DISTANCE k COLORINGS OF HAMMING GRAPHS

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Abstract. A coloring of the vertices of a graph G is a *distance k coloring* of G if and only if any two vertices lying on a path of length less than or equal to k are given different colors. Hamming graphs are Cartesian (or box) products of complete graphs. In this paper, we will consider the interaction between coding theory and distance k colorings of Hamming graphs.

1. Introduction

In this paper, we consider two types of powers of graphs. First, there is the Cartesian (or box) product of a graph with itself some number n of times. The vertices of the *Cartesian power* are n-tuples of vertices of G. Second, there is a notion of power arising from distance. The kth *distance power* of a graph G has the same vertex set as G but with all pairs of vertices at a distance k or less in G joined by an edge in the kth power. A coloring of the vertices of a graph G is a *distance k coloring* of G if and only if any two (distinct) vertices lying on a path of length less than or equal to k are given different colors. Notice that a distance k coloring of G is just a proper coloring of its kth distance power.

The *Hamming graphs* are Cartesian powers of complete graphs. Adapting the notational style of finite geometries (of which the Hamming graphs are generalizations), we will let $H(q,n)$ denote the n-fold Cartesian product of a complete graph K_q with itself. Note that a vertex of $H(q,n)$ is just an n-tuple whose entries come from a fixed set of q symbols – the vertices of K_q. In other words, $H(q,n)$ is the set of all words of length n from a q-ary alphabet. Two words are adjacent if and only if they differ in exactly one place. Two words are at distance k (the *Hamming distance*) if and only if they differ in exactly k places.

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The k-fold distance power of $H(q,n)$ will be denoted by $H^k(q,n)$. Notice that the vertices of $H^k(q,n)$ are again words of length n from a q-ary alphabet. However, in $H^k(q,n)$ two words are adjacent if and only if they differ in at most k places. Distance k colorings of $H(q,n)$ correspond to proper colorings of $H^k(q,n)$.

The minimum number of colors required by a distance k coloring of $H(q,n)$ may thus be denoted by $\chi\left(H^k(q,n)\right)$. Distance k colorings in general graphs are motivated by the channel assignment problem [7] and the study of the scalability of optical networks [13, 16]. Much work has been devoted to finding or obtaining bounds on the distance k chromatic number [2, 3, 4, 5, 6, 11, 12, 16, 17].

In this paper, we are interested in bounding and, whenever possible, finding $\chi\left(H^k(q,n)\right)$. When $q = 2$, the Hamming graphs $H(2,n)$ are precisely the well-known hypercubes. We recall some earlier results on hypercubes proved in [8].

Proposition 1.1. [8]

(1) *Given any integer* n,
$$n+1 \leq \chi\left(H^2(2,n)\right) \leq 2n+1.$$
(2) *If* $n = 2^r - 1$ *for some integer* $r > 1$, *then* $\chi\left(H^2(2,n)\right) = n+1$.
(3) *The distance* 2 *chromatic number of the* n*-dimensional cube* $H(2,n)$ *satisfies*
$$\chi\left(H^2(2,n)\right) \geq \left\lceil \frac{2^n}{\left\lfloor \frac{2^n}{n+1} \right\rfloor} \right\rceil = \begin{cases} n+1 & \text{if } n+1 \text{ divides } 2^n \\ n+2 & \text{otherwise.} \end{cases}$$

Our goal is to obtain analogues for these results for more general q and k.

This paper is organized as follows. In Section 2, we use notions from coding theory to bound $\chi\left(H^k(q,n)\right)$. An analysis of one of these bounds for $2 \leq k \leq 5$ is given in Section 3. In Section 4, we provide some exact values for $\chi\left(H^k(q,n)\right)$ in the case where k is either small or large with respect to n.

2. Bounds from coding theory

In this section, we use some ideas from the theory of error-correcting codes to provide bounds on the distance k chromatic number of Hamming graphs. This approach has been employed by several authors (see, for example, [4, 5, 11, 12, 17]). We begin by recalling the necessary terminology.

Let $\mathbb{F}$ denote a finite alphabet with q elements. An $(n, \geq d)_q$-code is a subset of $\mathbb{F}^n$ in which any two words differ in at least d positions. If an $(n, \geq d)_q$-code contains two words which differ in exactly d positions, it is called an $(n, d)_q$-code. Let $A_q(n, d)$ denote the maximum size of an $(n, d)_q$-code. If q is a power of a prime number we often take $\mathbb{F}$ to be the finite field with q elements. In this case, we can consider codes which are subspaces of $\mathbb{F}^n$; such codes are said to be linear. If an $(n, d)_q$-code is linear, then it is called an $[n, d]_q$-code. Let $B_q(n, d)$ denote the maximum size of an $[n, d]_q$-code.

First, we use the maximum size of an $(n, d)_q$-code to obtain a lower bound on $\chi\left(H^{d-1}(q, n)\right)$. Suppose that ϕ is a distance $d-1$ coloring of $H(q, n)$ with color classes $S_1, \ldots, S_m$. Then each color class S_i is a $(n, \geq d)_q$-code with $|S_i|$ words. Moreover,

$$q^n = |V(H(q,n))| = \sum_{i=1}^{m} |S_i| \leq \sum_{i=1}^{m} A_q(n, \geq d) \leq m A_q(n, d).$$

As a result,

$$m \geq \frac{q^n}{A_q(n,d)} \quad \text{and} \quad \chi\left(H^{d-1}(q,n)\right) \geq \frac{q^n}{A_q(n,d)}.$$

Next we note that any upper bound on $A_q(n, d)$ provides a lower bound on $\chi\left(H^{d-1}(q, n)\right)$. As an illustration, we consider the sphere packing bound which states that

$$A_q(n,d) \leq \left\lfloor \frac{q^n}{\sum_{i=1}^{t} \binom{n}{i} (q-1)^i} \right\rfloor$$

where $t := \left\lfloor \frac{d-1}{2} \right\rfloor$. It follows that

$$\chi\left(H^{d-1}(q,n)\right) \geq \left\lceil \frac{q^n}{A_q(n,d)} \right\rceil \geq \left\lceil \frac{q^n}{\left\lfloor \frac{q^n}{\sum_{i=1}^{t} \binom{n}{i}(q-1)^i} \right\rfloor} \right\rceil \geq \sum_{i=1}^{t} \binom{n}{i} (q-1)^i.$$

We will revisit this bound for small values of d in Section 3.

Finally, we use the maximum size of an $[n, d]_q$-code to obtain an upper bound on $\chi\left(H^{d-1}(q, n)\right)$ where q is a prime power. Suppose that C is a $[n, d]_q$-code with q^k words. Then C has q^{n-k} cosets $c_1 + C, \ldots, c_{q^{n-k}} + C$ where $c_i \in \mathbb{F}^n$. Assign color i to the coset $c_i + C$. This gives a distance $d-1$ coloring of $H(q, n)$ with q^{n-k} colors. Therefore,

$$\chi\left(H^{d-1}(q,n)\right) \leq \frac{q^n}{B_q(n,d)}.$$

As a result, we see that if q is a prime power, then

$$\left\lceil \frac{q^n}{A_q(n,d)} \right\rceil \leq \chi\left(H^{d-1}(q,n)\right) \leq \frac{q^n}{B_q(n,d)}. \tag{1}$$

3. Analysis of the sphere packing bound for distance 2-5 chromatic numbers

In Section 2, we saw that the sphere packing bound gives a lower bound on the distance k chromatic number of a Hamming graph. In this section, we consider this bound for $2 \leq k \leq 5$.

For convenience, we define the *sphere packing bound* on the distance k chromatic numbers of $H(q,n)$ to be

$$SPB_k(q,n) := \left\lceil \frac{q^n}{\left\lfloor \frac{q^n}{\sum_{i=1}^{t} \binom{n}{i}(q-1)^i} \right\rfloor} \right\rceil \geq \sum_{i=1}^{t} \binom{n}{i} (q-1)^i$$

where $t := \left\lfloor \frac{k}{2} \right\rfloor$. So we have

$$SPB_2(q,n) = SPB_3(q,n) = \left\lceil \frac{q^n}{\left\lfloor \frac{q^n}{n(q-1)+1} \right\rfloor} \right\rceil$$

and

$$SPB_4(q,n) = SPB_5(q,n) = \left\lceil \frac{q^n}{\left\lfloor \frac{q^n}{\binom{n}{2}(q-1)^2+n(q-1)+1} \right\rfloor} \right\rceil.$$

We first look at the sphere packing bound on the distance 2 chromatic number (equivalently, on the distance 3 chromatic number). Though the next result follows from the material in Section 2, we provide an alternate proof here.

Lemma 3.1. *The distance 2 chromatic number of the Hamming graph $H(q,n)$ satisfies $\chi\left(H^2(q,n)\right) \geq SPB_2(q,n)$.*

Proof. Clearly,

$$\chi\left(H^2(q,n)\right) \geq \frac{q^n}{\alpha(H^2(q,n))}$$

where $\alpha\left(H^2(q,n)\right)$ denotes the independence number of $H^2(q,n)$. Any point in $H(q,n)$ has a closed neighborhood with $n(q-1)+1$ points. Thus

$\alpha(H^2(q,n)) \leq \frac{q^n}{n(q-1)+1}$. Since $\alpha(H^2(q,n))$ is an integer, this gives

$$\alpha(H^2(q,n)) \leq \left\lfloor \frac{q^n}{n(q-1)+1} \right\rfloor .$$

It follows that

$$\chi\left(H^2(q,n)\right) \geq \frac{q^n}{\left\lfloor \frac{q^n}{n(q-1)+1} \right\rfloor}.$$

Since $\chi\left(H^2(q,n)\right)$ is an integer, the desired bound is obtained. ∎

We wish to obtain a nicer expression for $SPB_2(q,n)$ in terms of n and q. To do this, let $\Re$ be the set of integral lattice points (q,n) of the following forms: $q \geq 2$ and $n \geq 8$; and, $q \geq 9$ and $n \geq 3$.

Lemma 3.2. *For all pairs of integers $(q,n) \in \Re$, the bound*

$$(n(q-1)+1)(n(q-1)+2) \leq q^n$$

holds.

Proof. We have

$$\begin{aligned}
(n(q-1)+1)(n(q-1)+2) &= n^2(q-1)^2 + 3n(q-1) + 2 \\
&= n^2q^2 - 2n^2q + n^2 + 3n(q-1) + 2 \\
&= n^2q^2 + (1-2q)n^2 + 3nq - 3n + 2 \\
&= n^2q^2 + (1-q)n^2 + (3-n)nq \\
&\quad +(2-3n) \\
&< n^2q^2.
\end{aligned}$$

The final inequality holds since $q > 1$ and $n \geq 3$. Thus the next goal is to show that $n^2q^2 \leq q^n$ for $(q,n) \in \Re$. Dividing by q^2 and taking the square root, one sees that this is equivalent to $n \leq q^{n/2-1}$.

Consider the function

$$f(x) := q^{x/2-1} - x$$

where q is a fixed integer. Its derivative is

$$f'(x) = \frac{q^{\frac{x}{2}-1} ln(q)}{2} - 1$$

which is obviously increasing. There are two cases to consider.

- For $q \geq 2$ and $x \geq 8$, we get $f'(8) = \frac{q^3 ln(2)}{2} - 1 \geq 2^3(0.3657) - 1 > 0$, so f is increasing for $x \geq 8$. Also, $f(8) = q^3 - 8 \geq 2^3 - 8 = 0$. Since f is increasing when $x \geq 8$, f is positive for all $x \geq 8$. Hence when $n \geq 8$ and $q \geq 2$, we have $q^{\frac{n}{2}-1} > n$ as desired.

- For $q \geq 9$ and $x \geq 3$, we have
$$f'(3) = \frac{q^{\frac{1}{2}} ln(q)}{2} - 1 \geq \frac{\sqrt{9} ln(9)}{2} - 1 > 3.2958 - 1 > 0.$$
Thus, f is increasing for $n \geq 3$. Moreover,
$$f(3) = q^{\frac{1}{2}} - n \geq \sqrt{9} - 3 = 0$$
when $q \geq 9$. Thus, $q^{\frac{1}{2}-1} > n$ when $n \geq 3$ and $q \geq 9$.

■

Using Lemma 3.2, we obtain a neater expression for $SPB_2(q, n)$ for those pairs $(q, n) \in \Re$.

Proposition 3.3. *For $(q, n) \in \Re$,*
$$SPB_2(q, n) = \begin{cases} n(q-1)+1 & \text{if } n(q-1)+1 \text{ divides } q^n \\ n(q-1)+2 & \text{otherwise.} \end{cases}$$

Proof. Set $l := n(q-1)+1$. Then there are integers r and s such that $q^n = ls + r$ and $0 \leq r < l$. It follows that
$$SPB_2(q, n) = n(q-1) + 1 + \left\lceil \frac{r}{s} \right\rceil.$$
We must show that $r \leq s$. To see this, note that $l \leq s$. Otherwise, Lemma 3.2 gives $q^n = ls + r < l^2 + l = l(l+1) \leq q^n$. ■

Proposition 3.3 yields a bound on the distance 2 chromatic number of the Hamming graph $H(q, n)$.

Theorem 3.4. *If $n(q-1)+1$ divides q^n, then*
$$\chi\left(H^2(q, n)\right) \geq n(q-1)+1.$$
Otherwise,
$$\chi\left(H^2(q, n)\right) \geq n(q-1)+2.$$
Moreover, $\chi\left(H^2(3, 3)\right) \geq 9$.

Proof. For $(q, n) \in \Re$, the result follows from Proposition 3.3. The remaining pairs were checked using Magma [1]. ■

We conclude our discussion of the sphere packing bound on the distance 2 chromatic number by noting that by taking $q = 2$ in Theorem 3.4, we recover Proposition 1.1 (2), (3).

Next, we provide a similar analysis for the sphere packing bound on the distance 4 chromatic number (equivalently, on the distance 5 chromatic number). Let $\mathcal{L}$ denote the set of integral lattice points (q,n) of the following forms: $q \geq 2$ and $n \geq 13$; $q \geq 3$ and $n \geq 10$; $q \geq 4$ and $n \geq 8$; $q \geq 7$ and $n \geq 7$; and, $q \geq 29$ and $n \geq 6$.

Proposition 3.5. *For* $(q,n) \in \mathcal{L}$,

$$SPB_4(q,n) = \begin{cases} \binom{n}{2}(q-1)^2 + n(q-1) + 1 & \text{if } \binom{n}{2}(q-1)^2 + n(q-1) + 1 \\ & \text{divides } q^n \\ \binom{n}{2}(q-1)^2 + n(q-1) + 2 & \text{otherwise.} \end{cases}$$

Proof. Set $l := \binom{n}{2}(q-1)^2 + n(q-1) + 1$. Then there are integers r and s such that $q^n = ls + r$ and $0 \leq r \leq l-1$. It follows that

$$SPB_4(q,n) = l + \left\lceil \frac{r}{s} \right\rceil.$$

We must show that $r \leq s$. To do this, we will prove that $l - 1 \leq s$.

Suppose that $l - 2 \geq s$. Then $q^n \leq l(l-1) - 1$. Notice that

$$\begin{aligned} l(l-1) &= n(q-1) + (n^2 + \tbinom{n}{2})(q-1) + 2n\tbinom{n}{2}(q-1)^3 + \tbinom{n}{2}^2(q-1)^4 \\ &\leq nq + \tfrac{3}{2}n^2q^2 + n^3q^3 + \tfrac{1}{4}n^4q^4 \\ &\leq \left(\tfrac{1}{64} + \tfrac{3}{32} + \tfrac{1}{4} + \tfrac{1}{4}\right) n^4q^4 \\ &= \tfrac{39}{64}n^4q^4 \end{aligned}$$

for $(q,n) \in \mathcal{L}$ since $n \geq 2$ and $q \geq 2$.

We claim that $\frac{39}{64}n^4q^4 \leq q^n$; that is, we claim that $q^n - \frac{39}{64}n^4q^4 \geq 0$. To see this, we use the same ideas as in the proof of Lemma 3.2. Consider the function

$$f(n) := q^n - \frac{39}{64}n^4q^4$$

where q is a fixed integer. Note that

$$f'(n) = \frac{ln(q) \cdot q^{\frac{n}{4}-1}}{4} - \sqrt[4]{\frac{39}{64}}$$

which is increasing (in n). If $q = 2$, then $f(21) > 0$ and $f'(21) > 0$. Hence, the claim holds for pairs $(2,n)$ with $n \geq 21$. Calculations using Magma [1] show that $r \leq s$ for the pairs $(2,n) \in \mathcal{L}$ with $n < 21$. For $q = 3$, we can see that $f(13) > 0$ and $f'(13) > 0$. This, together with [1], proves the claim for $(3,n) \in \mathcal{L}$. Similarly, it can be checked the claim holds for all remaining $(q,n) \in \mathcal{L}$.

Now we have that

$$q^n \leq l(l-1) - 1 < \frac{39}{64}n^4q^4 \leq q^n$$

for all pairs $(q,n) \in \mathcal{L}$ which is a contradiction. Therefore, for $(q,n) \in \mathcal{L}$,

$$SPB_4(q,n) = l + \left\lceil \frac{r}{s} \right\rceil \leq l+1.$$

■

Proposition 3.5 gives a lower bound on the distance 4 chromatic number of the Hamming graph $H(q,n)$.

Theorem 3.6. *Suppose that $n \geq 13$. If $\binom{n}{2}(q-1)^2 + n(q-1) + 1$ divides q^n, then*

$$\chi\left(H^4(q,n)\right) \geq \binom{n}{2}(q-1)^2 + n(q-1) + 1.$$

Otherwise,

$$\chi\left(H^4(q,n)\right) \geq \binom{n}{2}(q-1)^2 + n(q-1) + 2.$$

Moreover, this result holds in the following cases: $q \geq 3$ and $n \geq 10$; $q \geq 4$ and $n \geq 8$; $q \geq 7$ and $n \geq 7$; $q \geq 29$ and $n \geq 6$.

4. Some exact values for the distance k chromatic number

In this section, we use the ideas in Sections 2 and 3 to find the exact value of $\chi\left(H^k(q,n)\right)$ for certain pairs (q,n). We focus on large and small values of k.

We begin with the trivial observation that $\chi\left(H^n(q,n)\right) = q^n$. Next we consider $\chi\left(H^{n-1}(q,n)\right)$. By the same argument as in the proof of Lemma 3.1,

$$\chi\left(H^{n-1}(q,n)\right) \geq \frac{q^n}{\alpha\left(H^{n-1}(q,n)\right)}$$

where $\alpha(G)$ denotes the independence number of G. We claim that

$$\alpha\left(H^{n-1}(q,n)\right) = q.$$

Suppose S is an independent set in $H^{n-1}(q,n)$ containing $(a,\ldots,a)$ where $a \in \mathbb{F}$. Since any other vertex in S must differ from $(a,\ldots,a)$ in all n positions, we may assume that $(b,\ldots,b)$ in S for some $b \in \mathbb{F} \setminus \{a\}$. It follows that $S = \{(a,\ldots,a) : a \in \mathbb{F}\}$. As a result

$$\chi\left(H^{n-1}(q,n)\right) \geq \frac{q^n}{q} = q^{n-1}.$$

Noting that S is the $[n,n]_q$-repetition code, we see that

$$\chi\left(H^{n-1}(q,n)\right) = q^{n-1}.$$

A similar argument shows that

$$\chi\left(H^{n-2}(q,n)\right) = q^{n-1}.$$

Next, we consider small values of k. In [14, 15], it is shown that there are two families of codes with parameters meeting the sphere packing bound: the Hamming codes and the Golay codes. Both are families of linear codes and so can be used to provide distance k coloring. For more details on these codes as well as others mentioned here, see [10].

Theorem 4.1. *If* $n = \frac{q^r-1}{q-1}$, *then*

$$\chi\left(H^2(q,n)\right) = n(q-1)+1.$$

Proof. Set $n = \frac{q^r-1}{q-1}$. Then there is a $[n,3]_q$ Hamming code with q^{n-r} words. The result now follows from Equation (1). ∎

Using the $[23,7]_2$ and $[11,5]_3$ Golay codes with 2^{12} and 3^6 words respectively we see that

$$\chi\left(H^2(2,23)\right) = 2^{11} \text{ and } \chi\left(H^4(3,11)\right) = 3^5.$$

As a final application, we consider the Nordstrom-Robinson code. The Nordstrom-Robinson code is a $(16,6)_2$-code with 2^8 words. By the bound in Equation (1),

$$2^8 = \left\lceil \frac{2^{16}}{A_2(16,6)} \right\rceil \leq \chi\left(H^5(2,16)\right) \leq \frac{2^{16}}{B_2(16,6)} = 2^9.$$

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