ON NUMERICAL SEMIGROUPS GENERATED BY GENERALIZED ARITHMETIC SEQUENCES

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ABSTRACT. Given a numerical semigroup S, let $M(S) = S \setminus \{0\}$ and $(lM(S) - lM(S)) = \{x \in \mathbb{N}_0 : x + lM(S) \subseteq lM(S)\}$. Define associated numerical semigroups $B(S) := (M(S) - M(S))$ and $L(S) := \cup_{l=1}^{\infty}(lM(S) - lM(S))$. Set $B_0(S) = S$, and for $i \geq 1$, define $B_i(S) := B(B_{i-1}(S))$. Similarly, set $L_0(S) = S$, and for $i \geq 1$, define $L_i(S) := L(L_{i-1}(S))$. These constructions define two finite ascending chains of numerical semigroups $S = B_0(S) \subseteq B_1(S) \subseteq \cdots \subseteq B_{\beta(S)}(S) = \mathbb{N}_0$ and $S = L_0(S) \subseteq L_1(S) \subseteq \cdots \subseteq L_{\lambda(S)}(S) = \mathbb{N}_0$. It has been shown that not all numerical semigroups S have the property that $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$. In this paper, we prove that if S is a numerical semigroup with a set of generators that form a generalized arithmetic sequence, then $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$. Moreover, we see that this containment is not necessarily satisfied if a set of generators of S form an almost arithmetic sequence. In addition, we characterize numerical semigroups generated by generalized arithmetic sequences that satisfy other semigroup properties, such as symmetric, pseudo-symmetric, and Arf.

1. Introduction

A *numerical semigroup* is a submonoid of the moniod $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ of non-negative integers under addition. It is well known that each numerical semigroup is finitely generated. More precisely, given a numerical semigroup S, there exist $a_1, a_2, \ldots, a_\nu \in \mathbb{N}$ such that $S = \{\sum_{i=1}^{\nu} c_i a_i : c_i \in \mathbb{N}_0\}$. In this case, we say that $\{a_1, a_2, \ldots, a_\nu\}$ is a generating set for S and write $S = \langle a_1, a_2, \ldots, a_\nu \rangle$. We adopt the conventions of [1] and [4]. In particular, we will only consider those numerical semigroups S with the property that the set of elements of S has greatest common divisor 1. (Note that while not every numerical semigroup satisfies this property, every numerical semigroup is isomorphic to one that does.) Then each numerical semigroup S has a unique generating set $\{a_1, a_2, \ldots, a_\nu\}$ so that $S = \{\sum_{i=1}^{\nu} c_i a_i : c_i \in \mathbb{N}_0\}$, $a_1 < a_2 < \cdots < a_\nu$, $\gcd\{a_1, a_2, \ldots, a_\nu\} = 1$, and for $1 \leq j \leq \nu$, $a_j \notin \langle a_1, a_2, \ldots, a_{j-1}, a_{j+1}, \ldots, a_\nu \rangle$. In this case, we say that $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ is a *canonical form* description of S.

One can define a partial order $\leq_S$ on a numerical semigroup S by $s \leq_S t$ for $s, t \in S$ if and only if there exists $u \in S$ such that $s + u = t$. Then, given $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ in canonical form, the set of elements $\{a_1, a_2, \ldots, a_\nu\}$ is called the *minimal set of generators* for S since this set consists precisely of those elements

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of $S \setminus \{0\}$ which are minimal with respect to the partial order $\leq_S$. The *embedding dimension* of S, denoted $e(S)$, is the cardinality of the minimal set of generators for S; that is, $e(S) = \nu$. It can be shown that $e(S) \leq a_1$. Thus, a numerical semigroup S is said to be of *maximal embedding dimension* if $e(S) = a_1$. The assumption that $\gcd\{a_1, a_2, \ldots, a_\nu\} = 1$ ensures that $\mathbb{N}_0 \setminus S$ is finite. Let $g(S)$ denote the largest integer in $\mathbb{N}_0 \setminus S$. The number $g(S)$ is called the *Frobenius number* of S due to the fact that Frobenius, following the work of Sylvester [13], proposed the study of the largest integer not representable as a linear combination of positive integers $a_1, a_2, \ldots, a_\nu$ with non-negative integer coefficients [2].

Given a numerical semigroup S, let $M(S) := S \setminus \{0\}$ denote the *maximal ideal* of S. For an integer $l \geq 1$, define

$$(lM(S) - lM(S)) := \{x \in \mathbb{N}_0 : x + lM(S) \subseteq lM(S)\}.$$

One may consider associated numerical semigroups

$$B(S) := (M(S) - M(S)),$$

which is sometimes called the *dual of* $M(S)$ *with respect to* S, and

$$L(S) := \cup_{l=1}^{\infty}(lM(S) - lM(S)),$$

called the *Lipman semigroup of* S after [9]. Clearly, $S \subseteq B(S) \subseteq L(S)$. Moreover, if $S \neq \mathbb{N}_0$, then $S \cup \{g(S)\} \subseteq B(S) \subseteq L(S)$. As in [1], one may iterate the B and L constructions to obtain two ascending chains of numerical semigroups

$$B_0(S) := S \subseteq B_1(S) := B(B_0(S)) \subseteq \cdots \subseteq B_{h+1}(S) := B(B_h(S)) \subseteq \ldots \qquad \mathbf{(B(S))}$$

and

$$L_0(S) := S \subseteq L_1(S) := L(L_0(S)) \subseteq \cdots \subseteq L_{h+1}(S) := L(L_h(S)) \subseteq \ldots \qquad \mathbf{(L(S))}.$$

Since $\mathbb{N}_0 \setminus S$ is finite, there exist smallest non-negative integers $\beta(S)$ and $\lambda(S)$ such that $B_{\beta(S)}(S) = \mathbb{N}_0 = L_{\lambda(S)}(S)$. Thus, the B and L constructions give rise to finite strictly increasing chains of numerical semigroups. These two chains play a role in characterizing classes of certain local Noetherian domains (see [1] for more on relationships between properties of semigroups and corresponding properties of rings). For instance, a numerical semigroup is said to be *Arf* if the chains $\mathbf{(B(S))}$ and $\mathbf{(L(S))}$ coincide. Arf semigroups help to characterize Arf rings, an important class of rings in commutative algebra and algebraic geometry. Since $B_0(S) = S = L_0(S)$, $B_1(S) \subseteq L_1(S)$, and $B_{\beta(S)}(S) = \mathbb{N}_0 = L_{\lambda(S)}(S)$, it is natural to compare the two chains. In particular, it is natural to ask, as in [1], if $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$. In [4], we show that this containment does not hold in general and raise the question of determining classes of numerical semigroups for which $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$.

In this work, we focus on numerical semigroups of the form

$$S = \langle a, ha + d, ha + 2d, \ldots, ha + kd \rangle\,,$$

where a, d, h, k are positive integers such that $a \geq 2$ and a and d are relatively prime. Since the minimal set of generators of such a numerical semigroup S forms a generalized arithmetic sequence, we say that S is *generated by a generalized arithmetic sequence*. Numerical semigroups of this form were first studied by Roberts in the case $h = 1$ [11]. This study was carried on by Selmer [12], and more recently, by Ritter [10]. Lewin [8] considered numerical semigroups with minimal sets of generators that form more general sequences, such as almost arithmetic sequences.

An *almost arithmetic sequence* is a sequence in which all but one of the elements form an arithmetic sequence. Thus, a generalized arithmetic sequence is certainly an almost arithmetic sequence. We show that if S is generated by a generalized arithmetic sequence, then $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$. However, this containment does not necessarily hold for all $i \geq 2$ if S is generated by an almost arithmetic sequence. An example is given to illustrate this. In addition, we characterize those numerical semigroups generated by generalized arithmetic sequences that are symmetric (rediscovering the main result of [5]), pseudo-symmetric, and Arf.

For more background on numerical semigroups, see [6], [1].

2. Results

It is convenient to collect here some results that will be used in this section.

Proposition 2.1. [1] *Let $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ be a numerical semigroup expressed in canonical form. Then:*
(a) $g(B(S)) = g(S) - a_1$.
(b) S is of maximal embedding dimension if and only if $B(S) = L(S)$.
(c) $L(S) = \langle a_1, a_2 - a_1, \ldots, a_\nu - a_1 \rangle$.

Proposition 2.2. [4, Theorem 2.6] *Let S be a numerical semigroup of embedding dimension $e(S) = 2$; that is, $S = \langle a_1, a_2 \rangle$, where a_1 and a_2 are relatively prime positive integers greater than 1. Then $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$.*

To simplify the exposition, we first consider those numerical semigroups generated by generalized arithmetic sequences that are either doubly-generated or of maximal embedding dimension. As an immediate consequence, we obtain a characterization of Arf semigroups that are generated by generalized arithmetic sequences.

Lemma 2.3. *Let $S = \langle a, ha + d, ha + 2d, \ldots, ha + kd \rangle$, where a, d, h, k are positive integers such that a and d are relatively prime and $a \geq 2$. If $k = 1$ or $k = a - 1$, then $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$.*

Proof. If $k = 1$, then $S = \langle a, ha + d \rangle$, and the result follows immediately from Proposition 2.2.

If $k = a - 1$, then $e(S) = k + 1 = a$, and S is of maximal embedding dimension. By Proposition 2.1(b),(c), we have that

$$B_1(S) = L_1(S) = \langle a, (h-1)a + d, (h-1)a + 2d, \ldots, (h-1)a + kd \rangle .$$

Repeated applications of Proposition 2.1(b),(c) yield

$$B_i(S) = L_i(S) = \langle a, (h-i)a + d, (h-i)a + 2d, \ldots, (h-i)a + kd \rangle$$

for $0 \leq i \leq h$. In particular, $B_h(S) = L_h(S) = \langle a, d \rangle$. The fact that $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$ now follows from Proposition 2.2. □

Proposition 2.4. *Let $S = \langle a, ha + d, ha + 2d, \ldots, ha + kd \rangle$, where a, d, h, k are positive integers such that a and d are relatively prime and $a \geq 2$. Then S is Arf if and only if $a = 2$ or S is of maximal embedding dimension and $d = 2$.*

Proof. Suppose $S = \langle a, ha + d, ha + 2d, \ldots, ha + kd \rangle$ is an Arf semigroup. Then, according to Proposition 2.1(b), $B_i(S)$ is of maximal embedding dimension for all $i \geq 0$. In particular, $B_0(S) = S$ is of maximal embedding dimension, and so $k = e(S) = a - 1$. By the proof of Lemma 2.3, this implies $B_h(S) = L_h(S) = \langle a, d \rangle$ is of maximal embedding dimension. Thus, $a = 2$ or $d = 2$.

Let $S = \langle a, ha+d, ha+2d, \ldots, ha+kd \rangle$. If $a = 2$, then $S = \langle 2, 2h+d \rangle$ which is an Arf semigroup according to [1, Theorem I.4.2]. If $d = 2$ and S is of maximal embedding dimension, the fact that S is Arf is a consequence of the proof of Lemma 2.3, as $B_i(S) = L_i(S)$ for $0 \leq i \leq h$ and $B_h(S) = L_h(S) = \langle 2, a \rangle$ which is Arf by [1, Theorem I.4.2]. □

We fix the following notation that will be used in the remainder of this section. Given a numerical semigroup S generated by the generalized arithmetic sequence $a, ha+d, ha+2d, \ldots, ha+kd$, we will assume that $2 \leq k \leq a-2$. Set $c := \lfloor \frac{a-2}{k} \rfloor$ and $r := a - 2 - ck$. Then, according to [8, Theorem 5.2], the Frobenius number of S is

$$g(S) = (ch + h - 1)a + (a-1)d.$$

In the following proposition, we obtain a useful description of the elements of S. When $h = 1$, this gives [11, Lemma 1].

Proposition 2.5. *Let $S = \langle a, ha+d, ha+2d, \ldots, ha+kd \rangle$, where a, d, h, k are positive integers such that a and d are relatively prime and $a \geq 2$. Then $S = \{la + jd : 0 \leq l, 0 \leq j \leq \lfloor \frac{l}{h} \rfloor k\}$.*

Proof. Let $s \in S$. Then $s = l_0 a + \sum_{i=1}^{k} l_i(ha + id) = (l_0 + h\sum_{i=1}^{k} l_i)a + (\sum_{i=1}^{k} il_i)d$ for some $l_0, \ldots, l_k \in \mathbb{N}_0$. Since $\sum_{i=1}^{k} il_i \leq k\sum_{i=1}^{k} l_i \leq k\lfloor \frac{l_0 + h\sum_{i=1}^{k} l_i}{h} \rfloor$, we have that $s \in \{la + jd : 0 \leq l, 0 \leq j \leq \lfloor \frac{l}{h} \rfloor k\}$. Thus, $S \subseteq \{la + jd : 0 \leq l, 0 \leq j \leq \lfloor \frac{l}{h} \rfloor k\}$.

Now let $x = la + jd$ with $0 \leq l$ and $0 \leq j \leq \lfloor \frac{l}{h} \rfloor k$. Write $l = qh + r$ and $j = q'k + r'$ with $q, q', r, r' \in \mathbb{N}_0$ such that $0 \leq r < h$ and $0 \leq r' < k$. Then $x = q'(ha + kd) + ra + (q - q')ha + r'd \in S$ since $r' > 0$ only if $q' < q$. Therefore, $\{la + jd : 0 \leq l, 0 \leq j \leq \lfloor \frac{l}{h} \rfloor k\} \subseteq S$, completing the proof. □

Given a numerical semigroup S and $a \in S$, let $S(a) = \{s \in S : s - a \notin S\}$. This set was introduced in [3] and used in [6] to give the following description of the numerical semigroup $B(S)$.

Proposition 2.6. [6, Proposition 7] *Let S be a numerical semigroup and $a \in S \setminus \{0\}$. For a positive integer t, the following are equivalent:*
(a) $t - a \in B(S) \setminus S$.
(b) t is maximal in $S(a)$ with respect to the partial ordering $\leq_S$.

Using Proposition 2.6, we obtain a description of the elements of $B(S)$ when the generators of S form an generalized arithmetic sequence. Taking $h = 1$ gives [7, Corollary 5].

Lemma 2.7. *If $S = \langle a, ha+d, ha+2d, \ldots, ha+kd \rangle$, where a, d, h, k are positive integers such that a and d are relatively prime, $a \geq 2$, and $2 \leq k \leq a-2$, then $B(S) = S \cup \{(ch + h - 1)a + jd : ck < j \leq a - 1\}$.*

Proof. Given $S = \langle a, ha+d, ha+2d, \ldots, ha+kd \rangle$ where $2 \leq k \leq a-2$, we claim that

$$S(a) = \{lha + jd : 0 \leq l \leq c+1, (l-1)k < j \leq \min\{lk, a-1\}\}.$$

Let $x := lha + jd$ with $0 \leq l \leq c+1$ and $(l-1)k < j \leq \min\{lk, a-1\}$. Then $x \in S$ and $x - a = (lh-1)a + jd \notin S$ by Proposition 2.5. Thus, $x \in S(a)$.

Now let $s \in S(a)$. Since $s \in S$, we may write $s = (lh + m)a + jd$ with $0 \leq l$, $0 \leq m < h$, and $0 \leq j \leq lk$. Notice that $s - a \notin S$ implies $m = 0$ and $(l-1)k < j \leq lk$. If $l \geq c+2$, then $s - a = (lh-1)a + jd \geq ((c+2)h - 1)a + jd \geq$

$(ch+2h-1)a+(a-1)d > g(S)$ and so $s-a \in S$. Thus, $l \leq c+1$. If $l = c+1$, then $ck < j \leq a-1$. Otherwise, $s-a = (ch+h-1)a+jd > g(S)$ which implies $s-a \in S$. Thus, $s = lha+jd$ with $0 \leq l \leq c+1$ and $(l-1)k < j \leq \min\{lk, a-1\}$. Therefore, $S(a) \subseteq \{lha+jd : 0 \leq l \leq c+1, (l-1)k < j \leq \min\{lk, a-1\}\}$ and the claim holds.

Using this description of $S(a)$, it is clear that the elements of $S(a)$ which are maximal with respect to $\leq_S$ are $\{(c+1)ha+jd : ck < j \leq a-1\}$. Applying Proposition 2.6 yields $B(S) \setminus S = \{(ch+h-1)a+jd : ck < j \leq a-1\}$. □

Recall that a numerical semigroup S is said to be *symmetric* (respectively, *pseudo-symmetric*) if the Frobenius number $g(S)$ is odd (respectively, even) and the map

$$S \cap \{0,1,\ldots,g(S)\} \to (\mathbb{N}_0 \setminus S) \cap \{0,1,\ldots,g(S)\} : s \mapsto g(S)-s$$

(respectively,

$$S \cap \{0,1,\ldots,g(S)\} \to (\mathbb{N}_0 \setminus S) \cap \{0,1,\ldots,g(S)\} \setminus \{\frac{g(S)}{2}\} : s \mapsto g(S)-s)$$

is a bijection. As a consequence of Lemma 2.7, we obtain a characterization of those symmetric and pseudo-symmetric semigroups that are generated by generalized arithmetic sequences. In the symmetric case, this is a restatement of the main result of [5].

Proposition 2.8. *Let $S = \langle a, ha+d, ha+2d, \ldots, ha+kd\rangle$, where a,d,h,k are positive integers such that a and d are relatively prime and $a \geq 2$. Then:*
(a) S is symmetric if and only if $a = 2$ or $k \in \{1, a-2\}$.
(b) S is pseudo-symmetric if and only if $a = 3$, $k = 2$, $h = 1$, and $d = \frac{g}{2}$, where $g \geq 2$ is an even integer and $g \equiv 1,2 \pmod 3$; that is, S is pseudo-symmetric if and only if $S = \left\langle 3, 3+\frac{g}{2}, 3+g\right\rangle$ where $g \geq 2$ is an even integer and $g \equiv 1,2 \pmod 3$.

Proof. According to [1, Lemma I.1.8], a numerical semigroup S with Frobenius number $g(S)$ odd (respectively, even) is symmetric (respectively, pseudo-symmetric) if and only if $B(S) = S \cup \{g(S)\}$ (respectively, $B(S) = S \cup \{\frac{g(S)}{2}, g(S)\}$). By Lemma 2.7, if $2 \leq k \leq a-2$, then S is symmetric if and only if $|\{(ch+h-1)a+jd : ck < j \leq a-1\}|=1$ if and only if $r = 0$ if and only if $k = a-2$. If $k = 1$, then $S = \langle a, ha+d\rangle$ is symmetric being of embedding dimension 2 [2], [13]. If $k = a-1$, then S is of maximal embedding dimension and so by [1, Theorem I.4.2] is symmetric if and only if $a = 2$.

By Lemma 2.7, if $2 \leq k \leq a-2$, then S is pseudo-symmetric if and only if $r = 1$ and $(ch+h-1)a+(ck+1)d = \frac{(ch+h-1)a+(ck+2)d}{2}$ if and only if $h = 1$ and $c = 0$, which cannot be the case since $k \leq a-2$. If $k = 1$, then $S = \langle a, ha+d\rangle$ which has odd Frobenius number $g(S) = (a-1)(ha+d)-a$ by [2] (or [13]) and so is not pseudo-symmetric. If $k = a-1$, then S is of maximal embedding dimension. By [1, Theorem I.4.4], S is pseudo-symmetric of maximal embedding dimension if and only if $S = \left\langle 3, 3+\frac{g}{2}, 3+g\right\rangle$ for some even integer $g \geq 2$ such that $g \equiv 1,2 \pmod 3$. Notice that this implies $a = 3$, $h = 1$, $d = \frac{g}{2}$, $k = 2$, and $g = g(S)$. □

Theorem 2.9. *If $S = \langle a, ha+d, ha+2d, \ldots, ha+kd\rangle$, where a,d,h,k are positive integers such that a and d are relatively prime and $a \geq 2$, then $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$.*

Proof. Let $S = \langle a, ha+d, ha+2d, \ldots, ha+kd\rangle$, where $a, d, h, k \in \mathbb{N}$ with $a \geq 2$ and $\gcd(a,d) = 1$. According to Lemma 2.3, $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$ in the cases $k = 1$ and $k = a-1$. Thus, in the following, we assume that $2 \leq k \leq a-2$. As before, set $c = \lfloor \frac{a-2}{k} \rfloor$ and $r = a-2-ck$. Then $0 \leq r < k$. For convenience, we will write B_i and L_i for $B_i(S)$ and $L_i(S)$ respectively. Much of the proof is devoted to establishing the claim that if $1 \leq i \leq c+1$, then $B_i =$

$$B_{i-1} \cup \left\{ \begin{array}{ll} (ch-lh-i+l+1)a+jd : & -1 \leq l \leq i-2, \\ & (c-l-1)k < j \leq \min\{(c-l)k, a-1\} \end{array} \right\}.$$

The claim holds in the case $i = 1$ by Lemma 2.7. We now proceed by induction on $i \geq 1$.

Let $1 \leq i \leq c+1$ and suppose the claim holds for all $1 \leq t \leq i-1$. Then $B_t =$

$$B_{t-1} \cup \left\{ \begin{array}{ll} (ch-lh-t+l+1)a+jd & : -1 \leq l \leq t-2, \\ & (c-l-1)k < j \leq \min\{(c-l)k, a-1\} \end{array} \right\}$$

for $1 \leq t \leq i-1$; that is, $B_t =$

$$S \cup \left\{ \begin{array}{ll} (ch-lh-m+l+1)a+jd & : 1 \leq m \leq t,\ -1 \leq l \leq m-2, \\ & (c-l-1)k < j \leq \min\{(c-l)k, a-1\} \end{array} \right\}$$

for $1 \leq t \leq i-1$. In addition, $g(B_{i-1}) = (ch+h-i)a + (a-1)d$, according to Proposition 2.1(a). Recall that $B_{i-1}(a) = \{s \in B_{i-1} : s-a \notin B_{i-1}\}$. By the induction hypothesis, $B_{i-1}(a) = \{s \in S : s-a \notin B_{i-1}\} \cup$

$$\left\{ \begin{array}{ll} (ch-lh-(i-1)+l+1)a+jd & : -1 \leq l \leq i-3, \\ & (c-l-1)k < j \leq \min\{(c-l)k, a-1\} \end{array} \right\}.$$

Then Proposition 2.6 implies $B_i \setminus B_{i-1} \subseteq \{s-a : s \in S, s-a \notin B_{i-1}\} \cup$

$$\left\{ \begin{array}{ll} (ch-lh-i+l+1)a+jd & : -1 \leq l \leq i-3, \\ & (c-l-1)k < j \leq \min\{(c-l)k, a-1\} \end{array} \right\}.$$

Let $x := (ch-lh-i+l+1)a+jd$, where $-1 \leq l \leq i-3$ and $(c-l-1)k < j \leq \min\{(c-l)k, a-1\}$. To show that $x \in B_i$, we will prove that $x+a$ is maximal in $B_{i-1}(a)$ with respect to the partial order $\leq_{B_{i-1}}$. According to [6, Lemma 6], it suffices to show that $(x+a)+b \notin B_{i-1}(a)$ for any $b \in B_{i-1}(a) \setminus \{0\}$; that is, it suffices to show that $x+b = ((x+a)+b)-a \in B_{i-1}$ for all $b \in B_{i-1}(a) \setminus \{0\}$. Clearly, $x+a = (ch-lh-(i-1)+l+1)a+jd \in B_{i-1}$. For $1 \leq n \leq k$, $x+ha+nd \in B_{i-1}$ since $(c-l-1)k+1 < j+n \leq \min\{(c-l)k, a-1\}+k$ and $x+ha+nd$ can be expressed as

$$x+ha+nd = (ch-lh-(i-1)+l+1)a+(j+n)d+(h-1)a \in B_{i-1}$$

if $(c-l-1)k+1 < j+n \leq (c-l)k$ and as

$$x+ha+nd = (ch-(l-1)h-(i-1)+(l-1)+1)a+(j+n)d \in B_{i-1}$$

if $(c-l)k < j+n \leq (c-(l-1))k$. This leads to $x+(S \setminus \{0\}) \subseteq B_{i-1}$. To see that $x+(B_{i-1}(a) \setminus S) \subseteq B_{i-1}$, let

$$y := (ch-ph-(i-1)+p+1)a+wd,$$

where $-1 \leq p \leq i-3$ and $(c-p-1)k < w \leq \min\{(c-p)k, a-1\}$. If $c \geq l+p+3$, then Proposition 2.1(a) gives

$$x+y \geq g(B_{i-1})+((h-1)a+kd)(c-(l+p+1))+(c+2-i)a-(k+r+1)d > g(B_{i-1}).$$

Hence, $x+y \in B_{i-1}$. Thus, we may assume that $c-2 \leq l+p$. Write $l+p = c-2+v = i-3+u+v$ for some $0 \leq v \leq i-3$ and $u \geq 0$. Notice that $(c-v)k < j+w \leq (c-v+2)k$. Then

$$x+y = (ch-(v-1)h-(i-1)+(v-1)+1)a+(j+w)d+(h+u-1)a \in B_{i-1}$$

if $(c-v)k < j+w \leq (c-v+1)k = \min\{(c-v+1)k, a-1\}$, and

$$x+y = (ch-(v-2)h-(i-1)+(v-2)+1)a+(j+w)d+ua \in B_{i-1}$$

if $(c-v+1)k < j+w \leq (c-v+2))k$ and $v \neq 0$. If $(c-v+1)k < j+w \leq (c-(v+1))k$ and $v=0$, then $x+y > g(B_{i-1})$ as $j+w \geq a-1$. This shows that $x+y \in B_{i-1}$. It follows that $x+b \in B_{i-1}$ for all $b \in B_{i-1}(a) \setminus \{0\}$, which implies that $x+a$ is maximal in $B_{i-1}(a)$. Therefore, $x \in B_i \setminus B_{i-1}$.

Next, suppose $s-a \in B_i \setminus B_{i-1}$ where $s \in S$. Since $s-a \notin B_{i-1}$, we certainly have $s-a \notin B_1$. Thus, there exists $ha+nd \in S$, with $1 \leq n \leq k$, such that $s-a+ha+nd \in B_{i-1} \setminus S$. By the induction hypothesis,

$$s-a+ha+nd = (ch-lh-m+l+1)a+jd$$

where $1 \leq m \leq i-1$, $-1 \leq l \leq m-2$, and $(c-l-1)k < j \leq \min\{(c-l)k, a-1\}$. Thus,

$$s-a = (ch-(l+1)h-(m+1)+(l+1)+1)a+(j-n)d$$

with $(c-l-2)k < j-n < \min\{(c-l)k, a-1\}$. If $-1 \leq l+1 \leq i-3$ and $(c-l-2)k = (c-(l+1)-1)k < j-n \leq \min\{(c-(l+1))k, a-1\}$, then $s-a \in B_i$ by the argument above. Similar calculations lead to $s-a \in B_i$ in the case $l+1 = i-2$ and $(c-l-2)k = (c-(l+1)-1)k < j-n \leq \min\{(c-(l+1))k, a-1\}$.

To complete the proof of the claim, it remains to show that if $(c-l-1)k < j-n \leq \min\{(c-l)k, a-1\}$ then $s-a \notin B_i \setminus B_{i-1}$. By definition,

$$s = s-a+a = (ch-lh-h-m+l+2)a+(j-n)d \in S.$$

By Proposition 2.5, $0 < j-n < a$ implies $0 \leq j-n \leq \lfloor \frac{ch-lh-h-m+l+2}{h} \rfloor k = ((c-l-1)-\lfloor \frac{m-l+2}{h} \rfloor)k$. Since $l \leq m+2$, this contradicts the fact that $(c-l-1)k < j-n \leq \min\{(c-l)k, a-1\}$. Therefore, $s-a \notin B_i \setminus B_{i-1}$. This completes the proof of the claim.

Since the claim holds, we have that $B_{c+1} = B_c \cup$

$$\left\{ \begin{array}{ll} (ch-lh-(c+1)+l+1)a+jd & : -1 \leq l \leq c-1, \\ & (c-l-1)k < j \leq \min\{(c-l)k, a-1\} \end{array} \right\}.$$

Taking $l = c-1$ above gives $\{a, (h-1)a+d, (h-1)a+2d, \ldots, (h-1)a+kd\} \subseteq B_{c+1}$. Then Proposition 2.1(c) implies

$$L_1 = \langle a, (h-1)a+d, (h-1)a+2d, \ldots, (h-1)a+kd \rangle \subseteq B_{c+1}.$$

Consider $z := (ch-lh-m+l+1)a+jd$, where $1 \leq m \leq c+1$, $-1 \leq l \leq m-2$, and $(c-l-1)k < j \leq \min\{(c-l)k, a-1\}$. Write $m = c+1-t$ so that $0 \leq t \leq c$. Then $z = ((h-1)(c-l)+t)a+jd \in L_1$ by Proposition 2.5. Therefore,

$$B_{c+1} = L_1 = \langle a, (h-1)a+d, (h-1)a+2d, \ldots, (h-1)a+kd \rangle \, .$$

As a result, $B_i \subseteq L_i$ for all $0 \leq i \leq c+1$.

Since $B_{c+1} = L_1$ is generated by a generalized arithmetic sequence, it now follows by induction that

$$B_{j(c+1)} = \langle a, (h-j)a+d, (h-j)a+2d, \ldots, (h-j)a+kd \rangle = L_j$$

for $0 \le j \le h$, and $B_i \subseteq L_i$ for $0 \le i \le h(c+1)$. In particular, we have that

$$B_{h(c+1)} = L_h = \langle a, d \rangle .$$

Then, since $B_{h(c+1)}$ is doubly-generated, Proposition 2.2 implies that $B_i \subseteq L_i$ for all $i \ge 0$. $\square$

While we have shown that if S is a numerical semigroup generated by a generalized arithmetic sequence then $B_i(S) \subseteq L_i(S)$ for all $i \ge 0$, this is not the case for all semigroups generated by almost arithmetic sequences. We conclude with an example to illustrate this.

Example 2.10. Consider the numerical semigroup $S := \langle 5, 7, 11, 13 \rangle$. Notice that S is generated by an almost arithmetic sequence since $5, 7, 9, 11, 13$ is an arithmetic sequence. We have that $S = \{0, 5, 7, 10 \rightarrow\}$ where the symbol "$\rightarrow$" indicates that all integers greater than 10 are elements of S. Then $B_1(S) = \{0, 5 \rightarrow\}$ and $B_2(S) = \mathbb{N}_0$. However, $L_1(S) = \langle 5, 2, 6, 8 \rangle = \langle 2, 5 \rangle$ and $L_2(S) = \langle 2, 3 \rangle$. Therefore, $B_2(S) = \mathbb{N}_0 \not\subseteq \langle 2, 3 \rangle = L_2(S)$.

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