REFORM MATHEMATICS EDUCATION IS COUNTERPRODUCTIVE FOR HIGH-TECH CAREERS (AND IT'S NOT THE TEACHERS)

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Contents

1. Conclusion

The Reform approach to mathematics education is counterproductive for high-tech careers, even for high-achieving students. It should be considered a terminal track for students interested in such careers.

This is a sad conclusion. When the Reform movement started making waves at the college level in the 1990s I, like so many others, found the ideas bold and

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exciting. I taught from a Reform text for a few years and it didn't work, but I presumed this was something about me rather than the text.

In the late 1990s I became involved in another bold and exciting project: computer-based education in the Math Emporium at Virginia Tech[1]. I did a great deal of one-on-one diagnostic work with students, helping them learn rather than teaching them. In retrospect I see that I was learning too; *about* learning, and the difference between teaching and learning. My students had troubling deficiencies, and I was drawn by this into K-12 education. I've had many adventures in education since then, and each one increased my anxiety and sense of urgency. This essay is the result.

The goal here is to explain why educational problems matter, give some nitty-gritty detail about learning deficits, and trace out some of their causes. The main conclusion is described above, and cannot be avoided. Several others are described in §6 and I mention two here. First:

It's not the teachers

It is fashionable to blame the current mess on incompetent teachers, but this is misplaced: it is the curriculum and methodology that are incompetent. Teachers following them faithfully *should* get poor outcomes. Worse, those calling for better teachers also call on schools of education—the real perpetrators—to provide them. Bad plan. I urge those wise enough to see that we are in a hole to also be wise enough to stop digging.

In fact, it seems likely to me that our current teacher corps could do the job if provided with competent methods, materials, and training. Hard, yes, but 'not impossible' would be very good news: we wouldn't have to wait for governments to raise taxes for higher salaries and better working conditions. We wouldn't have to wait for a complete replacement of teachers by a teacher-education system that currently can't even keep up with the attrition rate. Fixing the program will be hard, replacing the teachers would be impossible.

The other point is:

We need separate tech-oriented mathematics tracks

If a child wants to play the violin with any proficiency, private lessons are necessary; school music programs are not enough. If a child wants to be a professional violinist then more-disciplined lessons are necessary, to avoid habits that might support proficiency but block professional development. The professional must be able to focus on the music, and fully-correct use of the instrument must become nearly transparent.

Mathematics is the instrument on which science and engineering are played. Someone who has to think about the instrument will have difficulty being more than mediocre. Achievement at the highest levels usually requires basic math skills so correct and so well-learned that they are both effective and transparent. Unlike the private tutoring system in music, responsibility for mathematical development has been given to the schools. But to effectively meet this responsibility, *schools* must provide a range of instruction and discipline analogous to the range from standard school music programs to professional violin tutors.

The use of tracking has declined substantially in the last few decades. If we want high-level performance back, then we have to bring back tracking. It will,

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[1] http://www.emporium.vt.edu/

however, be necessary to address some of the drawbacks that drove the de-tracking movement, see §6.2.

2. Why weak high-tech outcomes are a problem

On the individual level, failing to develop technical talent raises concerns about wasting talent and not appropriately serving some students. On the other hand if we block access to technical careers, clever people can become lawyers or bankers or something. The inescapable problems are on the societal level.

We are producing fewer well-prepared scientists and engineers than our universities and high-tech industries need, and make up the difference with people from China, India, eastern Europe, etc. Many come to the US for advanced training, and they dominate elite graduate programs because Americans can't compete. They aren't smarter than Americans, just better prepared. But having our universities and high-tech industries dependent on a steady flow of well-educated foreigners (the "educational parasite" strategy, for short) is dangerous in the long run.

This was how the "Nation at Risk" commission described the situation *thirty years ago* [12], and why they identified weak education as a threat to national prosperity. After thirty years our universities and high-tech industries show the effects: they are heavily populated with immigrants and a great deal of the innovation in the US is being done by people from China and India. The *fact* of domestic under-production should be obvious even if the cause is not. Unfortunately, our current K-12 graduates have even more deficits than before, and we at the college level are even less successful getting them up to speed. This analysis also suggests that the new Common Core State Standards in Mathematics (CCSSM) will lock the problems in place for another generation. We will depend, more than ever, on the educational-parasite strategy. How long can this possibly last?

New since thirty years ago: graduate programs and high-tech industries in China and India have made great strides and are hungry for dominance. Many of those well-prepared students are now staying home, and high-tech jobs are already starting to go where the people are. Leadership in specialized target areas has already shifted, and this trend will accelerate as our head start erodes, our imported leaders age out, and our loss of leadership attracts fewer intellectual imports. My guess is that it is too late to prevent loss of US leadership in most technical areas, even if we address our shortfall immediately. If we put it off too much longer we may end up struggling to stay ahead of Brazil and Mexico.

3. Report from the front lines

The diagnosis begins with a report on math for engineering and science at Virginia Tech. The specific context is a course for second-year science and engineering majors, on multivariable differential and integral calculus, and infinite series.

3.1. **Discipline from the real world.** Virginia Tech has strong programs in science and engineering, and we get the best students in the state who don't go somewhere like Cal Tech or MIT. I am one of the people responsible for developing the math skills needed for engineering. Our goals for student learning are set by what it takes to deal effectively with the real world, and can't be redefined.

The problem is that, as compared with fixed real-world goals, useful preparation of incoming students has been declining for thirty years. The decline accelerated

10-15 years ago and the bottom has almost dropped out in the last five years[2]. Ten years ago I could still get most of my students up to speed by demanding better work (these are the bright ones, remember), but this year the old goals are simply impossible for many of them. I tell them "if you are an English major I will give partial and extra credit, drop bad scores, and grade on a curve, but if you are an engineer you actually need to be able to do this." Ten years ago this was motivation to do better; now it is a hint that they should switch majors.

Recently a student came to try to get more partial credit. He had put a plus instead of a comma in an expression, turning it from a vector to a real number. "But there was an example that looked like this, and anyway it is only one symbol and almost all the others are right." He had never heard the words 'conceptual' and 'error' used together; it made no sense to him and he would not accept it as a justification for a bad grade. I tried my usual "are you an engineer? ...", but he would not say and seemed unmoved. He was sympathetic with my frustration with poor preparation, but felt I should go with the flow and downscale my expectations. Looked him up afterwards: math major. Ouch.

This student was unusual in his determination to make the argument, but not unusual in the mistakes he made or in being more interested in partial credit than learning what he had done wrong. After this experience I told my classes that I was happy to help them with problems (I offer about 8 hours a week of office hours) but would not negotiate grades. Unfortunately, instead of coming for better reasons, they quit coming.

3.2. **Infinite series didn't converge.** The last third of this course concerns infinite series, and is the first point in our sequence where logic and mathematical structure cannot be avoided. The answer to 'does it converge' is either 'yes' or 'no', but serious reasoning is needed to reliably determine which. Instructional concern shifts accordingly, from getting the answer right to getting the reasoning right. Fifteen years ago students struggled with this and few were able to see it on a structural level, but most were at least able to deal with the logically-complex rules. Now it is like hitting a brick wall. What is going on?

To find out I announced a new grade scheme: a blank problem would be worth 30%, and they could lose this by writing something I felt they should know is nonsense. One day I will learn not to do things like this. Many of them honestly did not understand what did or didn't make sense, and I think I would rather not have known. Outcome: of 60 students, 55 failed and quite a few scores were lower than the blank-paper 30%. Highest score 99/100 (architecture student, high school near Bejing), next was 77 (private school that did not allow calculators). The following semester I started with a test on middle-school skills (most failed), serious quizzes, and frequent warnings about expectations. Many dropped the course but 30 of the 47 remaining still failed the test on series. There was, however, one perfect paper (high school taught in a monastery).

3.3. **Dismayed.** One day a student asked, "This is what I was taught to do in my tenth-grade class. Why did you give me a 0?[3]" Alarmed, I took a poll: 60% had seen infinite series in high school! As with the integral calculus they had been taught

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[2]Some of this acceleration is probably due to No Child Left Behind. Reform educators who opposed NCLB should not be blamed for all of this.

[3]This should be a question for the high-school teacher.

some simple tricks ("sit", "roll over", "fetch", "use the root test", ...). The precise conceptual preparation necessary to go beyond the tricks was incomprehensible to them, but rather than trying to learn, most of them were trying to survive using the tricks. As with serious calculus, high-school exposure is a barrier that has to be overcome, and many students don't make it.

4. Learning deficiencies

This section gives more detail on my students' cognitive and skill deficits, and a description of their sources in reform practice. Bear in mind that these are relatively high-performers, mostly second-year engineering students at Virginia Tech, and that 'learning deficiency' is defined functionally in terms of what high-performers need for technical careers. To keep this distinction clear I use 'tech deficiency', 'tech-appropriate standards', etc.

An overarching problem is that reform courses have very low functionality goals. This means tech deficiencies are either 'not part of our mission', or go unnoticed. Tech deficiencies that are visible but dismissed as unimportant are illustrated in the next section, §4.1.

Curriculum innovations have led to many new tech deficiencies. §4.2 gives a disturbingly long list of problems due to the way calculators are used. Making tests "calculator-friendly" obviously makes these deficiencies invisible, see §5.1.

Deficiencies in reasoning and precision are described in §§4.3–4.6. Another curriculum innovation, discovery-based learning, seems to have contributed substantially to this, see §4.3.

4.1. **Basic knowledge and skills.** I began a recent semester with a skills test on what I think of as middle-school or early high-school material. The first question was "Solve $bx^2 + (b-1)x + 1 = 0$ for x." This is supposed to trigger a reflex to use the quadratic formula, and take about ten seconds[4]. Only 30% could do it at all.

I explain why this is important. Many procedures in mathematics require solving for something. There are very few families of relations that students can solve quickly and reliably and—because of the quadratic formula—quadratic polynomials are the most useful. It would be great if we could also use cubics, trig or exponential solution procedures, but these are complex enough even for well-prepared students that they would seriously distract from the main topic. Consequently we contrive a great many of our examples and problems to depend on the quadratic formula, and we need for students to be able to use it quickly and transparently. Students who do not have this skill (around 70% now) are at a serious disadvantage.

Development of fast, accurate, and transparent skills has been explicitly deemphasized in the reform curriculum (cf. "drill and kill"). Rather than requiring memorization of the quadratic formula, "barriers are lowered" by using quadratics with integer roots that can be easily factored. The focus is almost exclusively on polynomials with numerical coefficients, so the more realistic examples with symbolic coefficients seem completely alien.

The next question was "Describe the c that satisfy $|2 - c| \leq 3$, in interval notation". Only 20% got this completely right, though a bit over half got it well enough that it would not have killed a line of reasoning.

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[4]Really ten seconds. This is supposed to indicate the level of 'automaticity' needed.

The question "Find the coefficient on y^2 in the product

$$\big((a+2)y^2 + 3y + 7\big)\big(2y^2 - ay + 1\big),$$

when it is written as a polynomial in y", is not hard if one understands the structure of polynomials well enough to pick out only the parts of the product that contribute to the y^2 term. About 35% managed to get the answer but, alas, most multiplied the whole thing out and discarded what they did not need. About 30% were essentially clueless.

Again the school focus is almost exclusively on polynomials with numerical coefficients, and many students simply did not know what to do with the symbols. Products of polynomials are usually limited to the simplest case (two binomials). The mnemonic "FOIL" gives intelligence-free guidance in this case but often becomes a barrier to dealing with larger examples.

4.1.1. *Summary.* Routine skills that should be established in middle school and well-exercised in high school are weak or missing. Teaching calculus and statistics without these skills is like teaching writing without a firm grasp of the alphabet.

4.2. **Calculator-related deficiencies.** Calculators are problematic for direct instruction as well but, having demonized drill, reform educators are more deeply committed to them and less likely to be aware of damage. They also tend to identify calculator use per se as mathematical, because they have little awareness of the mathematical sophistication needed for *effective* calculator use.

The following examples of calculator-related deficiencies come from extensive one-on-one work with students and are discussed in detail in other essays [16].

4.2.1. *Weak symbolic and abstraction skills.* It is now common for students to work numerical word problems quickly and easily with a calculator, but be unable to write the corresponding arithmetic expression and be stumped by the same problem with a symbol in the coefficients. In written work I see parenthesis errors that reflect dependence on encapsulation by sequential evaluation in calculator use. Symbols have gone from "things that act like numbers" to qualitatively different and alien, because numbers are processed with calculators and symbols aren't. The reason seems to be that the more complex tasks in K-12 are now usually formulated numerically and done with calculators, with a corresponding reduction in symbolic work, and that the resulting calculator expertise does not transfer to algebraic and symbolic skills.

This analysis also gives a useful perspective on what actually happens in manual arithmetic. We represent numbers by symbols. Manual arithmetic has symbolic and organizational subtasks that involve algebraic structure of addition and multiplication, and subtasks in which operations are carried out. It seems very likely that the non-numerical subtasks provide templates and subliminal preparation for symbolic work. As a result, de-emphasizing manual arithmetic probably contributes significantly to symbolic skill deficits. On the other hand, these benefits are accidental side effects rather than explicit objectives of manual arithmetic. A good understanding of them should lead to more effective ways to get them back than simply reinstating traditional arithmetic. This is explored at length elsewhere.

4.2.2. *Weak number sense.* In manual arithmetic the structure of numbers is used to carry out, or sometimes to avoid, calculation. Awareness of structure is functional. In calculator work, all numbers are qualitatively the same, so there is no payoff for awareness of structure. Students don't develop it. For instance multiplying by a power of ten can be accomplished by moving the decimal point. This is powerful in hand work, but useless on a calculator because moving the decimal point is accomplished by multiplying by a power of ten.

Weak number sense has two consequences. First, symbolic expressions often have analogous structure. Awareness of it in numbers makes it easier to see in other situations. Second, we need a certain amount of transparent mental arithmetic to be able to illustrate new procedures. Specifically, we want to be able to contrive examples in which the arithmetic does not break focus on the main task, but also without misleading numerical coincidences. Traditional students usually have skills adequate for this, but students who need a calculator to divide by ten do not.

4.2.3. *Weak geometric sense.* Qualitative sketches are essential in many areas of science and engineering, and qualitative function graphs give easy ways to organize a great deal of information. We used to take it for granted that students would have internalized libraries of shapes, including exponentials, logarithms, lines, quadratics, and basic trig functions. However, the skills test included the question:

"Roughly sketch graphs of $y = 3x^2 - 5$, and the logarithm $y = \ln(x)$. Label intersections with x and y axes, and please make the drawing large enough to be easily understood."

Only about a quarter of the class could draw something that resembled the logarithm. Around 2/3 managed an acceptable quadratic, though it often looked more like a "V" than "U". Many of the drawings were replicas of a graphing-calculator display, complete with poor choice of scale and too small to be useful.

Engineers complain that students can no longer do the rough sketches needed to organize many engineering problems. A clue as to why: when I ask them to pick up a pencil to draw a graph, some of them act as if I was asking them to pick up a snake. Apparently the eye-hand coordination aspect of actual drawing plays an important role in internalization of qualitative geometric structure. If they draw it they might get it, but just looking at it doesn't stick. High-school programs that use calculators for graphing, and that test visually ("which of the above four curves…"), produce little real learning. Assessments used by K-12 educators do not reveal this.

4.2.4. *Summary.* Calculators improve outcomes when the measures are adjusted to be 'calculator-accessible' (purge symbols, numerical answers, test graphs visually). Calculators are also part of the modern high-tech image, so they enable students to *look* like engineers and scientists. K-12 educators take pride in these superficial gains and are ignorant of (or strongly deny) evidence that this has undercut the learning needed for *effective* calculator use, let alone traditional symbolic and numerical skills.

4.3. **Reasoning deficiencies.** This section provides context for reasoning deficiencies described in following sections. It took me a long time to recognize and make sense of these, so I start with diagnosis rather than symptoms.

There are (roughly) three levels of student work in mathematics:

  (1) Follow patterns inferred from examples;

(2) Use systematic methods and algorithms that, among other things, account for the patterns in examples; and

(3) Exploit the mathematical structures that lie behind algorithms.

Each level provides substantially more flexibility, range and accuracy in applications than the one before it. Learning at each level is faster, more powerful, and more transferrable than at the one before it. Basic features are described below, and specific illustrations are given in §§4.4–4.6.

Advanced mathematics takes place at the structural level. Failure to make the transition to this level makes advanced study in mathematics impossible, and is a serious disadvantage in most technical professions. This has been the main bottleneck in high-tech preparation for at least a century. Reform attempts to open the bottleneck by redefining 'mathematical structure', 'understanding' etc. in more-accessible ways are necessarily ineffective: power comes from functionality, not words, and the redefinitions disconnect the words from functionality.

4.3.1. *Patterns and discovery.* Natural, or discovery-based learning takes place at the pattern-in-examples level. Patterns are abstracted as rules of thumb or heuristic ideas adapted to the examples given. This is done by the same cognitive processes that produce habits and superstitions so, naturally, the results are unreliable and poorly related to mathematical structure.

It is possible to get students to refine pattern-in-examples learning, but it is tedious and difficult because students have to be told repeatedly that they are wrong, and have to be willing to spend a lot of time trying to figure out why and how to fix it[5]. The discipline and engagement necessary to get functional outcomes are rare in the general student population. The level of discipline needed is unacceptable in reform education, and the expertise necessary is rare among teachers of any stripe. In practice, therefore, the tech deficiencies of discovery-based learning are not corrected.

The conclusion is that, in a tech-oriented curriculum, essential tools such as the quadratic formula must be taught directly. The formula for roots of cubic polynomials, on the other hand, is not an essential tool. Sub-optimal discovery outcomes would not be harmful, and it seems a good topic for a discovery investigation in a tech course.

Students taught mathematics at the pattern-in-examples level often get stuck there. Advancing to the systematic methods and algorithms requires not only unlearning the dysfunctional heuristics they discovered for themselves, but unlearning the approach to learning that produced these dysfunctional heuristics.

4.3.2. *Methods and algorithms.* Learning directly at the method-and-algorithm level takes advantage of accumulated experience to bypass the more-primitive level. Students get functional tools quickly and reliably and are less likely to get stuck at the lower level. Most students enjoy exercising their skills if given interesting problems, and they are more receptive to the mathematical structure behind well-learned algorithms.

In principle, traditional K-12 mathematics education takes place at the methods-and-algorithms level. In practice there are difficulties: the algorithms are usually

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[5]In mathematics this is known as the "Moore method", cf. Douglas, [4]. As an undergraduate I took such a course with Gordon Whyburn at the University of Virginia. A good experience, but very slow and disciplined.

historical relics, often not well-adapted to mathematical structure, and the standard range of problems does not provide interest or satisfaction as rewards for good skills. Word problems usually have completely trivial mathematical components.

4.3.3. *Mathematical structure.* Work at the structural level replaces a mass of rules and facts with logical reasoning and a much smaller list of more-basic rules and facts. Naturally, this only works if the reasoning is highly disciplined and the inputs are used with complete precision.

Our calculus courses are mostly rule-driven because attempts to introduce explicit structure have met with resistance and little success. Infinite series is usually the first topic with a structural orientation, because the reasoning is relatively transparent and the questions are qualitative. But it is accessible only to students already functioning effectively at the method-and-algorithm level.

In the very old days, math and math-heavy science majors got a sink-or-swim introduction to the structural level in advanced calculus, baby real or complex analysis, abstract algebra, and topology courses. Most programs now have an "Introduction to Proofs" course to ease the transition. But this means that even math majors don't see the real thing until their fourth or fifth semester in college. Modernizing the methods and algorithms in earlier courses should make this transition much easier.

This completes the overview. The following sections provide detail and explicit examples.

4.4. **Pattern-matching.** I can't remember any students in engineering calculus working at the pattern-in-examples level fifteen years ago, but there are quite a few now. Their rule skills are so poorly developed that sometimes just changing a letter, for instance asking for the integral or derivative of bx^2 with respect to b, throws them off. Another example concerns derivatives of the expression x^y. Everyone knows the derivative with respect to x. Many not only don't know how to take the derivative with respect to y, but seem unable to *learn* how. This is partly because the high-school approach to exponentiation is not functional in calculus, so this is not covered in high-school type courses. But when the correct pattern is missing, human pattern searches usually return the best available match (in this case, derivative with respect to the wrong variable) rather than no match. They get an error, and without recourse to disciplined reasoning it is hard to identify these, or correct them when they are identified.

Another example concerned a test question on calculation with derivatives of vector-valued functions. The answer was supposed to be a vector of the form (A, B, C). About 15% of the class did the computation (including a matrix product) correctly up to the last step, but then ruined the whole thing by converting the vector to a sum $A + B + C$. They explained that there was an example like this in which the outcome was a sum, not a vector. I checked, and found an example with a real-valued function quite similar to the sum of the components of the vector function. The outcome in the example was, accordingly, a sum. The point is that when confronted with a conflict between the outcome of a careful calculation and a pattern recalled from an example, they gave priority to the pattern.

The opposite mistake, following the pattern of an example to write an outcome as a vector rather than a sum, occurred on another test.

The last example concerns studying for a multiple-choice common final exam. The department posts exams from previous semesters, but does not post answer keys. The point of mathematics is, after all, to get the right answer without knowing it in advance. Students working at the method-and-algorithm level can get close enough to this ideal to confidently identify the correct answer from a small number of choices, and can also identify why the incorrect answers are wrong. But objections from some students revealed that they could not study without an answer key. To them, "studying" means discovering patterns of connections between questions and correct answers, not exercising the methods that produce correct answers!

4.4.1. *Summary.* Reform-trained students often rely more on pattern-in-example associations ('discoveries') than on careful step-by-step reasoning and use of algorithms. Their reasoning skills are rarely strong enough to refine or even filter the pattern-matching process, and are inadequate for extended or unfamiliar problems. They are doomed to fail advanced courses. The most painful aspect for a teacher is that they can't even identify why they fail, and go away thinking they must be stupid. But this isn't true: anyone who can get to second-year engineering calculus using pattern-in-example learning is not stupid.

4.5. **Disciplined reasoning.** Mathematical reasoning in real life is like walking a tightrope: if you make a single misstep, you fall. Functional skills therefore require careful step-by-step reasoning, learning to spot errors (especially their own), and learning how to fix them. The payoff is the ability to work long and complex problems with confidence. Years ago near the end of a semester I could say to my students, "this problem took almost an hour and required a variety of techniques that we did not anticipate. Getting it right is a real accomplishment. You should write home: 'Mom, look what I can do!'." But if I give an extended example now, most students will lose traction after one or two steps and its not something to write home about.

The most telling errors occur in infinite series. Most students do well with simple problems, and at first glance their work seems to show appropriate limit calculations. But the success rate drops sharply as problems become more complex, and on closer inspection the limit arguments are often bogus. There are frequent exponent errors such as $(\frac{1}{a})^{1/n} = \frac{1}{a^n}$, $(a(b^n))^{1/n} = ab$, and $(\frac{1}{a+b})^n = \frac{1}{a^n+b^n}$. Mistakes with factorials are common, for example canceling the '!' to get $\frac{(n+1)!}{n!} = \frac{n+1}{n}$. Parenthesis errors when incrementing the index in the ratio test are common. They know that $\lim_{n\to\infty} 5/n = 0$ because "$1/\infty = 0$", but some use the same heuristic argument to conclude $\lim_{n\to\infty} 5^n/n = 0$. Bizarre things are attributed to L'Hŏpitál's rule. Sometimes they simply pull limits out of the air. Finally, many cannot distinguish between statements such as 'the nth term test fails to show the series diverges', and 'the nth term test shows the series fails to diverge'.

There are two immediate questions: how can reasoning like this be successful at all, and where did it come from?

Limits and series are actually good opportunities for heuristic work. Simple limit calculations tend to be insensitive to errors, and this translates to simple patterns for outcomes. There are few enough tricky limits, (eg. $\lim_{n\to\infty}(1 + 1/n)^n$), that they can be memorized. There are only two possible answers to "does it converge". As a result, pattern-oriented students can usually determine the outcome of simple, routine problems.

Where such reasoning comes from? I mentioned above that almost 2/3 of my students had a heuristic introduction to limits and series in high school, nearly all in reform-oriented classes. They have been taught that algorithms are inferior to 'understanding', and that proofs—including limit calculations—are things mathematicians do after the fact to justify conclusions from intuition or 'understanding'. They do not expect conclusions to *emerge* from proofs, and their imitations of limit arguments are not reliable enough to do this for them anyway.

This explanation leads to the larger question: why do reform educators avoid disciplined reasoning? As a practical matter, discipline is unfashionable and children are increasingly unwilling to accept it. Discovery-based learning does not develop it, but most educators don't notice. Some educators offer a theoretical justification: standards for reasoning have changed a lot over the millennia, and if they prefer the less-disciplined approach of the fifteenth century, or even the basically-visual approach of the third century BCE, it should still count as 'mathematics'. Some promote intuitive approaches, to emulate the apparently effortless intuitions of experts. Quite a few educators think of logic as a cultural artifact; magical incantations with no essential function. Finally, all this is supported by low expectations and what amounts to a quantifier error. 'Teach quadratics' is taken to mean 'teach some examples' rather than 'teach a flexible and effective tool'. 'Teach series' means 'do tricks that work for a few simple examples' rather than 'lay a foundation for the subject'.

Finally, why is disciplined reasoning necessary? The discussion here provides a good context for an answer. When someone reaches his personal limits of heuristic reasoning and intuition, the reasons for failure are obscure and there is not much that can be done about it. This is why advanced mathematics was limited to a few extraordinary people up through the nineteenth century, and why students feel stupid when they reach their limits today. The great discovery of the early twentieth century [14] was that basing mathematics on disciplined reasoning rather than intuition makes it accessible to ordinary people. When people reach the limits of good basic logical skills then the failures are localized and can usually be identified and fixed. There is a clear, though disciplined and rigorous, way forward. Experts do eventually develop powerful intuitions, but these can now be seen as a battery, charged by thousands of hours of disciplined reasoning and refinement. Without the thousand hours you have a dead battery, and without training in disciplined reasoning you have no way to charge it.

4.5.1. *Summary.* Reasoning skills learned in reform courses are rarely strong enough to refine or even filter the pattern-matching process, and are completely inadequate for successful work at the mathematical-structure level. If real mathematics is like walking on a tightrope, reform math is like giving partial credit for steps anywhere near a limp rope on a pavement. Or like a spelling bee in which prizes are awarded for getting 80% of the letters right. Describing the outcomes as 'mathematics' is like describing air guitar and lip-synching as 'music'.

4.6. **Functional concepts and precision.** There are key facts and definitions that took professionals centuries to perfect. Many of these, including the quadratic formula, convergence, continuity, derivatives, and completeness of the real numbers, can be stated in a single carefully crafted sentence. They can be thought of as cognitive seeds, or highly condensed understanding, and as such are precious and

powerful legacies from our predecessors. They have also been optimized for effective use in disciplined reasoning.

Professionals begin by essentially memorizing such statements. Preliminary discussions may give clues about what to expect, but they are *brief* and *not* intended to develop understanding. Understanding and skill emerge from use. Beginning with a highly-optimized formulation enables rapid development of fully-precise intuitions and highly-reliable skills. This approach has been extremely successful, and most of us urge our students to follow this pattern in hopes that they will experience similar success.

In the reform "dialogic" approach, understanding is supposed to precede skill or precise concepts. Students are given heuristic explanations, examples and other hints, and are invited to synthesize or discover their own concepts. But the results have almost no chance of being functional: remember that it took gifted professionals centuries to extract themselves from all the subtle dead ends. I saw this in action in the late nineties when I taught from a reform text for several semesters. Most of the students failed to synthesize much of anything, and what they did come up with rarely supported material in the next course. My current students have been indoctrinated to believe memorization is 'wrong', so they find it almost impossible to take even the first step of the process that professionals find effective. They lack the logical skills needed to recognize and fix dysfunctional concepts, so even good students arrive at the college level with deeply embedded confusions about elementary material.

Embedded conceptual errors are difficult to fix. I first experienced this myself in a graduate seminar I taught at Yale. I scanned the paper we would study, to get an "understanding" to use to orient the students. This understanding turned out to be wrong. We spent a semester grinding through details and I saw my error clearly, but what did I remember decades later? The erroneous first impression. Recent neuroscience studies (cf. [5] for the physics version) paint a darker picture: efforts to correct conceptual errors often never actually succeed, and they are suppressed rather than fixed. Moreover, this suppression requires effort and attention. Some of my students persistently make middle-school errors. I used to say "you must be more careful", but if they are still doing it when they get to me, their automatic facilities are not recognizing the mistakes as wrong. "Careful" is not enough. Now I say "you have a problem with this; you must watch your work closely to catch it if it happens". For some it is too late. Others can, with effort, learn to consciously suppress these problems, but the cognitive overhead will keep them from ever working at full capacity.

The point is that the professional memorize-first approach is not a philosophical preference, but a hard-found way to work around limitations of the human brain. The educational understand-first approach *is* a philosophical preference, and maybe even a beautiful one, but no argument for it can change the fact that it is incompatible with human cognitive structure and inflicts damage.

4.6.1. *Summary.* To teach someone to use a hammer, have them drive a lot of nails. Asking them to 'understand' or 'discover' hammers essentially guarantees confusion and poor skills.

5. Mechanisms

I describe factors that have played a role in the decline of learning needed for high-tech careers.

5.1. **Problems with testing.** Like it or not, high-stakes testing heavily influences course goals and outcomes. Misunderstanding what they measure essentially guarantees that the influence will be bad. Believing that tests are neutral assessments essentially guarantees that the influence will be overlooked. Current practices, in short, pretty much guarantee that high-stakes testing will be influential *and* counterproductive *and* this will be unnoticed.

5.1.1. *Learning missed by tests.* Tests measure things we think should be correlated with learning, not learning directly. Correlations are speculative, and what is identified as 'learning' depends heavily on educational theory and ideology.

For example: multiplying two three-digit numbers by hand requires quite a bit of neural activity. Multiplying with a calculator requires almost no neural activity, but improves accuracy. Using performance (the number obtained) as a measure completely misses the difference. If the neural activity has important correlates (eg. providing templates and subliminal preparation for algebra) then conclusions drawn from test outcomes will be seriously flawed.

5.1.2. *Goal distortion by tests.* Tests have time constraints, and to accommodate this, tests traditionally spot-check and focus on easy cases. This may be harmless when students know nothing about the test, but if representative problems or tests from previous years are made available then these features become visible. Easy cases then tend to become course goals. Examples that have caused problems downstream include restricting quadratics to ones with integer roots, restricting polynomial multiplication to binomials, and focusing on numerical problems.

To avoid goal dilution, standardized tests must be noticeably harder than blind tests. The advance information about test contents can offset this, but it is a delicate balance.

5.2. **Self-serving goal distortion.** Systems with no responsibility for downstream consequences will set goals for their own comfort and convenience. K-12 educators, for example, have replaced precise statements with vague 'understandings', replaced disciplined reasoning with vague 'explanations', and replaced skills with calculators.

Systems like NCLB that focus on performance at the lowest levels, particularly when enforced with high-stakes penalties, need standards and tests that give schools a reasonable chance to avoid penalties. In other words, weak. Minimum standards will come close to being proficiency standards, and outcomes will be uniformly weak because resources are also focused on low-performing students.

All of these effects are easily visible in our schools today.

5.3. **Short-term goals without accountability.** Mathematics is cumulative. Children learn addition in the first grade, and professionals thirty years down the road are still adding. Ideally the approach used for children should be informed by, and compatible with, later use. Current practice, however, is that the curriculum is compartmentalized. Each level devises its own "age-appropriate" methods, interprets long-term goal statements in its own terms, and has essentially no subject-related accountability for the consequences. As a result the most common

approaches to addition in first grade do not even support activity in the middle grades. In fact nearly all of the creative and "child-friendly" methods in early grades are seriously out of step with needs later in the curriculum.

The worst failure of subject-related accountability is in the articulation between high school and college. K-12 educators' understanding of the needs of college students comes from within their own tradition, from free interpretation of general principles, and from the AP calculus exam. They have enough confidence in this that they are not fazed by critical feedback from college faculty, and there is no accountability that might impose discipline.

It should be emphasized that college faculty *are* accountable, to client departments in science and engineering, and through them to success in the real world. College criticism of K-12 programs is not a matter of opinion or optional cultural differences, but a message passed down from the real world: what you are doing is preparing students for remedial courses in community colleges, not the real thing. And many are crippled beyond recovery.

Finally, systems without subject-related accountability are vulnerable to political "solutions". College professors, for instance, were unwilling to develop subject-related accountability for their teaching. As a result they got stuck with student evaluations, even though the correlation with actual learning is insignificant or negative, see Clayton [2]. It seems to me that the reform movement is another instance. It provides a politically attractive solution (improving grades by de-emphasizing skills) that is irresistible in the absence of effective subject-related accountability.

6. Further conclusions

Here I sketch a few other conclusions from this line of investigation.

6.1. **Stop slandering teachers.** As noted in the introduction, it seems to be fashionable to blame bad outcomes on incompetence of teachers, cf. the *Science* editorial by Burris, [1] and the AMS *Notices* column by Kra [7]. The conclusion here and in [13] is that it is the *methodology* that is incompetent. Teachers using these methods *should* get poor results. Outstanding teachers may sometimes get better outcomes by partly compensating for methodological flaws, but remember these are "better" by flaccid standard measures and still far short of meeting real-life needs.

My belief is that the current teacher corps is capable of doing much better, given competent methodology and training on how to use it. The obstacles may be unsurmountable: the people we would ordinarily turn to for methodology and training are the ones who got us in this mess. Even if they acknowledged the problem they would be certain to botch the repair. But this is not a teacher problem, and it is unconstructive and dishonest to blame teachers for it.

A few comments on retraining: First, this should not be presented as an ideological winner-take-all struggle in which teachers would have to denounce their old beliefs as evil. Different children have different needs, and the real evil is insisting on a one-size-fits-all approach. We need different tracks for different interests (see the next section) and the reform approach may be satisfactory for the general track. This leads to the second comment: it is not necessary to retrain the entire teacher corps, just enough to implement the sort of tech track outlined next.

6.2. **Tech tracks.** We desperately need *some* technically well-prepared people, but it is both inappropriate and impractical to try to prepare all students for high-tech careers. Separate tech tracks are needed no matter what happens with the rest of the curriculum.

Tracking has declined substantially in the last few decades, see Loveless [9, 11]. It promoted skill development, but it was demonized as "elitist" and "undemocratic" by the same people who demonized skill development as mindless rote activity. On the other hand the charges have some foundation, and any attempt to re-introduce tracking must address the problems directly and honestly.

For example, placement must be based *only* on interest and performance, *not* ability. There will always be profoundly gifted students with no interest in technical topics, and it is a disservice to everyone to confuse a lack of interest with a lack of ability. Similarly, technical teachers will be unable to engage such students, and it would be inappropriate to describe this as a "failure" of the teacher.

Another common concern about tracking is that limited mobility between tracks will unduly constrain student opportunities. In principle there are straightforward and graceful ways to address this [15]. However it must be recognized that it would be very difficult to go from a tech-terminal track to a tech-oriented one. Good mobility might require intermediate tracks that are at least tech-neutral.

6.3. **Alternatives.** The approach here has been to start with what is needed for an effective career in science or engineering, and work backward to see what is needed from various levels of education. This is a high standard; real success needs a lot of preparation. The reform approach is terminal from this perspective, but are alternatives any better?

6.3.1. *'Traditional', and best practices.* 'Traditional' is usually not tech-terminal. It might serve as a starting point, but it was considered unsatisfactory thirty years ago (see §2), and while 'reform' may make 'unsatisfactory' look good, it is still unsatisfactory. One reason is that in the US in the early twentieth century, the Progressive movement set the stage for our current predicament by weakening content and introducing reform-like ideas, see Klein [6] and Lagemann [8]. Many of these changes are still visible in what we think of as 'traditional' today.

Best practices in other countries are also not the solution. The problem is that professional practice changed substantially early twentieth century, to become better adapted to both the subject and human cognition [14]. Extracting highly-precise understanding from definitions and concise theorems, as described in §4.6, is one of the most effective adaptations. With the brief exception of "new math", none of these changes were incorporated into education anywhere. The result is that even the most successful programs are still getting (at best) nineteenth-century outcomes. The students still have to deal with a huge conceptual gap between their training and the methodology that enabled explosive growth in mathematics itself over the last century. This will not be satisfactory in the twenty-first century.

6.3.2. *New math.* The "new math" episode tried to bring some modern methodology into elementary education, but was seriously unsuccessful in many ways, see Loveless [10] for a comparative analysis. It produced a pulse of well-prepared students that, by comparison, made the situation in 1980 look so bad to the "Nation at Risk" commission [12]. These well-prepared students also contributed substantially to US success in the late twentieth century. But the failures far overshadowed these

accomplishments and it is essential that the mistakes not be repeated. It seems to me that the main lessons are:

- The need is for modern treatments of old topics, *not* modern topics;
- Don't inflict it on students *not* interested in technical careers;
- Provide training and support for teachers, and *listen and respond to* their feedback.

The K-12 education community drew the conclusion "modern is bad for kids", but this is looking more and more like a one-way ticket into the third world.

7. SUMMARY

Thirty years ago the "Nation at Risk" commission concluded

> If an unfriendly foreign power had attempted to impose on America the mediocre educational performance that exists today, we might well have viewed it as an act of war. As it stands, we have allowed this to happen to ourselves. [12]

As Pogo put it, "we have met the enemy, and he is us". But Pogo is a comic book character and real people are less wise. Instead of recognizing the problem, the reform movement acted out Einstein's definition of insanity: doing more of the same and expecting different results. Mathematics outcomes have worsened in the last thirty years and are still declining, and how do they propose to respond? Redouble their efforts to do more of the same.

Is there any way to break out of this death spiral, given that US math-education is dominated by people committed to the reform agenda? Take the act-of-war view seriously and have treason trials? Adapt Shakespeare's suggestion "first, kill all the lawyers"? A compromise that accommodates both skill and non-skill approaches through tracking might be acceptable. In any case some sort of rigorous account-ability is long overdue, and as a start we could at least openly acknowledge the reality of the situation: reform-oriented math courses are tech-terminal.

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