On comparing two chains of numerical semigroups and detecting Arf semigroups

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Abstract

If T is a numerical semigroup with maximal ideal N, define associated semigroups $B(T) := (N - N)$ and $L(T) = \cup\{(hN - hN) : h \geq 1\}$. If S is a numerical semigroup, define strictly increasing finite sequences $\{B_i(S) : 0 \leq i \leq \beta(S)\}$ and $\{L_i(S) : 0 \leq i \leq \lambda(S)\}$ of semigroups by $B_0(S) := S =: L_0(S)$, $B_{\beta(S)}(S) := \mathbb{N} =: L_{\lambda(S)}(S)$, $B_{i+1}(S) := B(B_i(S))$ for $0 < i < \beta(S)$, $L_{i+1}(S) := L(L_i(S))$ for $0 < i < \lambda(S)$. It is shown, contrary to recent claims and conjectures, that $B_2(S)$ need not be a subset of $L_2(S)$ and that $\beta(S) - \lambda(S)$ can be any preassigned integer. On the other hand, $B_2(S) \subseteq L_2(S)$ in each of the following cases: S is symmetric; S has maximal embedding dimension; S has embedding dimension $e(S) \leq 3$. Moreover, if either $e(S) = 2$ or S is pseudo-symmetric of maximal embedding dimension, then $B_i(S) \subseteq L_i(S)$ for each i, $0 \leq i \leq \lambda(S)$. For each integer $n \geq 2$, an example is given of a (necessarily non-Arf) semigroup S such that $\beta(S) = \lambda(S) = n$, $B_i(S) = L_i(S)$ for all $0 \leq i \leq n-2$, and $B_{n-1}(S) \subsetneqq L_{n-1}(S)$.

1 Introduction

All semigroups considered below are *numerical semigroups*, that is, submonoids of the natural numbers $\mathbb{N} := \{0, 1, 2, 3, \dots\}$ under addition. We adopt the conventions of [4] and [2]. In particular, except for the degenerate cases $S = \mathbb{N}$ and $S = \{0\}$, we have a *canonical form* description of a semigroup $S = \langle a_1, \dots, a_\nu \rangle$ so that $\nu \geq 2$; $a_i < a_{i+1}$ for $1 \leq i \leq \nu - 1$; $\mathrm{GCD}(a_1, \dots, a_\nu) = 1$ (equivalently, $\mathbb{N} \setminus S$ is finite); and $a_i \notin \langle \{a_j : 1 \leq j \leq \nu, j \neq i\} \rangle$ for $1 \leq i \leq \nu$. Given such a generating set of S, we call ν the *embedding dimension* of S and denote it by $e(S)$; and we call a_1, the least positive element of S, the *multiplicity* of S and denote it by $\mu(S)$. In general, $e(S) \leq \mu(S)$. Besides $\mathbb{N}$, semigroups S satisfying $e(S) = \mu(S)$ are said to be of *maximal embedding dimension.* Their role in characterizing a class of Noetherian integral domains of maximal embedding dimension is given in [2, Proposition II.2.10].

Let S be a semigroup (other than $\mathbb{N}$, $\{0\}$). The *maximal ideal* of S is $M(S) := S \setminus \{0\}$. The largest element of $\mathbb{N} \setminus M(S)$ is called the *Frobenius number* of S and is denoted by $g(S)$. For instance, if $S := \langle 4, 7 \rangle = \{0, 4, 7, 8, 11, 12, 14, 15, 16, 18, \rightarrow\}$ (where the symbol "$\rightarrow$" means that all subsequent natural numbers belong to S), then $g(S) = 17$. More generally, given any doubly-generated semigroup $S = \langle a, b \rangle$ in canonical form, one knows that $g(S) = ab - a - b$ (cf. [3]). It is also known [3] that each doubly-generated semigroup S is a *symmetric semigroup*, in the sense that $s \mapsto g(S) - s$ determines a bijection $S \cap \{0, 1, \dots, g(S)\} \rightarrow (\mathbb{N} \setminus S) \cap \{0, 1, \dots, g(S)\}$. Evidently, each symmetric semigroup has odd Frobenius number. Several characterizations of symmetric semigroups are given in [2, Lemma I.1.8], including their maximality among semigroups having a given odd number as Frobenius number. The role of symmetric semigroups in characterizing a class of Gorenstein domains is due to Kunz (cf. [2, Proposition II.1.1(b)]). In case of even Frobenius number, analogous considerations of maximality lead to the class of pseudo-symmetric semigroups characterized in [2, Lemma I.1.9]; these serve to characterize the so-called Kunz domains [2, Proposition II.1.12].

Let S be a semigroup with maximal ideal M. It is useful to consider the associated semigroups $B(S) := (M - M) = \{x \in \mathbb{N} : x + M \subseteq M\}$ and $L(S) := \cup_{h=1}^{\infty}(hM - hM)$. In honor of [5], $L(S)$ is called the *Lipman semigroup* of S. Evidently, $B(S) \subseteq L(S)$. In fact, S is of maximal embedding dimension if and only if $B(S) = L(S)$ (see Proposition 2.2(d)).

By iterating the B and L constructions, one arrives at an interesting class of semigroups of maximal embedding dimension called the Arf semigroups. (See [2, Theorem I.3.4] for fifteen characterizations of Arf semigroups and [2, Theorem II.2.13] for their role in characterizing Arf rings, an important class of rings studied in algebraic geometry and commutative algebra: cf. [1], [5].) In general, for any semigroup S, we obtain two ascending chains of semigroups

$$B_0(S) := S \subseteq B_1(S) := B(B_0(S)) \subseteq \ldots \subseteq B_{h+1}(S) := B(B_h(S)) \subseteq \ldots,$$
$$L_0(S) := S \subseteq L_1(S) := L(L_0(S)) \subseteq \ldots \subseteq L_{h+1}(S) := L(L_h(S)) \subseteq \ldots .$$

We say that S is an *Arf semigroup* in case $B_i(S) = L_i(S)$ for each $i \geq 0$. For an arbitrary semigroup S, we define $\beta(S)$ and $\lambda(S)$ to be the least integers such that $B_{\beta(S)}(S) = \mathbb{N} = L_{\lambda(S)}(S)$. Of course, if S is an Arf semigroup, then $\beta(S) = \lambda(S)$.

We come now to the focus of this paper. Since any semigroup S satisfies $B_0(S) = L_0(S)$ and $B_1(S) \subseteq L_1(S)$, it is tempting to conjecture that $B_i(S) \subseteq L_i(S)$ for all $i \geq 0$ and for arbitrary S. Indeed, this conjecture is claimed as fact for $i = 2$ in [2, p. 14]. However, that claim is mistaken, for Example 2.3 produces a semigroup S such that $B_2(S) \nsubseteq L_2(S)$. In that example, $e(S) = 4$. On the other hand, by re-examining the work underlying the erroneous claim in [2], we are led to Theorem 2.4(c): if $e(S) \leq 3$, then $B_2(S) \subseteq L_2(S)$. The same conclusion holds if S is symmetric or of maximal embedding dimension: see Theorem 2.4(a),(b).

In the context of their erroneous claim, the authors of [2] remark that "there is much calculational evidence to suggest that $\lambda(S) \leq \beta(S)$" [2, p. 14]. However, the semigroup S in Example 2.3 shows otherwise, as it satisfies $\beta(S) < \lambda(S)$. Moreover, the speculative inequality fails dramatically, for Proposition 2.5 constructs semigroups S showing that $\beta(S) - \lambda(S)$ can be any integer (positive, negative, or 0).

Let us return to the "tempting" (but false) conjecture that $B_i(S) \subseteq L_i(S)$ for each $i \geq 0$. The success in Theorem 2.4 leads one naturally to ask for classes of semigroups S for which the conjecture has an affirmative answer. We present two such classes, the doubly-generated (hence symmetric) semigroups (see Theorem 2.6) and the pseudo-symmetric semigroups of maximal embedding dimension (see Corollary 2.8).

One may consider a related question, namely, whether a semigroup S is necessarily Arf provided that $B_i(S) \subseteq L_i(S)$ for each $i \geq 0$, with $B_i(S) =$

$L_i(S)$ for "most i". The paper ends on a sobering note, as Example 2.10 gives a negative answer, with examples S for which $\beta(S) = \lambda(S)$ can be any integer ≥ 2.

2 Results

We begin with an elementary result that settles the case $i = 1$.

Proposition 2.1 *(a) $B_1(S) \subseteq L_1(S)$ for each (numerical) semigroup S.*
(b) There exists a semigroup S, for instance $S := \langle 3, 4 \rangle$, such that $B_1(S) \neq L_1(S)$.

Proof. (a) If M denotes the maximal ideal of S, then $B_1(S) = (M - M) = (1M - 1M) \subseteq \cup\{(hM - hM) : h \geq 1\} = L_1(S)$.

(b) Consider the semigroup $S := \langle 3, 4 \rangle = \{0, 3, 4, 6, \rightarrow\}$, with maximal ideal $M = \{3, 4, 6, \rightarrow\}$. Then $B_1(S) = (M - M) = \{0, 3, \rightarrow\} \neq \mathbb{N}$. However, $\mathbb{N} \supseteq L_1(S) \supseteq (2M - 2M) = \mathbb{N}$ since $2M = \{6, \rightarrow\}$. Hence, $L_1(S) = \mathbb{N} \neq B_1(S)$. $\square$

In a sense, the semigroup $S := \langle 3, 4 \rangle$ gives a minimal example of the behavior described in Proposition 2.1(b). Specifically, consider the semigroup $T := \langle 2, h \rangle$, where $h \geq 3$ is any odd positive integer. Then $B_i(T) = L_i(T)$ for *all* i (that is, T is an Arf semigroup), by [2, Theorem I.4.2, (v) $\Rightarrow$ (i)].

The example $S := \langle 3, 4 \rangle$ in Proposition 2.1(b) also serves to motivate a number of later results. Note that $B_2(S) = B(B_1(S)) = B(\{0, 3, \rightarrow\}) = \mathbb{N} = L_1(S) = L_2(S)$. It is shown, more generally, in Theorem 2.6 that for any 2-generated semigroup $T = \langle a_1, a_2 \rangle$ where $a_1 < a_2$ and a_1 is relatively prime to a_2, we have $B_i(T) \subseteq L_i(T)$. Note also that $S = \langle 3, 4 \rangle$ is a symmetric semigroup. Theorem 2.4(a) establishes, in fact, that $B_2(T) \subseteq L_2(T)$ for *any* symmetric semigroup T. However, S is not entirely typical of semigroups T such that $B_2(T) \subseteq L_2(T)$. Indeed, S is not of maximal embedding dimension, since $e(S) = 2 < 3 = \mu(S)$, while Theorem 2.4(b) ensures that $B_2(T) \subseteq L_2(T)$ for each semigroup T of maximal embedding dimension.

It is convenient next to collect some results from [2] that will be used frequently.

Proposition 2.2 *Let $S = \langle a_1, a_2, \ldots, a_\nu \rangle$ be a semigroup (written according to the canonical form conventions in the Introduction). Then:*
(a) $g(B_1(S)) = g(S) - a_1$.
(b) $L(S) = \langle a_1, a_2 - a_1, \ldots, a_\nu - a_1 \rangle$.
(c) If S is symmetric, then $B_1(S) = \langle a_1, a_2, \ldots, a_\nu, g \rangle$.
(d) S is of maximal embedding dimension if and only if $B_1(S) = L_1(S)$.

We next produce, contrary to what was claimed in [2, p. 14], a semigroup S such that $B_2(S) \nsubseteq L_2(S)$. In so doing, we also disprove the conjecture that $\lambda(S) \leq \beta(S)$. We return to the latter matter in Proposition 2.5.

Example 2.3 There exists a semigroup S, for instance $S := \langle 5, 7, 11, 13 \rangle$, such that $e(S) = 4$ and $B_2(S) \nsubseteq L_2(S)$. It can also be arranged that $\beta(S) < \lambda(S)$.

Proof. Since $S := \langle 5, 7, 11, 13 \rangle = \{0, 5, 7, 10, \rightarrow\}$, $B_1(S) = \{0, 5, \rightarrow\}$, so that $B_2(S) = \mathbb{N}$. In particular, $\beta(S) = 2$. On the other hand, it follows from Proposition 2.2(b) that $L_1(S) = L(S) = \langle 5, 2, 6, 8 \rangle = \langle 2, 5 \rangle$, $L_2(S) = L(\langle 2, 5 \rangle) = \langle 2, 3 \rangle \neq \mathbb{N}$, and $L_3(S) = L(\langle 2, 3 \rangle) = \mathbb{N}$. In particular, $B_2(S) = \mathbb{N} \nsubseteq \langle 2, 3 \rangle = L_2(S)$ and $\lambda(S) = 3 > 2 = \beta(S)$. $\square$

The next result includes some assertions that were promised following the proof of Proposition 2.1, while also showing the minimality of the condition "$e(S) = 4$" in Example 2.3.

Theorem 2.4 *Let $S = \langle a_1, \ldots, a_\nu \rangle$ be a semigroup (written according to the canonical form conventions in the Introduction). Then $B_2(S) \subseteq L_2(S)$ in each of the following four cases:*
(a) S is symmetric;
(b) S is of maximal embedding dimension;
(c) $e(S) \leq 3$;
(d) $a_2 > 2a_1$.

Proof. (a) Put $g := g(S)$. Since S is symmetric, $B_1(S) = \langle a_1, \ldots, a_\nu, g \rangle$ by Proposition 2.2(c). We claim that $B_2(S) \subseteq T := \langle a_1, \ldots, a_\nu, g - a_1, \ldots, g - a_\nu \rangle$ (where the generator $g - a_j$ is considered if and only if $a_j < g$). Take

$x \in B_2 \setminus B_1$. Let $y := g - x$. Then $y \in M(S)$ by the symmetry of S. Write $y = \sum_{i=1}^{\nu} c_i a_i$, with $c_i \in \mathbb{N}$. If $\sum_{i=1}^{\nu} c_i > 1$, then $g - a_i \in g - y + M(S) = x + M(S) \subseteq B_1(S)$ for some a_i by definition of $B_2(S)$. This is a contradiction in view of Proposition 2.2(c), since $g - a_i < g$, and $g - a_i \notin S$ is a consequence of the symmetry of S. Therefore, $\sum_{i=1}^{\nu} c_i = 1$ and $x = g - a_j \in T$ for some j. Thus, $B_2 \subseteq T$.

Next we show that $T \subseteq L_2(S)$. Clearly $g - a_1 = g(B_1(S)) \in B(B_1(S)) \subseteq L(B_1(S)) \subseteq L_2(S)$. Let $1 < j \leq \nu$. Then $g - a_j = (g - a_j + a_1) - a_1 \in L_1(S) \subseteq L_2(S)$ since $g - a_j + a_1 \in S$ by the symmetry of S and the irredundancy of the generating set. This gives $B_2(S) \subseteq T \subseteq L_2(S)$, completing the proof of (a).

(b) By Proposition 2.2(d), $B_1(S) = L_1(S)$. Therefore, by Proposition 2.1(a), $B_2(S) = B(B_1(S)) = B_1(L_1(S)) \subseteq L_1(L_1(S)) = L_2(S)$, thus proving (b).

(d) Suppose $x \in B_2(S)$. Then $x + 2a_1 \in S$. Thus, $x = s - 2a_1$ for some $s \in S$. Since $2a_1 < a_2$, iterating the formula in Proposition 2.2(b) gives $L_2(S) = \langle a_1, a_2 - 2a_1, \ldots, a_\nu - 2a_1 \rangle$. Then clearly $x = s - 2a_1 \in L_2(S)$.

(c) Assume that $e(S) \leq 3$. If $a_2 > 2a_1$, the result holds by (d) above. Thus, there is no loss of generality in assuming $a_2 < 2a_1$. Suppose $x \in B_2(S)$ such that $x \notin L_2(S)$. By definition of $B_2(S)$, $x + 2a_1, x + a_1 + a_2 \in B_2 + 2M(S) \in M(S)$. Thus, there exist $c_1, c_2, c_3, d_1, d_2, d_3 \in \mathbb{N}$, such that $x = c_1a_1 + c_2a_2 + c_3a_3 - 2a_1 = d_1a_1 + d_2a_2 + d_3a_3 - a_1 - a_2$ (where $c_3 = d_3 = 0$ in case $e(S) = 2$). Since $x \notin L_2(S)$, it follows from Proposition 2.2(b) that $c_1 + c_2 + c_3 = 1$. If $e(S) = 2$, this forces $x = a_2 - 2a_1 < 0$, which is a contradiction. Thus, we may restrict ourselves to the case $e(S) = 3$. Then, by Proposition 2.2(b), $L_1(S) = \langle a_2 - a_1, a_1, a_3 - a_1 \rangle$ and $L_2(S) = \langle a_2 - a_1, 2a_1 - a_2, a_3 - a_2 \rangle$. These descriptions of $L_1(S)$ and $L_2(S)$ together with the fact that $x \notin L_2(S)$ imply that $c_1 = d_1 = c_2 = d_2 = 0$ and $c_3 = d_3 = 1$; that is, $x = a_3 - 2a_1 = a_3 - a_1 - a_2$. Then $a_1 = a_2$, which is a contradiction. $\square$

By pursuing the calculations figuring in the proof of Theorem 2.4(c), one finds a number of necessary conditions on the generators a_i of any semigroup satisfying $B_2(S) \not\subseteq L_2(S)$. Such considerations led us to discover the semigroup in Example 2.3 (which, by Theorem 2.4(c), is now seen to have minimal embedding dimension). Extending the final assertion of Example 2.3, we next show that the conjecture that $\lambda(S) \leq \beta(S)$ (cf. [2, p. 14]) fails in every possible way.

Proposition 2.5 *For each integer n, there exists a semigroup S such that $\beta(S) - \lambda(S) = n$.*

Proof. We consider first the case $n \leq -1$. Let $a := 3 - 2n$, an odd integer ≥ 5. We shall show that $S := \{0, a, a+2, 2a, \rightarrow\}$ satisfies $\beta(S) - \lambda(S) = n$. It is straightforward to verify that $B_1(S) = \{0, a, \rightarrow\} \neq \mathbb{N}$, and so $B_2(S) = \mathbb{N}$, whence $\beta(S) = 2$. It remains to prove that $\lambda(S) = 2 - n = \frac{a+1}{2}$. Since a is odd, it follows from Proposition 2.2(b) that $L_1(S) = \langle 2, a \rangle$. Applying this formula repeatedly, we obtain that $L_i(S) = \langle 2, a - 2(i-1) \rangle$ for all $1 \leq i \leq \frac{a-1}{2}$. In particular, $L_{\frac{a-1}{2}}(S) = \langle 2, 3 \rangle \neq \mathbb{N}$, and so $L_{\frac{a+1}{2}}(S) = L_1(\langle 2, 3 \rangle) = \mathbb{N}$, whence $\lambda(S) = \frac{a+1}{2}$.

Suppose next that $n \geq 0$. Let $S := \langle n+2, n+3 \rangle$. By Proposition 2.2(b), $L_1(S) = \mathbb{N}$, and so $\lambda(S) = 1$. It remains to show that $\beta(S) = n + 1$. This will follow immediately from the proof of Theorem 2.6. Observe that this can be verified independently as well. □

In view of Theorem 2.4, it seems reasonable to ask if some "naturally occurring" classes of semigroups S satisfy $B_i(S) \subseteq L_i(S)$ for *each* i. Theorem 2.6 identifies such a class of symmetric semigroups. Corollary 2.8 addresses this issue within the universe of pseudo-symmetric semigroups.

Theorem 2.6 *Let S be a semigroup of embedding dimension $e(S) = 2$; that is, $S = \langle a, b \rangle$, where a and b are relatively prime integers such that $2 \leq a < b$. If $i \geq 0$, then $L_i(S) = B_j(S)$ for some $j \geq i$ and, consequently, $B_i(S) \subseteq L_i(S)$ and $\lambda(S) \leq \beta(S)$. Indeed, $L_i(S) = B_{\sum_{j=0}^{i-1}(\mu(L_j(S))-1)}(S)$.*

Proof. It will be convenient here to write B_i and L_i instead of $B_i(S)$ and $L_i(S)$, respectively. Most of the proof is devoted to establishing the claim that $B_{a-1} = L_1$; that is, $B_{a-1} = \langle a, b-a \rangle$. Then the result follows by induction, since either $L_1 = \langle a, b-a \rangle$ has embedding dimension 2 or $L_1 = \mathbb{N}$.

It remains to prove that $B_{a-1} = \langle a, b-a \rangle$. Without loss of generality, $a \geq 3$, for if $a = 2$, then S is an Arf semigroup [2, Theorem I.4.2], in which case the assertions are immediate. Observe that the hypothesis on S ensures that S is a symmetric semigroup with Frobenius number $g = ab - a - b$ [3]. We shall show that if $1 \leq i \leq a-2$, then

$$B_i = \langle a, b, g - (i-1)b, g - a - (i-2)b, g - 2a - (i-3)b, \ldots, g - (i-1)a \rangle .$$

By Proposition 2.2(c), $B_1 = \langle a, b, g\rangle$. This settles the case $i = 1$.

We now proceed by induction on $i \geq 1$. In particular, assume that

$$B_j = \langle a, b, g - (j-1)b, g - a - (j-2)b, g - 2a - (j-3)b, \ldots, g - (j-1)a\rangle\,,$$

for all $j, 0 \leq j \leq i-1$. Suppose for now that $i \leq a-2$. We claim that $g(B_j) = g - ja$ for all $0 \leq j \leq i-1$. Using the fact that $i < a-1$, one can check that $a < b < g - (j-2)b$, the smallest generator of B_{j-1} other than a and b. Thus, $\mu(B_k) = a$ for all $0 \leq k \leq j-1$. By j applications of Proposition 2.2(a), $g(B_j) = g - (\sum_{k=0}^{j-1} \mu(B_k)) = g - ja$.

By the induction hypothesis and Proposition 2.1(b),

$$\begin{aligned} B_{i-1} &= \langle a, b, g-(i-2)b, g-a-(i-3)b, \ldots, g-(i-2)a\rangle \subseteq B_i = B(B_{i-1}) \\ &\subseteq L(B_{i-1}) = \langle a, b-a, g-a-(i-2)b, g-2a-(i-3)b, \ldots, g-(i-1)a\rangle\,. \end{aligned}$$

Let $0 \leq j \leq i-1$. We will show that $g - ja - (i-j-1)b \in B_i$. It is apparent that $g - ja - (i-j-1)b + a \in B_{i-1}$ and $g - ja - (i-j-1)b + b \in B_{i-1}$. It remains to verify that $p := (g - ja - (i-j-1)b) + (g - ka - (i-k-2)b) \in B_{i-1}$ for all $0 \leq k \leq i-2$. Observe that if $j = i-1$ or $k = i-2$, then $p > g(B_{i-1})$ and so $p \in B_{i-1}$. Thus, we may restrict ourselves to the case $0 \leq j \leq i-2$ and $0 \leq k \leq i-3$. First, suppose that $0 \leq j+k \leq i-3$. Then

$$\begin{aligned} p &= (g - ja - (i-j-1)b) + (g - ka - (i-k-2)b) \\ &= (g - (j+k+1)a - (i - (j+k+1) - 2)b) + (a-i-1)b \\ &\in B_{i-1} + M(S) \subseteq B_{i-1}. \end{aligned}$$

In the remaining case, $i-2 \leq j+k \leq 2i-5$. Write $j+k = i-2+r$, $0 \leq r \leq i-3$. Then

$$\begin{aligned} p &= g - (j+k)a + g - (2i - (j+k) - 3)b \\ &= (g - (i-2)a) + (g - ra - (i-r-1)b) > g(B_{i-2}) \end{aligned}$$

since $g - ra - (i-r-1)b = (a-(i+1))b + (r+1)(b-a) > 0$. Therefore, $p \in B_{i-2} \subseteq B_{i-1}$. As a result,

$$\begin{aligned} &\langle a, b, g-(i-1)b, g-a-(i-2)b, g-2a-(i-3)b, \ldots, g-(i-1)a\rangle \\ &\subseteq B_i \subseteq \langle a, b-a, g-a-(i-2)b, g-2a-(i-3)b, \ldots, g-(i-1)a\rangle\,. \end{aligned}$$

It suffices to prove that if $x \in \langle a, b-a\rangle \cap (B_i \setminus S)$, then

$$x \in \langle a, b, g-(i-1)b, g-a-(i-2)b, g-2a-(i-3)b, \ldots, g-(i-1)a\rangle\,.$$

By definition of B_i, $x+a \in B_{i-1}$. If $x+a \in B_{i-1} \setminus S$, then there exist j, $0 \le j \le i-2$, and $m \in B_{i-1}$ such that $x+a = g - ja - (i-j-2)b + m$. Thus, $x = g-(j+1)a-(i-j-2)b+m$, and the assertion holds. In the other case, $x+a \in S$. Thus, we may write $x+a = c_1a + c_2b$, where $c_1, c_2 \in \mathbb{N}$. As $x \notin S$, this forces $c_1 = 0$ and $x = c_2b - a$. First suppose $x \ge g-(i-1)b$. It follows that $c_2 = a-i+k$ for some k, $0 \le k \le i-2$. Then $x = (a-i+k)b - a = g-(i-1)b+kb$, and the assertion holds. It only remains to consider $x < g-(i-1)b$. Since $x \in B_i$, $x+b = (c_2+1)b-a \in B_{i-1}$. In particular, $(c_2+1)b-a \in S$ with $c_2 \le a-2$, as $x+b < g-(i-2)b$, the smallest generator of B_{i-1} other than a and b. This implies $g = ((c_2+1)b-a)+(a-c_2-2)b \in S$, which is a contradiction. Therefore, the assertion holds in all cases.

This proves that

$$B_i = \langle a, b, g-(i-1)b, g-a-(i-2)b, g-2a-(i-3)b, \ldots, g-(i-1)a\rangle$$

for all $0 \le i \le a-2$. In particular, when $i = a-2$, this yields

$$B_{a-2} = \langle a, b, g-(a-3)b, g-a-(a-4)b, g-2a-(a-5)b, \ldots, g-(a-3)a\rangle .$$

We claim that the above is a canonical form description of B_{a-2}. Let $0 \le j \le a-3$. Suppose that $g-ja-(a-j-3)b \in W$ where

$$W := \langle a, b, g-(a-3)b, g-a-(a-4)b, \ldots, g-(j-1)a-(a-(j-1)-3)b\rangle .$$

It follows that $(j-k)b-(j-k)a = (g-ja-(a-j-3)b)-(g-ka-(a-k-3)b) \in W \subseteq B_{a-2}$ for some $0 \le k \le j-1$. By definition of B_{a-2}, $(j-k)b-a \in (j-k)b-(j-k)a+(j-k-1)M(S) \subseteq B_{a-j+k-1}$. This implies $(j-k)b-a \in S$ since $(j-k)b-a < g-(a-j+k-2)b$, the smallest generator of $B_{a-j+k-1}$ other than a and b. Then $g = (a-(j-k)-1)b+(j-k)b-a \in S$, which is a contradiction. Thus, the claim holds.

Observe that B_{a-2} is of maximal embedding dimension, as $e(B_{a-2}) = a = \mu(B_{a-2})$. Then, by Proposition 2.2(b),(d), we have

$$\begin{aligned} B_{a-1} &= B(B_{a-2}) = L(B_{a-2}) \\ &= \langle a, b-a, g-a-(a-3)b, g-2a-(a-4)b, \ldots, g-(a-2)a\rangle . \end{aligned}$$

Notice that $g-(j+1)a-(a-j-3)b = (j+2)(b-a) \in \langle a, b-a\rangle$ for all $0 \le j \le a-3$. Therefore, $B_{a-1} = \langle a, b-a\rangle = L_1$, as desired.

Repeated application of the above procedure gives $B_{\sum_{j=0}^{i-1}(\mu(L_j(S))-1)}(S) = L_i$ for all i, $0 \le i \le \lambda$. Thus, $B_i \subseteq L_i$ for all i, $0 \le i$. $\square$

The next two results may be viewed as companions of Theorem 2.6.

Proposition 2.7 *Let $S = \langle a_1, \ldots, a_\nu \rangle$ be a semigroup (written according to the canonical form conventions of the Introduction). If $i \geq 0$ and $ia_1 < a_2$, then $B_j(S) \subseteq L_j(S)$ for all $0 \leq j \leq i$.*

Proof. Let $0 \leq j \leq i$. Suppose $x \in B_j(S)$. Then $x + ja_1 \in x + jM(S) \subseteq M(S)$. Thus, $x = s - ja_1$ for some $s \in S$. We claim that $x \in L_j(S)$. Since $ia_1 < a_2$ and $j \leq i$, iterating the formula in Proposition 2.2(b) gives $L_j(S) = \langle a_1, a_2 - ja_1, \ldots, a_\nu - ja_1 \rangle$. Therefore, $x \in L_j(S)$ and so $B_j(S) \subseteq L_j(S)$ □

The sharpness of Proposition 2.7 may be illustrated by the semigroup S in Example 2.3. There, for $i := 2$, we have $(i-1)a_1 < a_2 < 2a_1 = ia_1$, $B_j(S) \subseteq L_j(S)$ for $0 \leq j \leq i-1$, and $B_i(S) \nsubseteq L_i(S)$.

Recall what is arguably the deepest result in [2], namely, the characterization of the pseudo-symmetric semigroup S of maximal embedding dimension and Frobenius number g [2, Theorem I.4.4]: $S = \left\langle 3, \frac{g}{2} + 3, g + 3 \right\rangle$, where g is a positive even integer and $g \equiv 1, 2 \pmod 3$.

Corollary 2.8 *If S is a pseudo-symmetric semigroup of maximal embedding dimension, then $B_i(S) \subseteq L_i(S)$ for each $i \geq 0$.*

Proof. If $i = 0$, the assertion holds by definition of $B_0(S)$ and $L_0(S)$. By the above remarks, $S = \left\langle 3, \frac{g}{2} + 3, g + 3 \right\rangle$, where $2 \leq g \equiv 1, 2 \pmod 3$. Hence, by Proposition 2.2(b), $L_1(S) = \left\langle 3, \frac{g}{2}, g \right\rangle = \left\langle 3, \frac{g}{2} \right\rangle$. On the other hand, the proof of [2, Theorem I.4.4, (ii) $\Rightarrow$ (iii)] establishes that $B_1(S) = \left\langle 3, \frac{g}{2} \right\rangle$. Hence, if $i \geq 2$, Theorem 2.6 yields that $B_i(S) = B_{i-1}(\left\langle 3, \frac{g}{2} \right\rangle) \subseteq L_{i-1}(\left\langle 3, \frac{g}{2} \right\rangle) = L_i(S)$. □

Remark 2.9 Recall that each symmetric semigroup of maximal embedding dimension is an Arf semigroup [2, Theorem I.4.2]. However, according to [2, Theorem I.4.5], the only psuedo-symmetric Arf semigroups are $\langle 3, 4, 5 \rangle$ and $\langle 3, 5, 7 \rangle$, corresponding to Frobenius numbers 2 and 4, respectively. Therefore, by [2, Theorem I.4.4], the pseudo-symmetric semigroup of maximal embedding dimension which is not an Arf semigroup and has minimal Frobenius number g is $S := \langle 3, 7, 11\rangle$. By the proof of Corollary 2.8, $B_1(S) = L_1(S) = \left\langle 3, \frac{g}{2} \right\rangle = \langle 3, 4\rangle$. Applying Proposition 2.2(b), we see that $L_i(S) = \mathbb{N}$ for each $i \geq 2$. On the other hand, it is straightforward to verify that $B_2(S) = \langle 3, 4, 5\rangle \subsetneqq L_2(S)$ and if $i \geq 3$, then $B_i(S) = \mathbb{N} = L_i(S)$. Thus, the failure of S to be Arf is detected by only one of the inclusions $B_i(S) \subseteq L_i(S)$ in Corollary 2.8 being a proper inclusion, namely, at $i = 2$.

The situation is different for the "next larger" pseudo-symmetric non-Arf semigroup of maximal embedding dimension, namely, $T := \langle 3, 8, 13\rangle$, with Frobenius number 10. Indeed, $B_1(T) = \langle 3, 5\rangle = L_1(T)$, but $B_2(T) = \langle 3, 5, 7\rangle \subsetneqq \langle 2, 3\rangle = L_2(T)$ and $B_3(T) = \langle 2, 3\rangle \subsetneqq \mathbb{N} = L_3(T)$. Thus, there are exactly two indices i for which the inclusions $B_i(T) \subseteq L_i(T)$ are proper inclusions: $i = 1, 2$.

Examples of the above type raise the question of whether a semigroup K is Arf provided that $\beta(K) = \lambda(K)$ and $B_i(K) \subseteq L_i(K)$ for all i, with these inclusions known to be equalities except possibly for a "few" values of i. We do not know of any such result. In particular, Example 2.10 shows, as in the case of S above, that the sole proper inclusion may occur at $i = \beta(K) - 1$.

Example 2.10 For each integer $i \geq 0$, the semigroup

$$S^i := \langle 6, 8 + 6i, 9 + 6i, 13 + 6i, 16 + 6i, 17 + 6i\rangle$$

satisfies $B_j(S^i) = L_j(S^i)$ for all $0 \leq j \leq i$, $B_j(S^i) = L_j(S^i)$ for all $j \geq i + 2$ (indeed, $\beta(S^i) = \lambda(S^i) = i + 2$), and $B_{i+1}(S^i) \subsetneqq L_{i+1}(S^i)$.

Proof. We begin by considering the case $i = 0$. Then $S^0 = \langle 6, 8, 9, 13, 16, 17\rangle = \{0, 6, 8, 9, 12, \rightarrow\}$ can be described in canonical form as $\langle 6, 8, 9, 13\rangle$. By Proposition 2.2(b), $L_1(S^0) = \langle 2, 3\rangle \neq \mathbb{N}$ and $L_j(S^0) = \mathbb{N}$ for all $j \geq 2$. In particular, $\lambda(S^0) = 2$. Moreover, $B_1(S^0) = \{0, 6, \rightarrow\} \subsetneqq \langle 2, 3\rangle = L_1(S^0)$ and $B_2(S^0) = \mathbb{N}$, so that $B_j(S^0) = \mathbb{N}$ for each $j \geq 2$. In particular, $\beta(S^0) = 2$.

Fix $i \geq 1$. It is straightforward to verify that S^i is of maximal embedding dimension, that is, that the given generating set of cardinality 6 for S^i is irredundant. (Of course, S^0 is *not* of maximal embedding dimension.) It follows from Proposition 2.2(b),(d) that $B_1(S^i) = L_1(S^i)$ consists of the nonnegative integers obtainable by subtracting 6 from an element of S^i. Therefore, $L_1(S^i) = S^{i-1}$.

We now proceed to a proof of the assertion by induction on $i \geq 1$. Consider $1 \leq j \leq i+2$. It was shown above that $B_1(S^i) = L_1(S^i) = S^{i-1}$, and so $B_j(S^i) = B_{j-1}(B_1(S^i)) = B_{j-1}(L_1(S^i)) = B_{j-1}(S^{i-1})$ and $L_j(S^i) = L_{j-1}(L_1(S^i)) = L_{j-1}(S^{i-1})$. Now, if $1 \leq j \leq i$, the induction hypothesis ensures that $B_j(S^i) = L_j(S^i)$, $B_{i+1}(S^i) \subsetneqq L_{i+1}(S^i) \subsetneqq \mathbb{N}$, and $B_{i+1}(S^{i-1}) = \mathbb{N} = L_{i+1}(S^{i-1})$ which can be rewritten as $B_{i+2}(S^i) = \mathbb{N} = L_{i+2}(S^i)$. In view of the definitions of β and λ, this completes the induction step and the proof. $\square$

Our work leaves open two problems concerning the property "$B_i(S) \subseteq L_i(S)$ for all $i \geq 0$": in the spirit of Theorem 2.6 and Corollary 2.8, to find additional classes of symmetric or pseudo-symmetric semigroups satisfying this property; and, despite Example 2.10, to find a role for this property in characterizing at least some Arf semigroups.

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