On the acyclic chromatic number of Hamming graphs

Robert E. Jamison[1], Gretchen L. Matthews[2*]

[1] Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975; Affiliated Professor, University of Haifa, rejam@clemson.edu

[2] Department of Mathematical Sciences, Clemson University, Clemson, SC 29634-0975, gmatthe@clemson.edu

Abstract. An *acyclic coloring* of a graph G is a proper coloring of the vertex set of G such that G contains no bichromatic cycles. The *acyclic chromatic number* of a graph G is the minimum number k such that G has an acyclic coloring with k colors. In this paper, acyclic colorings of Hamming graphs, products of complete graphs, are considered. Upper and lower bounds on the acyclic chromatic number of Hamming graphs are given.

Key words. acyclic coloring, Cartesian product of graphs, distance 2 coloring, Hamming graph

1. Introduction

A *k-coloring* of a graph G with vertex set $V(G)$ is a labeling $f : V(G) \to \{1, \ldots, k\}$. Such a coloring is said to be a *proper coloring* provided any two adjacent vertices have distinct colors. The chromatic number of a graph G, denoted $\chi(G)$, is the minimum number k such that G has a proper k-coloring. A more restrictive type of coloring is an acyclic coloring. A proper coloring of G is called *acyclic* if and only if the subgraph of G induced by any two color classes of G contains no cycles. The *acyclic chromatic number* of a graph G, denoted $AC(G)$, is the smallest number k such that G has an acyclic k-coloring. Acyclic colorings are hereditary in the sense that the restriction of an acyclic coloring to a subgraph is an acyclic coloring. Thus, the acyclic chromatic number is nondecreasing from subgraph to supergraph. An even more restrictive type of coloring is a distance 2 coloring. A *distance 2 coloring* of a graph G is a coloring in which any two vertices at distance at most 2 apart get distinct colors. The *distance 2 chromatic number* of G, denoted $\chi_2(G)$, is the minimum number k such that G has a distance 2 coloring with k colors. Note that a distance 2 coloring is necessarily acyclic. Thus $AC(G) \leq \chi_2(G)$.

Acyclic colorings were first studied by Grünbaum [9] who proved that a graph with maximum degree 3 has an acyclic 4-coloring. This was followed by work of Berman and Albertson [1] and Borodin [5] on acyclic colorings for planar graphs. In [6], Burnstein proved that a graph with maximum degree 4 has an acyclic 5-coloring. Later, the acyclic chromatic number for graphs on certain surfaces [3] was considered. More recently, acyclic colorings have been studied by Alon, McDiarmid, and Reed [2], Mohar [13], and Skulrattanakulchai [18]. Nowakowski and Rall have investigated the behavior of several graph parameters with respect to an array of different graph products [15].

* *The work of this author is supported by NSA H-98230-06-1-0008.*

In this paper, we study the acyclic chromatic numbers of Hamming graphs, the products of complete graphs. The product we are taking is the usual *Cartesian* (or *box*) product. The vertex set of $G\Box H$ is the Cartesian product $V(G)\times V(H)$ of the vertex sets of G and H. There is an edge between two vertices of the product if and only if they are adjacent in exactly one coordinate and agree in the other. This is an extension of the work of Ferrin, Godard, and Raspaud [7] where acyclic colorings of certain grids (products of paths) are studied, of the authors [10] and of the authors with Villalpando [12] where acyclic colorings of products of trees and cycles are studied.

Since we consider only Hamming graphs in this paper, we will write $\mathcal{H}(s_1, s_2, s_3, \ldots, s_t)$ to denote $K_{s_1}\Box K_{s_2}\Box K_{s_3}\Box \ldots \Box K_{s_t}$. The *dimension* of this Hamming graph is t, and we always normalize by always assuming $2 \leq s_1 \leq s_2 \leq s_3 \leq \cdots \leq s_t$. The lower bound is here is $2 \leq s_1$ since $s_1 = 1$ would effectively lower the dimension. To simplify the potentially cumbersome notation for the acyclic and distance 2 chromatic number, we write $AC(s_1, s_2, s_3, \ldots, s_t)$ for $AC(\mathcal{H}(s_1, s_2, s_3, \ldots, s_t))$ and $\chi_2(s_1, s_2, s_3, \ldots, s_t)$ for $\chi_2(\mathcal{H}(s_1, s_2, s_3, \ldots, s_t))$.

2. General bounds

In [10, Theorem 2.1], [7, Proposition 1], and [17, Lemma 10], it is shown that the acyclic chromatic number of a product of graphs $G_1, \ldots, G_t$ satisfies

$$AC(G_1\Box\cdots\Box G_t) \; > \; \sum_{i=1}^{t}\frac{|E(G_i)|}{|V(G_i)|} + 1.$$

If the graph G_i is r_i-regular, then $|E(G_i)| = \frac{r_i|V(G_i)|}{2}$. Hence, we have the following bound on the acyclic chromatic number of a product of regular graphs.

Proposition 1. *Consider the product $G_1\Box\cdots\Box G_t$ where G_i is r_i-regular for $1 \leq i \leq t$. Then*

$$AC(G_1\Box\cdots\Box G_t) \; > \; \frac{r_1+\cdots+r_t}{2} + 1.$$

Since the complete graph K_s is $(s-1)$-regular, Proposition 1 gives

$$AC(s_1, \ldots, s_t) \; > \; \frac{s_1+\cdots+s_t-t}{2} + 1. \tag{1}$$

Recall that $s_i \geq 2$ for all i, which yields

$$AC(s_1, \ldots, s_t) \; > \; \frac{2(t-1)+s_t-t+2}{2} \; = \; \frac{t+s_t}{2}.$$

Another simple but useful lower bound arises from the fact that $AC(G) \geq \chi(G)$. Since K_{s_t} is a clique in $\mathcal{H}(s_1, \ldots, s_t)$, we get

$$AC(s_1, \ldots, s_t) \; \geq \; s_t. \tag{2}$$

Now let $[s]^t := (s, \ldots, s)$ denote a string of t s's. In this case, Inequality (1) becomes

$$AC([s]^t) \; > \; \frac{t(s-1)}{2} + 1. \tag{3}$$

To obtain upper bounds on the acyclic chromatic number of a Hamming graph, we turn to distance 2 colorings. Recall that the *square* G^2 of a graph G has the same vertex set as G but has two vertices adjacent if and only if they are at most distance two apart in G. By definition, the distance 2 chromatic number is just the chromatic number of the square G^2, hence $\chi_2(G) = \chi(G^2)$. The bounds obtained here are quite crude, and, for simplicity's sake, we will not bother working through minor improvements. More significant improvements on these bounds will be given in Section 4. For any sequence $2 \le s_1 \le \cdots \le s_t$, set

$$\mathfrak{B}(s_1,\ldots,s_t) := \sum_{i<j} s_i s_j - (t-2)\left(\sum_{i=1}^{t} s_i\right) + \frac{t(t-3)}{2}.$$

Lemma 1. *The square $H = \mathcal{H}^2(s_1,\ldots,s_t)$ of the Hamming graph $\mathcal{H}(s_1,\ldots,s_t)$ is regular of degree $\mathfrak{B}(s_1,\ldots,s_t)$.*

Proof. Let $v = (v_1,\ldots,v_t)$ be a vertex of H. To find the degree of v in H, we determine the vertices at distance at most two from v in $\mathcal{H}(s_1,\ldots,s_t)$. Changing v_i to any one of $s_i - 1$ possible other values yields a set A_i of $s_i - 1$ vectors at distance one from v. Hence, there are exactly $\sum_{i=1}^{t}(s_i - 1)$ vertices at distance one from v in $\mathcal{H}(s_1,\ldots,s_t)$. Now for each vector in A_i, changing that vector in the j^{th} $(j \ne i)$ coordinate to any one of $s_j - 1$ possible new values yields a set $A_{i,j}$ of vectors at distance two from v. However, $A_{j,i} = A_{i,j}$, so we count these sets once by taking $i < j$. Thus the number of vertices at distance exactly two from v in $\mathcal{H}(s_1,\ldots,s_t)$ is

$$\sum_{i<j}(s_i-1)(s_j-1) \;=\; \sum_{i<j} s_i s_j - (t-1)\left(\sum_{i} s_i\right) + \binom{t}{2}.$$

Notice that on expanding $\sum_{i<j}(s_i-1)(s_j-1)$, each s_i will arise in a linear term from $(s_k-1)(s_i-1)$ for $i-1$ values of $k<i$ and from $(s_i-1)(s_j-1)$ for $t-i$ values of $i<j$, making a total of $t-1$ appearances for each i. The constant term 1 in each summand appears $\binom{t}{2}$ times. Therefore, the total number of vertices in $\mathcal{H}(s_1,\ldots,s_t)$ at distance at most two from v is

$$\left(\sum_{i=1}^{t} s_i\right) - t + \sum_{i<j} s_i s_j - (t-1)\left(\sum_{i=1}^{t} s_i\right) + \binom{t}{2} \;=\; \sum_{i<j} s_i s_j - (t-2)\left(\sum_{i=1}^{t} s_i\right) + \frac{t(t-3)}{2} \;= \mathfrak{B}(s_1,\ldots,s_t).$$

It follows that H is regular of degree $\mathfrak{B}(s_1,\ldots,s_t)$. ■

Theorem 1. *For $t \ge 3$, the acyclic chromatic number of the Hamming graph $\mathcal{H}(s_1,\ldots,s_t)$ satisfies*

$$\begin{aligned} AC(s_1,s_2,s_3,\ldots,s_t) &\le \mathfrak{B}(s_1,\ldots,s_t) \\ &\le \binom{t}{2} s_t^2. \end{aligned}$$

Moreover,

$$AC([s]^t) \le \binom{t}{2} s^2 - t(t-2)\left(s - \frac{1}{2}\right)$$

provided $t \ge 3$.

Proof. First note that $AC(s_1, s_2, s_3, \ldots, s_t) \leq \chi_2(s_1, s_2, s_3, \ldots, s_t)$ since any distance 2 coloring is acyclic. The distance 2 chromatic number of a graph is simply the chromatic number of its square. Let $\Delta(G)$ denote the maximum degree of a graph G. As is well-known [19], $\Delta(G) + 1$ is an upper bound on the chromatic number $\chi(G)$ of a graph G. According to Lemma 1, this yields

$$AC(s_1, s_2, s_3, \ldots, s_t) \leq \mathfrak{B}(s_1, \ldots, s_t) + 1.$$

Next, recall Brooks' Theorem [19] which states that a connected graph G satisfies $\chi(G) = \Delta(G) + 1$ if and only if G is either complete or an odd cycle. A product of 3 or more complete graphs never has a square that is an odd cycle or is complete. As a result, the first inequality holds.

Finally, to obtain the second inequality, note that the expression $\mathfrak{B}$ has three terms. The first of these, $\sum_{i<j} s_i s_j$, consists of $\binom{t}{2}$ summands, each bounded by s_t^2; that is,

$$\sum_{i<j} s_i s_j \leq s_t^2 \sum_{i=1}^{t} (t - i) \leq s_t^2 \frac{t^2 - t}{2}.$$

Hence, to establish the second inequality, we need only show that the last two terms make an overall negative contribution. Since $t - 2 > t - 3$ and $s_i \geq 2$ for all i, it follows that

$$(t - 2) \sum_{i=1}^{t} s_i \geq (t - 3)2t \geq (t - 3)\frac{t}{2},$$

showing the second term of $\mathfrak{B}$ is larger than the third. Thus, neglecting the difference leads to an upper bound for $\mathfrak{B}$.

Taking $s_1 = \cdots = s_t = s$ in the previous argument and observing that the third term of $\mathfrak{B}$ is less than $\frac{t(t-2)}{2}$ produces the bound on $AC([s]^t)$. ■

We conclude this section by summarizing the bounds we have obtained for the acyclic chromatic number of a Hamming graph.

Corollary 1. *If $t \geq 3$, then*

$$\max\left(s_t, \left\lceil \frac{t + s_t + 1}{2} \right\rceil\right) \leq AC(s_1, \ldots, s_t) \leq \frac{t(t-1)}{2} s_t^2$$

and

$$\max\left(s, \left\lceil \frac{t + s + 1}{2} \right\rceil\right) \leq AC([s]^t) \leq \frac{t(t-1)}{2} s^2 - t(t-2)\left(s - \frac{1}{2}\right).$$

3. Colorings of two-dimensional Hamming graphs by groups

In this section, we will study the 2-dimensional Hamming graphs $\mathcal{H}(m, n)$ where $2 \leq m \leq n$. From the previous section, we see that

$$AC(m, n) \geq n$$

from Inequality (2) and

$$AC(n, n) \geq n + 1$$

from Inequality (3). The upper bounds given in Theorem 1 do not necessarily apply to Hamming graphs of dimension two. Even when these do apply, they are quite bad.

Next, we obtain a constructive upper bound on the acyclic chromatic number of a Hamming graph. The following notation will be useful in the theorem and its applications:

- $\varsigma(N)$, the smallest prime dividing N;
- $\alpha(N) := N - \frac{N}{\varsigma(N)}$;
- $\beta(n) := \min\{N : n \leq \alpha(N)\}$; and
- $NP(n)$, the smallest prime larger than n.

Certainly,

$$\beta(n) \leq NP(n).$$

In fact, it may be the case that $\beta(n) = NP(n)$. This has been verified computationally for $n \leq 1,000,000$. Since this is not the focus of this investigation, we will not comment further on this. Instead, we return our attention to the task of obtaining an upper bound on the acyclic chromatic number of a 2-dimensional Hamming graph.

Theorem 2. *Suppose m, n, and N are positive integers. If $m \leq \alpha(N)$ and $n \leq N$, then $AC(m, n) \leq N$.*

Proof. Suppose p is the smallest prime divisor of N. First we may assume that $m = N - \frac{N}{p}$ and $n = N$, for if we prove the result in this case, then it follows for all $m' \leq m$ and $n' \leq n$ as $AC(m', n') \leq AC(m, n)$.

Consider the graph $\mathcal{H}(N, N) = K_N \square K_N$ with vertices indexed by the elements of $\mathbb{Z}_N \times \mathbb{Z}_N$. Notice that $K_m \square K_n$ may be viewed as the subgraph of $K_N \square K_N$ induced by those vertices with indices in $\left(\mathbb{Z}_N \setminus \left\{0, 1, \ldots, \frac{N}{p} - 1\right\}\right) \times \mathbb{Z}_N$. Color the vertices of $K_N \square K_N$ by assigning the color $i + j \bmod N$ to the vertex (i, j). (In this proof, arithmetic on colors is done modulo N.) This coloring is obviously proper. We now show that it is acyclic when restricted to $K_m \square K_n$.

Suppose there is a bichromatic cycle C in $K_m \square K_n$. Let (s, t) be a vertex on the cycle C. Then $s + t$ is one of the colors on the cycle. Let c be the other color on the cycle and set $a := c - (s + t)$. Note that $a \neq 0$. Walking around the cycle corresponds to alternately adding one of $(a, 0)$ or $(0, a)$ and then subtracting the other on the next step. Thus, the cycle C is one of two types,

$$C1 : (s,t), (s+a,t), (s+a,t-a), (s+2a,t-a), (s+2a,t-2a), (s+3a,t-2a), \ldots$$

or

$$C2 : (s,t), (s,t+a), (s-a,t+a), (s-2a,t+a), (s-2a,t+2a), (s-3a,t+2a), \ldots.$$

Consider the cycle (C1). Let $\langle a \rangle$ denote the subgroup of $\mathbb{Z}_N$ generated by a, and suppose a has order r. Then $\langle a \rangle$ consists of all multiples of $\frac{N}{r}$. Every coset of $\langle a \rangle$ has the form $k + \langle a \rangle$ where $k \in \left\{0, 1, \ldots, \frac{N}{r} - 1\right\}$. It is clear that a will be added to the x-coordinate in (C1) every second step. As (C1) is a cycle, every multiple of a will eventually occur added to s in the x-coordinate of some vertex in (C1). That is, the x-coordinates of (C1) form the coset $s + \langle a \rangle$. Thus this coset must contain a representative k with $0 \leq k \leq \frac{N}{r} - 1$. Since $p \leq r$, we have $\frac{N}{r} - 1 \leq \frac{N}{p} - 1$. Thus k lies in the interval between 0 and $\frac{N}{p} - 1$, which is impossible since these values were explicitly forbidden as x-values in our definition of $K_m \square K_n$.

Similarly, the case of cycle (C2) leads to a contradiction. ■

Table 1. $AC(m, n)$ for small m and n

$m \backslash n$	2	3	4	5	6	7	8	9	10	11	12
2	3	$\langle \mathbf{3} \rangle$	4	5	6	7	8	9	10	11	12
3		5	5	$\langle \mathbf{5} \rangle$	6	7	8	9	10	11	12
4			5	5	$(6,7)$	$\langle \mathbf{7} \rangle$	8	9	10	11	12
5				$(6,7)$	$(6,7)$	7	$(8,9)$	$\langle \mathbf{9} \rangle$	10	11	12
6					7	7	$(8,9)$	9	$(10,11)$	$\langle \mathbf{11} \rangle$	12
7						$(8,11)$	$(8,11)$	$(9,11)$	$(10,11)$	11	12
8							$(9,11)$	$(9,11)$	$(10,11)$	11	$(12,13)$

Theorem 2 immediately yields the following bounds on the acyclic chromatic number of a 2-dimensional Hamming graph.

Corollary 2. *For any positive integer n,*

$$n + 1 \leq AC(n, n) \leq \beta(n).$$

If $m \leq \alpha(n)$, then

$$AC(m, n) = n.$$

Notice that Corollary 2 implies

$$n + 1 \leq AC(n, n) \leq NP(n).$$

We also see that if p is prime and $m < p$, then

$$AC(m, p) = p.$$

Another particularly useful consequence of Corollary 2 is the following result.

Corollary 3. *If $n \geq 2m - 1$, then $AC(m, n) = n$.*

Proof. Suppose $n \geq 2m - 1$. Then

$$\alpha(n) = n - \frac{n}{\varsigma(n)} \geq n - \frac{n}{2} \geq \frac{n}{2} \geq m - \frac{1}{2}.$$

Since both $\alpha(n)$ and m are integers, this implies $\alpha(n) \geq m$. Now, by Corollary 2, $AC(m, n) = n$. ∎

Corollary 3 shows that any integer $n \geq 3$ is the acyclic chromatic number of some 2-dimensional Hamming graph. Table 1 displays what we know about small values of $AC(m, n)$. A single number gives an exact value of $AC(m, n)$ when known. Otherwise, the ordered pair gives upper and lower bounds. The notation $\langle a \rangle$ means that $AC(m, n) = a$ and $AC(m, n') = n'$ for all $n' \geq n$. Except for $AC(3, 3) = 5$, which was determined in [10, Theorem 3.2], all values follow from results established here.

We conclude this section by considering the asymptotic behavior of the acyclic chromatic number of 2-dimensional Hamming graphs. As mentioned earlier, $\beta(n) \leq NP(n)$. Applying Bertrand's Postulate, we see that $\beta(n) \leq 2n$. However, even more is true.

Theorem 3. *The limit* $\lim_{n\to\infty} \frac{\beta(n)}{n}$ *exists and equals 1.*

Proof. Since $\beta(n) \geq n$ by definition, we only need to show that for each $\varepsilon > 0$, there is an L_ε such that if $n > L_\varepsilon$, then $\beta(n) \leq (1+\varepsilon)n$.

Let q be a prime so large that $\frac{2}{q-1} < \varepsilon$. Let Q denote the product of all primes less than q and set $L_\varepsilon = q(Q+1)$. Suppose $n > L_\varepsilon$. Let $A := \left\lceil \frac{qn}{q-1} \right\rceil$. There is an integer N between A and $A+Q$ such that $N \equiv 1 \pmod{Q}$. The congruence condition says N and Q are relatively prime, so the smallest prime divisor p of N must be q or bigger. Therefore,

$$\alpha(N) = N - \frac{N}{p} \geq N - \frac{N}{q} = N\left(1 - \frac{1}{q}\right) \geq A\left(1 - \frac{1}{q}\right) \geq \frac{qn}{q-1}\left(\frac{q-1}{q}\right) = n.$$

Thus by definition $\beta(n) \leq N$. We now show that $N \leq (1+\varepsilon)n$. From $n > L_\varepsilon = q(Q+1)$, we have

$$N \leq A + Q \leq \frac{qn}{q-1} + 1 + Q < \frac{qn}{q-1} + \frac{n}{q} < n\left(1 + \frac{2}{q-1}\right) < n\,(1+\varepsilon)\,.$$

Hence, $n \leq \beta(n)(1+\varepsilon)n$ for all $n > L_\varepsilon$ which establishes the result. ∎

Corollary 4. *The limit* $\lim_{n\to\infty} \frac{AC(n,n)}{n}$ *exists and equals 1. For each fixed* m*, the limit* $\lim_{n\to\infty} \frac{AC(m,n)}{n}$ *exists and equals 1.*

Proof. Note that $n+1 \leq AC(n,n) \leq \beta(n)$. Thus, $\frac{AC(n,n)}{n}$ is trapped between two sequences converging to 1. Since we are taking a limit, we may as well suppose $m < n$. Then we have $n \leq AC(m,n) \leq \beta(n)$. Hence, $\frac{AC(m,n)}{n}$ is also trapped between two sequences converging to 1. ∎

It is interesting to note that a statement analogous to that of Corollary 4 holds for the distance 2 chromatic number of hypercubes and is the main result of [16].

4. Applications of two-dimensional results to Hamming graphs of higher dimension

To obtain improved upper bounds on the acyclic chromatic number of certain Hamming graphs, the following result is helpful.

Theorem 4. *For two graphs* G *and* H*,*

$$AC(G \square H) \leq AC(\chi_2(G), \chi_2(H)).$$

Proof. For convenience, let $m := \chi_2(G)$, $n := \chi_2(H)$, and $N := AC(m,n)$. Let $g : V(G) \to \{1,2,3,\ldots,m\}$ be a distance 2 coloring of G, and let $h : V(H) \to \{1,2,3,\ldots,n\}$ be a distance 2 coloring of H. Let $f : V(G\square H) \to \{1,2,3,\ldots,N\}$ be an acyclic coloring of $K_m \square K_n$. We now define a coloring φ of $G\square H$ by setting

$$\varphi(x,y) := f(g(x), h(y))$$

for $(x,y) \in V\,(G\square H)$.

First, we claim that φ is a proper coloring of $G \square H$. Consider two adjacent vertices, say (a, y) and (a, z), in $G \square H$. Since h is proper, $h(y) \neq h(z)$. Thus $(g(a), h(y))$ and $(g(a), h(z))$ are different points in $K_m \square K_n$, so f assigns them different colors. Thus

$$\varphi(x, y) = f(g(x), h(y)) \neq f(g(x), h(z)) = \varphi(x, z).$$

The same argument holds for adjacent vertices of the form (a, y) and (b, y). Hence, φ is proper.

Now we show that φ is acyclic. Suppose

$$\Gamma : (x_1, y_1), (x_2, y_2), (x_3, y_3), \ldots, (x_k, y_k)$$

is a bichromatic cycle in $G \square H$. Then

$$\Gamma^* : (g(x_1), h(y_1)), (g(x_2), h(y_2)), (g(x_3), h(y_3)), \ldots, (g(x_k), h(y_k))$$

is a bichromatic closed walk in $K_m \square K_n$. We say walk because it is conceivable that both vertices and edges are repeated in Γ^*. If Γ^* has no repeated vertices, then it is a bichromatic cycle in $K_m \square K_n$, contrary to f being an acyclic coloring of $K_m \square K_n$.

If Γ^* has repeated vertices, let $s \neq t$ be the cyclically closest indices with

$$(g(x_s), h(y_s)) = (g(x_t), h(y_t)).$$

We can assume that $s = 1$ (by a rotation of Γ^* if necessary). We can also assume that the arc of the cycle s to t is no longer than the opposite arc (running the cycle backwards if necessary). These standardizations together with minimal choice of s and t imply that between $s = 1$ to t, there are no other coincidences; that is, as i goes from 1 to t, the points $(g(x_i), h(y_i))$ in $K_m \square K_n$ are distinct. We must show that $t \geq 4$ in order to have a legitimate bichromatic cycle, and hence a contradiction.

By an argument similar to that above showing φ is proper,

$$(g(x_1), h(y_1)) \neq (g(x_2), h(y_2)).$$

Thus, $t > 2$. Now consider $(g(x_1), h(y_1))$ and $(g(x_3), h(y_3))$. The path from (x_1, y_1) to (x_3, y_3) in $G \square H$ can take four possible forms:

Type A:	(x_1, y_1)	to	(x_1, y_2)	to	(x_1, y_3)
Type B:	(x_1, y_1)	to	(x_2, y_1)	to	(x_3, y_1)
Type C:	(x_1, y_1)	to	(x_1, y_2)	to	(x_2, y_2)
Type D:	(x_1, y_1)	to	(x_2, y_1)	to	(x_2, y_2).

In Type A, y_1 and y_3 are distance 2 apart. Since h is a distance 2 coloring, $h(y_1) \neq h(y_3)$. Thus, $(g(x_1), h(y_1)) \neq (g(x_3), h(y_3))$. The same applies to Type B. In Types C and D, y_1 is adjacent to y_2. Since h is proper, $h(y_1) \neq h(y_3)$. Thus, $(g(x_1), h(y_1)) \neq (g(x_3), h(y_3))$. Hence, $t \geq 4$.

We have shown that if φ has a bichromatic cycle in $G \square H$, then f has a bichromatic cycle in $K_m \square K_n$, contrary to f being acyclic. Thus φ must be acyclic on $G \square H$. The set of colors used by φ is a subset of the set of colors used by f. It follows that φ is an acyclic coloring of $G \square H$ with at most N colors, thereby establishing the result. ∎

Theorem 4 may be combined with results on perfect codes to give a better upper bound on the acyclic chromatic number of a number of Hamming graphs. For details on the use of perfect codes in distance 2 colorings, see [8], [11], [14], [16], and [20]. For any prime power q and any positive integer r, there exists a $\left[\frac{q^r-1}{q-1}, \frac{q^r-1}{q-1} - r, 3\right]_q$ Hamming code; that is, there exists a Hamming code of length $\frac{q^r-1}{q-1}$ with $q^{\frac{q^r-1}{q-1}-r}$ words, any two of which differ in at least three coordinates. As a consequence, the distance 2 chromatic number of the Hamming graph $\mathcal{H}\left([q]^{\frac{q^r-1}{q-1}}\right)$ is

$$\chi_2\left(K_q^{\frac{q^r-1}{q-1}}\right) = q^r$$

as shown in [11, Theorem 4.1]. This, together with Theorem 4, gives the following result.

Theorem 5. *Let q be a power of a prime number and $a \leq b$. Then*

$$\frac{1}{2}\left(q^a + q^b\right) + 1 \leq AC\left(K_q^{\frac{q^a+q^b-2}{q-1}}\right) \leq \begin{cases} q^b & \text{if } a < b \\ \beta(q^b) & \text{if } a = b. \end{cases}$$

Proof. Recall the notation $\mathcal{H}([q]^t) = K_q^t$ and $AC([q]^t) = AC\left(K_q^t\right)$ introduced in Section 2. By Proposition 1,

$$AC\left([q]^{\frac{q^a+q^b-2}{q-1}}\right) \geq \frac{1}{2}\left(q^a + q^b\right) + 1.$$

To obtain the upper bound, take $G = \mathcal{H}\left([q]^{\frac{q^a-1}{q-1}}\right)$ and $H = \mathcal{H}\left([q]^{\frac{q^b-1}{q-1}}\right)$ in Theorem 4. This gives

$$\begin{aligned} AC\left([q]^{\frac{q^a+q^b-2}{q-1}}\right) &= AC\left(\mathcal{H}\left([q]^{\frac{q^a-1}{q-1}}\right) \square \mathcal{H}\left([q]^{\frac{q^b-1}{q-1}}\right)\right) \\ &\leq AC\left(\chi_2\left(\mathcal{H}\left([q]^{\frac{q^a-1}{q-1}}\right)\right), \chi_2\left(\mathcal{H}\left([q]^{\frac{q^b-1}{q-1}}\right)\right)\right) \\ &= AC\left(q^a, q^b\right). \end{aligned}$$

Applying Theorem 2 now gives the desired upper bound. ■

5. Acyclic chromatic numbers of some hypercubes

In this section, we consider the t-dimensional hypercube $Q_t := K_2 \square \cdots \square K_2$. Determining the acyclic chromatic number of the hypercube is mentioned as an open problem in [7] where it is shown that

$$\left\lfloor \frac{t}{2} \right\rfloor + 2 \leq AC(Q_t) \leq t + 1$$

(see [7, Theorem 4] and [12, Theorem 2.1]). There the authors state that the exact value may be equal to the lower bound. Here, we show that this is indeed the case if $t + 3$ is a Fermat prime. In addition, we obtain an improved upper bound in a number of other cases.

First, note that taking $q = 2$ in Theorem 5 allows one to derive bounds on the acyclic chromatic number of certain hypercubes. To widen the class of hypercubes to which this

Table 2. Acyclic chromatic numbers of some hypercubes related to Fermat primes

t	$AC(Q_t)$
6	5
30	17
510	257
131070	65537

Table 3. Bounds on acyclic chromatic numbers of hypercubes of small dimension

t	$AC(Q_t)$
2	3
$3,4$	4
6	5
7	$(5,8)$
$8,9$	$(6,8)$
10	$(7,8)$
11	$(7,11)$
$12,13$	$(8,11)$
14	$(9,11)$

result applies, we rely on the work of Ostergard [16] in which a result of Best and Brouwer [4] on shortened Hamming codes is used to prove that

$$\chi_2(Q_{2^r-i}) = 2^r$$

for $1 \leq i \leq 4$. Using the same ideas as in the proof of Theorem 5 yields the following fact.

Theorem 6. *Assume $a \leq b$ and $1 \leq c, d \leq 4$. Then*

$$AC\left(Q_{2^a+2^b-c-d}\right) \leq \begin{cases} 2^b & \text{if } a < b \\ \beta(2^b) & \text{if } a = b. \end{cases}$$

Next, we apply Theorem 6 to obtain the exact acyclic chromatic numbers for hypercubes of dimensions related to Fermat primes.

Corollary 5. *The acyclic chromatic number of the hypercube of dimension $t := 2^{r+1} - 2$ satisfies*

$$2^r + 1 \leq AC\left(Q_{2^{r+1}-2}\right) \leq NP(2^r).$$

In particular, if $2^r + 1$ is a Fermat prime, then

$$\left(Q_{2^{r+1}-2}\right) = 2^r + 1.$$

Finally, we close this section with two tables. The acyclic chromatic numbers found using Corollary 5 are found in Table 2. Table 3 illustrates the bounds obtained here for hypercubes of small dimension.

Acknowledgements. The authors thank the referee for useful comments.

References

1. M. O. Albertson and D. M. Berman, The acyclic chromatic number, Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory and Computing, Utilitas Mathematica Inc., Winnipeg, Canada, 1976, 51–60.

2. N. Alon, C. McDiarmid, and B. Reed, Acyclic colorings of graphs, Random Structures Algorithms **2** (1991), no. 3, 277–288.

3. N. Alon, B. Mohar, and D. P. Sanders, On acyclic colorings of graphs on surfaces, Israel J. Math. **94** (1996), 273–283.

4. M. R. Best and A. E. Brouwer, The triply shortened binary Hamming code is optimal, Discrete Math. **17** (1977), 235–245.

5. O. V. Borodin, On acyclic colorings of planar graphs, Discrete Math. **25** (1979), no. 3, 211–236.

6. M. I. Burnstein, Every 4-valent graph has an acyclic 5-coloring, Soobšč Akad. Nauk Gruzin SSR 93 (1979), 21–24.

7. G. Fertin, E. Godard, and A. Raspaud, Acyclic and k-distance coloring of the grid, Inform. Process. Lett. **87** (2003), no. 1, 51–58.

8. F.-W. Fu, S. Ling, and C.-P. Xing, New results on two hypercube coloring problems, preprint.

9. B. Grünbaum, Acylic colorings of planar graphs, Isreal J. Math. **14** (1973), 390–408.

10. R. E. Jamison and G. L. Matthews, Acyclic colorings of products of cycles, Bull. Inst. Combin. Appl., to appear.

11. R. E. Jamison and G. L. Matthews, Distance k colorings of Hamming graphs. Proceedings of the Thirty-Seventh Southeastern International Conference on Combinatorics, Graph Theory and Computing. Congr. Numer. 183 (2006), 193–202.

12. R. E. Jamison, G. L. Matthews, and J. Villalpando, Acyclic colorings of products of trees, Inform. Process. Lett. 99 (2006), no. 1, 7–12.

13. B. Mohar, Acyclic colorings of locally planar graphs. European J. Combin. **26** (2005), no. 3-4, 491–503.

14. H. Q. Ngo, D.-Z. Du, and R. L. Graham, New bounds on a hypercube coloring problem, Inform. Process. Lett. **84** (2002), 265–269.

15. R. Nowakowski and D. F. Rall, Associative graph products and their independence, domination and coloring numbers. Discuss. Math. Graph Theory **16** (1996), no. 1, 53–79.

16. Östergård, P. R. J. On a hypercube coloring problem. J. Combin. Theory Ser. A **108** (2004), no. 2, 199–204.

17. A. Pór and D. R. Wood, Colourings of the Cartesian product of graphs and multiplicative Sidon sets, 6th Czech-Slovak International Symposium on Combinatorics, Graph Theory, Algorithms and Applications, 33–40, Electron. Notes Discrete Math., 28, Elsevier, Amsterdam, 2007.

18. S. Skulrattanakulchai, Acyclic colorings of subcubic graphs, Inform. Process. Lett. **92** (2004), 161–167.

19. D. B. West, Introduction to Graph Theory, Prentice Hall 1996, second edition, 2001.

20. G. M. Ziegler, Coloring Hamming graphs, optimal binary codes, and the 0/1-Borsuk problem in low dimensions, in H. Alt (Ed.): Computational Discrete Mathematics, Lecture Notes in Computer Science **2122**, Springer-Verlag, Berlin, 2001, 159–171.

Received: May 8, 2008
Final version received: May 8, 2008