similarly, $z=g-r_1a_1-r_2a_2-da_3$ implies $g\in S$. Thus, $z\neq g-r_1a_1-r_2a_2-da_3$. It follows that $z\in\langle\{a_1,a_2,a_3\}\cup T_i\rangle$. This proves that $B_i\subseteq\langle\{a_1,a_2,a_3\}\cup T_i\rangle$. Therefore, $B_i=\langle\{a_1,a_2,a_3\}\cup T_i\rangle$. By induction, this completes the proof of the claim that $B_i=\langle\{a_1,a_2,a_3\}\cup T_i\rangle$ for $1\leq i\leq\frac{a_1}{d}+d-3$.

Since we have shown the claim holds for $i=\frac{a_1}{d}+d-3$, $B_{\frac{a_1}{d}+d-3}=\left\langle\{a_1,a_2,a_3\}\cup T_{\frac{a_1}{d}+d-3}\right\rangle$. Note that $\left\langle\{a_1,a_2,a_3\}\cup T_{\frac{a_1}{d}+d-3}\right\rangle$ gives a canonical form description of $B_{\frac{a_1}{d}+d-3}$. Hence $e(B_{\frac{a_1}{d}+d-3})=|\{a_1,a_2,a_3\}\cup T_{\frac{a_1}{d}+d-3}|=3+|T_{\frac{a_1}{d}+d-3}|$. Using the fact that $|T_{\frac{a_1}{d}+d-3}|$

$$=\left|\left\{\begin{array}{ll} g-r_1a_1-r_2a_2-r_3a_3: & r_1+r_2+r_3=\frac{a_1}{d}+d-4,\\ & 0\leq r_1\leq\frac{a_1}{d}+d-4,\\ & 0\leq r_2\leq\frac{a_1}{d}-1,\\ & 0\leq r_3\leq d-1\end{array}\right\}\right|$$

$$=\left|\left\{\begin{array}{ll} (r_1,r_2,r_3): & r_1+r_2+r_3=\frac{a_1}{d}+d-4,\\ & 0\leq r_1\leq\frac{a_1}{d}+d-4,\\ & 0\leq r_2\leq\frac{a_1}{d}-1,\\ & 0\leq r_3\leq d-1\end{array}\right\}\right|=a_1-3,$$

we conclude that $e(B_{\frac{a_1}{d}+d-3}) = 3 + (a_1 - 3) = a_1$. Therefore, $B_{\frac{a_1}{d}+d-3}$ is of maximal embedding dimension.

By Proposition 2.1(c), since $B_{\frac{a_1}{d}+d-3}$ is of maximal embedding dimension, we have that $B_{\frac{a_1}{d}+d-2} =$

$$B(B_{\frac{a_1}{d}+d-3}) = L(B_{\frac{a_1}{d}+d-3}) = \left\langle \{a_1, a_2 - a_1, a_3 - a_1\} \cup T'_{\frac{a_1}{d}+d-3} \right\rangle,$$

where $T'_{\frac{a_1}{d}+d-3} :=$

$$\left\{ \begin{array}{ll} g - r_1a_1 - r_2a_2 - r_3a_3 - a_1 : & r_1 + r_2 + r_3 = \frac{a_1}{d} + d - 3 - 1, \\ & 0 \le r_1 \le \frac{a_1}{d} + d - 3 - 1, \\ & 0 \le r_2 \le \frac{a_1}{d} - 1, \\ & 0 \le r_3 \le d - 1 \end{array} \right\}.$$

In particular,

$$L_1 = \langle a_1, a_2 - a_1, a_3 - a_1 \rangle \subseteq B_{\frac{a_1}{d}+d-2}.$$

We claim that $L_1 = B_{\frac{a_1}{d}+d-2}$. To show this, we must show that $T'_{\frac{a_1}{d}+d-3} \subseteq \langle a_1, a_2 - a_1, a_3 - a_1 \rangle = L_1$.

Let $z \in T'_{\frac{a_1}{d}+d-3}$. Then $z = g-(r_1+1)a_1-r_2a_2-r_3a_3$, where $r_1+r_2+r_3 = \frac{a_1}{d} + d - 4$, $0 \le r_1 \le \frac{a_1}{d} + d - 4$, $0 \le r_2 \le \frac{a_1}{d} - 1$, and $0 \le r_3 \le d - 1$. Substituting for g yields $z = (\frac{a_1}{d} - r_1 - 1)(a_2 - a_1) + (d - r_3 - 1)(a_3 - a_1) \in \langle a_1, a_2 - a_1, a_3 - a_1 \rangle$. This gives $T'_{\frac{a_1}{d}+d-3} \subseteq L_1$, which leads to

$$B_{\frac{a_1}{d}+d-2} = \left\langle \{a_1, a_2 - a_1, a_3 - a_1\} \cup T'_{\frac{a_1}{d}+d-3} \right\rangle = L_1.$$

Since $B_i(S) \subseteq B_{i+1}(S)$ for all $i \ge 0$ and $B_{\frac{a_1}{d}+d-2} = L_1$, we have

$$B_0 \subseteq B_1 \subseteq B_2 \subseteq \cdots \subseteq B_{\frac{a_1}{d}+d-2} = L_1.$$

Since $L_i(S) \subseteq L_{i+1}(S)$ for all $i \ge 0$, this implies that $B_i(S) \subseteq L_1(S) \subseteq L_i(S)$ for all $0 \le i \le \frac{a_1}{d} + d - 2$. □

Corollary 2.4. *If $S = \langle a_1, a_2, a_3 \rangle$ is a telescopic semigroup and $a_1 \in \langle a_2 - a_1, a_3 - a_1 \rangle$, then $B_i(S) \subseteq L_i(S)$ for all $0 \le i \le \beta(S)$.*

To prove this corollary, we will use the following result from [2].

Proposition 2.5. *Let S be a numerical semigroup of embedding dimension $e(S) = 2$; that is, $S = \langle a_1, a_2 \rangle$ where a_1 and a_2 are relatively prime natural numbers greater than 1. Then $B_i(S) \subseteq L_i(S)$ for all $0 \le i \le \beta(S)$.*

Proof of Corollary 2.4. As before, let $d := \gcd\{a_1, a_2\}$. Then $B_i(S) \subseteq L_i(S)$ for $0 \le i \le \frac{a_1}{d} + d - 2$ by Theorem 2.3. In addition, $B_{\frac{a_1}{d}+d-2}(S) = L_1(S)$. By definition,

$$L_1(S) = \langle a_1, a_2 - a_1, a_3 - a_1 \rangle = \langle a_2 - a_1, a_3 - a_1 \rangle$$

since $a_1 \in \langle a_2 - a_1, a_3 - a_1 \rangle$. The fact that $B_i(S) \subseteq L_i(S)$ for $\frac{a_1}{d} + d - 2 < i \leq \beta(S)$ now follows immediately from Proposition 2.5 since L_1 is doubly-generated. □

References

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Department of Mathematical Sciences
Clemson University
Clemson, SC 29634-0975
U.S.A.
e-mail: gmatthe@clemson.edu