

SUPPLEMENT TO ‘A SCIENCE-OF-LEARNING APPROACH TO MATHEMATICS EDUCATION’

FRANK QUINN

ABSTRACT. Provides additional examples, and comparisons to articles in the March 2011 Special Issue on Education of the *Notices* of the American Mathematical Society.

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This supplement gives material that, for space reasons, could not be included in *A science-of-learning approach to mathematics education* [6]. There are also direct comparisons to articles in the March 2011 Special Issue on Education of the *Notices* of the American Mathematical Society [1] that inspired the writing of [6].

1. MORE ABOUT LEARNING

Everyone agrees that learning should be the goal of teaching, but there is not much clarity on what constitutes “learning”. Mainstream approaches are heavily influenced by classroom experience, social theory, philosophical principles, and abstract psychological models of students. In the end “learning” is very nearly defined

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as “the result of teaching”, and this traps the system in a logical circle that makes change almost impossible.

The basic focus in [6] and in [8] is on students, mathematics, and learning explicitly *separated* from teaching, not because teaching is unimportant but in order to see learning as it really is. The results are often very different from current practice. An explicit example is given next.

1.1. Reading and writing complex expressions. Difficulty dealing with complex expressions can be a real barrier to later success. Careful attention to learning, especially through diagnosis of individual student problems, leads to the following analysis and suggestions:

Working with mathematical expressions requires that they be parsed (linearized), and there are a number of ways to do this.

- The logical structure encodes an “outside-in” parsing order. For example:

$$5(3y^2 - a) + (y - a)(y + 6a)$$

is the sum of two terms, and each of these terms is a product. Logically parsing to this level gives

$$5(\dots) + (\dots)(\dots).$$

Terms in parentheses are deeper in the logical structure.

- The left-to-right parsing used for reading gives “five times the quantity three wye squared ...”. The encapsulation by parentheses is hard to say and may get lost.
- There is also an “inside-out” parsing order reflecting the fact that processing usually begins with innermost fragments and works out. This leads many students to ignore large-scale structure until inner things are simplified. For instance they will want to expand the $(y - a)(y + 6a)$ term before even thinking about the rest of the expression.

Teaching students to use logical (outside in) parsing order extends the expressions they can process, but the greatest benefits are in the expressions they can write. The organizational step in the polynomial multiplication template in [6], for instance, uses outside-in parsing very explicitly by writing parentheses first and filling them in later. Another instance is the summation notation. If this is not parsed correctly it is hard to make sense of it, and almost impossible to write or manipulate correctly.

I have two concrete suggestions. First, logical structure can be emphasized when expressions are described by teachers. For example when writing the expression above, after writing “5(” write the corresponding *closing* parenthesis first, and then fill in. Students emulate what they see. Note that this requires a dynamic presentation: showing the end result does not show this structure. A corollary is that to be maximally effective, texts and reference materials really should show the process, not just the outcome (YouTube?!).

The other concrete suggestion has to do with parentheses. The customary notation is hard to parse, and elementary educators go to great lengths to avoid them. Indeed “simplify” is often used to mean “write without parentheses”. But avoiding parentheses makes simple expressions harder and complex ones impossible. In the discussion of polynomial products in [6], for example, profligate use of parentheses is used to separate organization and arithmetic. Teachers often change the order of terms and do arithmetic on the fly to avoid intermediate forms with parentheses.

These hidden operations often mystify students, and if students try to emulate this but come out with a form that is incorrect without parentheses, then they get it wrong and can't understand why. The suggestion: use parentheses from the very beginning of elementary mathematics, with a notation that is easy to parse. Paired parentheses could be joined by an overline, for example:

$$5 \left(\overline{3y^2 - a} \right) + 3 \left(\overline{\overline{(y - a)} \overline{(y + 6a)}} \right)$$

2. MORE EXAMPLES

2.1. Square roots. This illustrates that the approach used for fractions is a general-purpose template that works for square roots and some other implicit descriptions.

2.1.1. *Definition.* “ \square can be expressed as \sqrt{a} ” means $\square^2 = a$ and $\square \geq 0$.

(See [6] for explanation of the placeholder notation \square .) Exactly as with fractions, the structure of the definition implies that the default way to work with square roots is to square them. The usual rules are shortcuts that sometimes avoid this, but these shortcuts are verified by squaring, and if you can't see how to use shortcuts then square.

Examples

- (1) The decimal 2.5 can be expressed as $\sqrt{6.25}$ because $2.5^2 = 6.25$ and $2.5 \geq 0$.
- (2) Problem: express $\sqrt{a}\sqrt{b}$ as a square root. Response: $\sqrt{a}\sqrt{b} = \sqrt{\square}$ means $(\sqrt{a}\sqrt{b})^2 = \square$. Expanding the left side gives

$$(\sqrt{a}\sqrt{b})^2 = \sqrt{a}\sqrt{b}\sqrt{a}\sqrt{b} = \sqrt{a^2}\sqrt{b^2} = ab.$$

So $\square = ab$. Referring back to the meaning of “ \square ” gives: $\sqrt{a}\sqrt{b} = \sqrt{\square} = \sqrt{ab}$. Convenient shortcut, not an independent fact.

- (3) Suppose $a > 0$. Is $1 + a \stackrel{?}{=} \sqrt{1 + a^2}$? (See [6] for explanation of the $\stackrel{?}{=}$ notation.) To find out, square. This transforms the question to $(1 + a)^2 \stackrel{?}{=} 1 + a^2$. Expanding gives $(1 + a)^2 = 1 + 2a + a^2 \stackrel{?}{=} 1 + a^2$. Subtracting $1 + a^2$ from each side gives $2a \stackrel{?}{=} 0$. But $a > 0$ so this is not true, and the answer is ‘no’.
- (4) Problem: express $\sqrt{a} + \sqrt{b}$ as a square root. Response: $\sqrt{a} + \sqrt{b} = \sqrt{\square}$ means $(\sqrt{a} + \sqrt{b})^2 = \square$. Simplifying the left side gives $a + 2\sqrt{a}\sqrt{b} + b = \square$, so $(\sqrt{a} + \sqrt{b}) = \sqrt{a + 2\sqrt{ab} + b}$ (but see the caution below). Another shortcut, not convenient enough to justify remembering.
- (5) Problem (advanced): use the transformation $x = \frac{1}{2}(t - t^{-1})$ to express $\sqrt{1 + x^2}$ without square roots. Response:

$$\begin{aligned} \sqrt{1 + \left(\frac{1}{2}(t - t^{-1})\right)^2} &= \sqrt{1 + \frac{1}{4}t^2 - \frac{1}{2} + \frac{1}{4}t^{-2}} = \sqrt{\frac{1}{4}t^2 + \frac{1}{2} + \frac{1}{4}t^{-2}} = \\ &= \sqrt{\left(\frac{1}{2}(t + t^{-1})\right)^2} = \frac{1}{2}(t + t^{-1}) \end{aligned}$$

This transformation is useful for arc-length problems in calculus. It might be used in an upper-track algebra course to explore connections between square roots and polynomial fractions.

Caution: If a question has the form “Express (thing) as a square root” then $(\text{thing}) = \sqrt{(\text{thing})^2}$ is a technically correct answer. The usual intention is that $(\text{thing})^2$ should be simplified, but it is technically incorrect to mark an answer as wrong if it has not been simplified in the expected way. This is a source of confusion and I am not sure what to do about it. The intent is clearer in “Find a simplified expression for (thing) as a square root”. This is uncomfortable because “simplify” is mathematically ambiguous, especially if there is no dramatic reduction. On the other hand it is exactly the mathematical ambiguity of “simplify” that can cause confusion. Having it explicit in the statement might help keep us honest, and at least makes it available for discussion if the need arises.

2.2. Polynomial partial fractions. In [6] I suggest that an advantage of a precise approach to fractions is that partial fractions in the integers and in polynomials are seen as different instances of a single general idea. Different techniques are used to solve specific examples, however: modular arithmetic in the integers and linear algebra in polynomials. On the one hand this illustrates how conceptual unity at an abstract level often dissolves at the computational level (and how approaching them computationally hides the conceptual unity). On the other hand the unity suggests looking for analogs of each computational technique in the other domain. There is no integer analog of the linear algebra approach (because integer primes are not linearly independent). But there is a polynomial analog of modular arithmetic, and a special case is commonly used in school courses. I illustrate this analog by finding the coefficients c, d in the equation

$$(1) \quad 3x^3 + 2x + 9 = (ax + b)(x^2 - 2x + 3) + (cx + d)(4x^2 - 4x + 1)$$

from the example in the Polynomial Partial Fractions section of [6].

The plan is that we want to work modulo the polynomial factor on the a, b term, $x^2 - 2x + 3$, so set $x^2 - 2x + 3 \equiv 0$. Technically we are working in the quotient ring $\mathbb{R}[x]/(x^2 - 2x + 3)$. In practice we write the imposed identity as $x^2 \equiv 2x - 3$ and use this to reduce second and higher-order terms. For instance $x^3 = x(x^2) \equiv x(2x - 3) = 2x^2 - 3x \equiv 2(2x - 3) - 3x = x - 6$. Equation (1) reduces, modulo $x^2 - 2x + 3$, to

$$5x - 9 \equiv (cx + d)(2x - 3).$$

Expand the right side and reduce the second-order term again to get $5x - 9 \equiv (-3c + 4d)x + (-12c - 11d)$.

We now apply the fact that if two degree-one polynomials are equivalent modulo a degree-two polynomial then they must actually be equal. (The technical context is that this is a Gröbner basis for the polynomial quotient ring.) The coefficients on the two sides must therefore be equal and we get the system of equations

$$\begin{pmatrix} -3 & 4 \\ -12 & -11 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} 5 \\ -9 \end{pmatrix}$$

This implies $(c, d) = (-\frac{19}{81} + \frac{29}{27})$, as before.

A version of this is commonly used as a shortcut in simple cases: working modulo a degree-one polynomial $x - r$ is the same as evaluating at $x := r$. If the factors of the denominator are *distinct* linear terms (i.e. no complex or repeated roots), and if it is easy to do arithmetic with the roots (i.e. small rational), then the coefficients can be obtained by evaluations. Some high-school courses use this method exclusively. This is a bad idea, in the same way that restricting to quadratics with

integer roots is a bad idea: contrived problems that enable easy methods leave students helpless in more general (and realistic) cases.

In general the modular-arithmetic version seems to take longer than linear algebra. It is worth exploring to illustrate similarities between the integer and polynomial situations. It is also good to give students multiple tools, and sometimes hybrids can be used to good effect. For example, if there are n unknown coefficients in the partial-fraction expansion then the goal is to find n linear equations that determine them. The routine system comes from coefficients in the polynomial equation. However if there is a root at which polynomials can be easily evaluated (e.g. small integer) then the evaluation can be used to get one linear equation that is often simpler than the ones from coefficients. Replace one of the coefficient equations with this to get an easier system. This comes with a caution: the system from polynomial coefficients is guaranteed to be nondegenerate, so can *always* be solved. Replacing one of the equations may give a degenerate system. It is a good strategy because the failures are very rare, but students should know how to recognize a failure and know that if one occurs they should go back to the routine system.

3. MORE WORD PROBLEMS

Again, a *mathematical model* is a translation of a real-world or word problem to a symbolic form suitable for mathematical analysis. The key to making this effective is to separate the cognitively-different tasks: do little or no analysis during modeling, and no further modeling during analysis.

Four examples are given. The first and third come from the article of Heaton and Lewis [4] in the *Notices* special issue, and the approach used here is contrasted with theirs below in §4.3. The second and third illustrate difficulties with trick problems. The fourth illustrates breakdowns in the modeling step.

3.1. Example 1: Chicken nuggets. The “chicken nugget conundrum¹” is:

Chicken nuggets are available in three size boxes: six, nine, and twenty. What is the smallest number of nuggets that you *cannot* get by ordering combinations of these three sizes?

3.1.1. Mathematical model. The mathematical model for the chicken nugget problem is:

What is the smallest integer that *cannot* be expressed as $6a + 9b + 20c$, with a, b, c nonnegative integers?

Note that this is a direct translation, without any mathematical processing at all. The work from now on is entirely mathematical, with no cognitive overhead or confusion about food or boxes.

The model makes the mathematical issues much clearer. The greatest common divisor of 6, 9, and 20 is 1 so *any* integer can be expressed as combination if negative coefficients are allowed. The restriction on realization comes from the restriction to *nonnegative* coefficients. The assumption is not explicit in the problem so it is easy to miss, especially if the problem is not modeled. Someone impressed with the “debit” explanation of negative numbers might even question its validity: “can one get 3 nuggets by getting a box of 9 and an empty 6-nugget box, putting 6 in the empty box and selling it back to the restaurant? You did not *say* this is

¹A ‘puzzler’ from the National Public Radio program *Car Talk*.

not allowed.” In any case this makes the question more subtle than a congruence problem. Another complication is that while the whole set of three coefficients is “relatively prime”, some of the pairs are not relatively prime as pairs. Specifically, primes 2, 3, and 5 are involved and both of the smaller ones appear in two of the given coefficients.

3.1.2. *Solution.* Set $k = 6a + 9b + 20c$. We want to estimate the coefficients in terms of information about k . Students should see (or easily discover) that modular arithmetic is the best approach; see the discussion in §4.3.

To get information about the coefficient c , work modulo the greatest common divisor of the other two denominators, namely 3. Mod 3 the equation becomes $k \equiv 20c \equiv 2c$. Multiplying by 2 (the inverse of 2, mod 3) gives $2k \equiv c$. This determines $c \pmod 3$, and since 3×20 can be expressed as a (nonnegative) combination of 6 and 9, we can assume c is 0, 1, or 2. This gives the first conclusion: if $k \equiv 20c$ for $0 \leq c < 3$, then $k \geq 20c$.

Next suppose $k \equiv 20c$ with $c = 0, 1, \text{ or } 2$, and $k \geq 20c$. Then $k - 20c = 3m \geq 0$, and k can be realized if and only if $3m$ can be realized as a nonnegative combination of 6 and 9. Or equivalently, if m can be realized as $2a + 3b$ for nonnegative a, b . This is a smaller version of the original problem, and can be done the same way (i.e. reduce mod 2 or 3 to relate m and one of the coefficients). Eventually we see that 1 is the only nonnegative integer that cannot be realized in the smaller problem, so the largest in the original problem is $2 \times 20 + 1 \times 3$. We could also easily list all non-realizable integers.

3.2. **Example 2: quarter-full fuel tank.** This² illustrates a difficulty with trick problems.

A trucker’s fuel gage is broken, and rather than fix it he puts a stick in the tank and measures the length that gets wet by the fuel. The tank is a circular cylinder with horizontal axis. He knows that when the wet length is half the diameter then he has a half tank of fuel. What length corresponds to a quarter tank of fuel?

This sounds like a typical geometry problem that can be solved by a clever trick or insight. However there is no trick, and one can waste a lot of time looking for it. To *prove* there is no trick we find an equivalent problem. A nontrivial integration and trig identities show this is equivalent to the following:

- Find $s > 0$ so that $\cos(s) = s$
- define d by $(\frac{s}{2})^2 = d^2 - d^4$, then
- the length of wet stick corresponding to 1/4 tank is $1 - d$ (approximately 0.5960) times the radius of the tank.

The quadratic formula can be used to express d explicitly in terms of s using square roots, and conversely knowing d solves $\cos(s) = s$, so the two problems have equivalent difficulty. But $\cos(s) = s$ is not an elementary problem.

The point is that trick problems are artificially contrived, and small changes give identical-looking problems that are impossible with elementary methods. Trick questions give misleading impressions of the power of the methods and the nature

²Yet another ‘puzzler’ from *Car Talk*.

of mathematics. Euclidean-style geometry particularly suffers from this drawback. For further discussion see §4.5, below.

3.3. Example 3: Crossing the river. Another example paraphrased from [4]:

A group of adults and children on a camping trip come to a river. They find a boat that can hold one adult or two children. Anyone in the group can safely row across the river by themselves. If there are four adults and two children on the trip, is it possible to get all of them across the river? If yes, how many one-way trips across the river will it take?

3.3.1. Model. Denote the number of adults and children by a, c respectively. We model the problem using ‘moves’:

- A corresponds to an adult taking the boat from the first shore to the second, and A^{-1} is the inverse move. C, C^2 correspond to one or two children taking the boat.
- A sequence of crossings corresponds to a list (or “word”) in which moves and inverses alternate. For instance AA^{-1} corresponds to an adult taking the boat across and then an adult coming back.
- A sequence is “allowable” if every initial segment has total exponent on A in $[0, a]$ and total exponent on C in $[0, c]$. For instance $AA^{-1}C^2C^{-1}$ is an allowable sequence if a, c are not too small, but $C^2A^{-1}C^{-1}A$ is not because the initial segment C^2A^{-1} has -1 total exponent on A .
- The mathematical model is: when is it possible to have an allowable sequence with total exponents a, c ?
- What is the minimal length of such a sequence?

There are so few allowable sequences without immediate cancellations that the solution to the original problem can be found essentially by trying all possibilities, even without setting up the model. The model clarifies what is going on and gives access to the general solution.

First note that if $c \geq 2$ then C^2C^{-1} is an allowable sequence of even length with net effect C . Iterating this gives an allowable sequence with total exponent $c - 1$, and ends with C^{-1} . Omit this final move to get a sequence with exponent c . The total length of this sequence is $2c - 3$.

Returning to the word problem, this means if there are at least two children then all of the children can get across the river.

Now look for a similar even-length sequence with total effect A . It must start with C^2 or the next move will just cancel it. The second move must be C^{-1} for the same reason. The next move can be A . The fourth move must be C^{-1} : A^{-1} would defeat the purpose and C^{-2} is not allowable. We are thus led to $C^2C^{-1}AC^{-1}$. Iterating this gives an allowable sequence with total A exponent a , and C exponents between 0 and 2.

Combining the moves above shows: if $c \geq 2$ then any pair a, c can be realized, by a sequence of moves of length $4a + 2c - 3$. This is the minimal length. Minimality can be proved by induction on a but this is not really elementary.

3.3.2. Discussion. This problem does not connect with much of anything. Most modifications of the rules, eg. to have the boat carry one adult and one child, or up to three children, make the problem trivial. Variations with three passenger types,

one of which might eat one of the others, are even more contrived and still fail to illustrate mathematical structure. Like so many trick problems, this seems to be a clever dead end.

3.4. Example 4: Problems with modeling. These illustrate the model/reality disconnect, and would be fun in a discussion of modeling as an independent activity. This expands on the pencil problem in [6].

3.4.1. *Alcohol tax.* Bubba makes 50 gallons of moonshine per month at the still behind his barn in Giles county. The Virginia state tax on alcohol is \$20.25 per gallon. How much tax does Bubba pay in a year?

3.4.2. *Pig tax.* A peasant has 5 pigs. The High Sheriff comes by to collect the tax of 3 copper coins per pig. How much tax does the High Sheriff collect from the peasant?

- (1) 7.324 copper coins, due to special pig-counting rules in the tax code.
- (2) 6 copper coins, for the two pigs the High Sheriff was able to find.
- (3) None, because the peasant killed the pigs. They then became ‘ham’, and there is no tax on ham.
- (4) None, because the peasant killed the High Sheriff.

3.4.3. *Hard time.* Joe is sentenced to 5 years in prison for embezzling ten thousand dollars. How many years will Bernie get for embezzling two *billion* dollars?

4. COMPARISONS

In this section I discuss some of the articles in the *Notices* special issue. They are all seriously flawed, but I focus on articles with enough explicit detail to support analysis.

4.1. **H. Wu.** There is a great deal to admire in Wu’s work, but here we must be concerned with flaws. In his article *The Mis-Education of Mathematics Teachers* [10] he observes that there is a chasm between advanced coursework in mathematics and what is needed for contemporary K-12 teaching. This is certainly true, but he sees this as a problem with advanced coursework rather than contemporary teaching.

First, Wu points out that fractions are usually presented without a definition, and described in three incompatible ways all of which are wrong. He then asserts that this is an instance where advanced training is irrelevant to elementary education, because the precise mathematical definition is too complicated. I discuss the general relevance issue in the section “Advanced study *is* necessary?” in [6]. Here I explain that what he describes as the mathematical definition is also wrong, and this is part of the reason it seems too complicated.

Wu says that fractions are equivalence classes of ordered pairs, but this is obviously wrong: a fraction is an ordered pair, not an equivalence class. This imprecision is sometimes convenient, as I explain next, but this doesn’t make it less confusing to beginners.

The first convenience of confusing pairs with equivalence classes comes from the desire to identify fractions with rational numbers. There are many fractions that represent any given rational. This is not a problem if we think of these as *names* for the number, but if we don’t want to distinguish between names and

things then we have to think of fractions as equivalence classes. This confusion is more-or-less harmless in a college class when fractions are used to construct the rational numbers. It is not harmless in elementary education where most of the difficulty concerns the *difference* between names and numbers (e.g. determining if two fractions represent the same number).

A deeper motivation for the imprecision is that an explicit description of a thing (e.g. a decimal) automatically shows the thing exists and is well-defined. Both of these must be established for things like fractions that are defined implicitly. Moreover, problems with things like $\frac{2}{0}$ show that this is not completely straightforward. Wu's "a fraction is an equivalence class" reflects the usual way to address this: it is short for "things satisfying the defining property of fractions do exist, in the ring of equivalence classes of fractions". This is inappropriate for elementary education in the same way as " $\sqrt{2}$ exists in the quotient polynomial ring $\mathbb{Q}[x]/(x^2 - 2)$ " but, fortunately, neither is necessary.

Uniqueness, not existence, is the key issue for fractions, square roots, and certain other implicit definitions. Roughly speaking, the reasons for uniqueness are so robust that they imply existence in an extended context (see *Proof projects for teachers of mathematics* in [8] for more about this). The point is that the feature that Wu sees as making the "real" definition excessively complicated can be omitted without harm, and when students are sophisticated enough to notice something is missing it can be upgraded quickly and easily.

Another problem Wu cites is the irrelevance of modern geometry courses to the Euclidean-style geometry taught in schools. It seems to me that a connection is impossible because Euclidean geometry no longer qualifies as mathematics. It certainly does not meet Wu's "fundamental principles" ([10] p. 378): the objects are not precisely defined; the arguments are not precise and can barely be considered logical; the methods are certainly not coherent with modern mathematics, and it is not particularly goal-oriented. Defects are analyzed in detail in [7] (see the index); I mention only one here. Many Euclidean arguments are essentially proof-by-example. People are supposed to choose "generic" examples that somehow implement the universal quantifier ("for every triangle...") but there is no criterion for when this is successful. In fact it sometimes fails, and it works most of the time only because the subject is so simple. The whole thing is a very poor model for mathematical reasoning.

In his discussion of the need for inservice training Wu writes:

It is time for us to break out of the vicious cycle by exposing teachers to a mathematically principled version of the mathematics taught in K-12.

But there is *no* mathematically principled version of much of what is now taught in K-12: most of it is ineffective, obsolete, or plain wrong. Breaking out of the vicious cycle will require profound changes in K-12 teaching. Wu actually gives the argument for this, particularly in his section on "Fundamental Principles", but somehow draws the opposite conclusion.

The above may seem like an attack on Wu's article, but in fact I think he is right more than he is wrong. His "Fundamental Principles of Mathematics", for instance, could be more precise and purposeful but they are sensible and refreshingly free of philosophy.

I would like to expand on some of his objections. First, fractions are *not* mathematically the same as ratios. In particular, ratios interact poorly with negative numbers, and if one wants to do vigorous arithmetic then one more-or-less has to give up one or the other (or be very careful and sophisticated). Descartes accepted ratios as the correct division-like operation, and consequently found negative numbers so problematic that he referred to them as “false numbers”. Imaginary numbers were less problematic for him! Ratios are *related* to fractions, but to equate the two is a fairly serious falsehood. Similarly, the ‘parts-of-a-whole’ approach to fractions is both dysfunctional and specialized to the integers. It might be seen as an *application* of integer fractions, but to present it as the definition seriously misrepresents the concept.

4.2. **Ira J. Papick.** Papick’s article [5] is titled *Strengthening the Mathematical Content Knowledge of Middle and Secondary Mathematics Teachers*. On p. 390 he gives a long list of student questions that “teachers should be prepared to address in a mathematically meaningful way.” But almost all of these questions reflect serious confusions, misrepresentations, or outright errors in the standard curriculum. Rather than preparing teachers to straighten out students who happen to notice them, shouldn’t we straighten out the curriculum so they don’t occur? Isn’t this better seen as a list of things that need to be fixed in contemporary K-12 mathematics? Some specifics:

- Question 2 reflects confusion due to sloppy and ambiguous use of “=”; see the section on notation in [6].
- Question 3 reflects the confusion between fractions and the rational numbers they represent; see the discussion in the previous section.
- Question 5 reflects egregious errors in educational use of guessing the next term in a sequence.
- Question 6 on $2^{\sqrt{2}}$ reflects problems that Wu [10] p. 376 tries to address with his “Fundamental assumption of school mathematics”: formulas for rationals extend (by continuity) to real numbers. This ensures things gotten this way won’t be false, but it is not functional as a definition. At some point the miraculous exponential function should be introduced, and the genuinely functional definition $A^B = \exp(A \log(A))$ given. Contemplating $2^{\sqrt{2}}$ reveals why this is a good thing.
- Question 7, on the difference between $\frac{(x+3)(x-2)}{x-2}$ and $x + 3$, reveals an imprecision that teachers often abuse (and reinforce) in testing. Claiming these are the same, and that the nonsingular version is the “right form”, enables teachers to mark as wrong an answer that doesn’t include the cancellation. Most interpretations of “simplify” are similarly problematic. The best path through this confusion seems to be that the two expressions are the same as polynomial fractions, but different as functions because they have different domains. This also explains why “rational functions” is a bad name for polynomial fractions.
- Question 9, on the relevance of the quadratic formula in the age of calculators, reflects serious problems in the educational community. First, calculators belong in the curriculum, but it is much more difficult than generally appreciated to use them in ways that do not undercut long-term learning goals. This is an instance. Second, the question arises because the

current curriculum is almost entirely focused on numerical problems. Why not include things like:

Problem: Describe the solutions of $x^2 = 3ax + 2$ as functions of a .

Papick describes a number of courses developed to help teachers deal with such issues. The Algebra for Algebra Teachers mentioned on p. 392, for instance, has a relevant list of topics. But it cannot connect with teaching until educational methodology becomes more precise and mathematical, see §5 in [6].

4.3. Ruth M. Heaton and W. James Lewis. Their article is *A Mathematician-Mathematics Educator Partnership to Teach Teachers*. Lewis (the mathematician) describes his goal as

... to help teachers become productive mathematical thinkers with a toolbox of skills and knowledge to use to experiment, conjecture, reason, and ultimately solve problems.

They describe his use of the “chicken nugget conundrum”, used as an example here in §3.1, in one of these courses. However the description raises questions about what they mean by “mathematical thinker”, “toolbox of skills”, etc.

Modeling is an essential part of the professional toolbox, but contemporary elementary-education philosophy rejects it and these students did not use a model. This problem has a subtle mathematical core, so massive confusion was a predictable consequence. Another consequence is that the students missed important structure. The key ingredient mathematically is that the coefficients are nonnegative. In the chicken formulation this is an implicit property of boxes of nuggets, and since the solutions were still phrased in terms of chicken (see Susan’s solution, p. 398) this was never made explicit. In effect they learned something about chicken nuggets rather than something about integers. A further consequence was that (as the authors observed) the students’ explanations were long and wordy. Professionals quit writing things out in word form in the seventeenth century when modeling became widespread.

A second concern is that the students did not use the appropriate mathematical tools. The discussion in §3.1 begins with “The greatest common divisor of 6, 9, and 20 is 1 so *any* integer can be expressed as combination if negative coefficients are allowed. The restriction on realization comes from the restriction to *nonnegative* coefficients.” It seems inappropriate to me to give this problem to anyone who does not know the common-divisor (relatively prime) fact and can use it to home in on the core issue. Moreover the appropriate tool for the analysis is modular arithmetic. Eventually Susan used divisibility to justify the answer, but this is unwieldy and the extract of her work makes it doubtful she could have found it on her own. Why was she not steered toward modular arithmetic during the mentoring sessions? A possibility is that the description of modular arithmetic Lewis sees as appropriate for school use is not actually functional. This is true of Dubinsky; see the next section.

To summarize: the authors present this as an example where success was salvaged from unexpected disaster. But it seems to me disaster should have been expected, and the outcome might be better described as *failure* salvaged from disaster. Apparently it looks better from within mainstream educational theory.

4.4. Ed Dubinsky and Robert P. Moses. Dubinsky and Moses present a synthesis of everything relevant, and the Civil Rights movement as well.

Dubinsky writes that at one point he realized that to significantly improve his students' learning he would have to better understand the process of student learning. But instead of studying students, he studied the education literature. When he found Piaget he says "... I knew I had come home" [3], p. 402. Some of Piaget's insights are impressive but they are abstract constructs that almost invite abuse at micro levels. For instance

If one has built appropriate [mental] structures, very early concepts can be grasped easily ... through normal life experiences ... Later, with such structures, more advanced concepts can be learned without undue difficulty via any pedagogical method that relates the concept to the structures. If, however, one does not possess structures appropriate for a concept, it is nearly impossible to learn it.

This implicitly describes two different approaches, and has a very strong hidden implication that they are compatible. First, it logically establishes "learning without undue difficulty" as a criterion for "appropriate structure", and conversely "nearly impossible to learn it" as a criterion for the lack thereof. Going up a level, an essential criterion for good learning is success in later courses. Putting these together implies that what qualifies as "appropriate mental structure" should be highly constrained by the need to support higher levels. I find this entirely reasonable. But the statement also claims "very early concepts can be grasped easily through normal life experiences". The hidden implication is that early concepts grasped this way will support later learning. Educators, including the authors and Beckmann in [2], take this hidden implication as an article of faith. Indeed their faith is so strong that rather than looking downstream they *compartmentalize* levels and encourage "age-appropriate" formulations in each level. Unfortunately, rather superficial examination shows this belief to be false. I illustrate this with Dubinsky's examples.

On page 403 he discusses strategies to help students reconcile the views of $2/3$ as a *process* (parts of a whole) and an *object* that encapsulates the process. But as Wu [10] p. 374, points out, neither of these views is mathematically sound. Perhaps they can be "grasped through normal life experiences" but neither supports later work, e.g. with $(x + 3)/(x - 4)$. Shouldn't they be discarded and replaced with something functional, rather than reconciled?

On page 407 the authors describe a way to help students grasp modular arithmetic, in the context of division-with-remainder:

$$a = qb + r, \text{ with } 0 \leq r < b$$

They use a game modeled on a clock, with a student walking a units around a circle of length b . The number of cycles is q , and the final position on the circle gives r . The first problem is that this shows "grasp" is interpreted as the "relate to and admire" of philosophy rather than the "exploit as a tool" of mathematics. In particular the "grasping" provided by this game does not provide an "appropriate structure" to support applications of modular arithmetic described here and in [6]. A more mathematical objection to this approach is that modular arithmetic is concerned with equivalence classes obtained by identifying b with 0. This does not mean the equivalence-class concept must be used, but whatever we do must be consistent with it and there are two problems here. First, the remainder is a

particularly nice representative of the equivalence class, but confusion is inevitable if it is used as a definition. Second, the quotient q is *not* part of modular arithmetic, and including it makes applications significantly more difficult.

On page 407 Bob describes how “people talk”, “feature talk”, and questions about trips on the Red Line in Cambridge relate to mathematics:

... students mathematizing these trips acquire powerful metaphors and concepts for addition and subtraction very different from their arithmetic metaphors for these operations ...

Again the objective seems to be the “relate to and admire” of philosophy rather than the “exploit as a tool” of mathematics. Further, he cannot have looked downstream: these metaphors interfere with algebra or even fluent work with numbers, so they are more likely to be barriers to be overcome than “mental structures” that support later learning.

Summary: many educators have unquestioning faith that any way they interpret “grasping early concepts through normal life experiences” will automatically support later learning. In most cases this faith is unjustified, and downstream responsibilities are not being met.

4.5. Mark Saul. The International Mathematical Olympiad is the subject of Saul’s article [9]. It focuses more on the associated community and some of the mathematicians who have participated than on actual content, so it is not clear why this was included in the special issue. However, it provides an opportunity to express concerns about the trick problems used in such competitions.

Trick problems depend on a clever insight or special feature that if missed makes them hard, and if seen makes them easy. But this gives a misleading view of the nature and goals of mathematics, and the activities of mathematicians. First:

- Trick problems are contrived. Small variations typically give identical-looking problems that are impossible with elementary methods; see §3.2 above for an example. These are contests between students and problem designers, not between students and nature.
- Many K-12 teachers take the use in competitions to mean that these problems are next step up, and talented students should be challenged with trick problems. Longer and more involved but genuinely illuminating mathematical opportunities go unused.
- Many talented students do not have the necessary quick cleverness, or don’t like tricks, and are turned off by this view of mathematics.

Saul takes pride in the fact that some outstanding mathematicians were first identified through their performance in competitions, and presents competitions as a recruiting tool. But the trick-problem aspect makes this seriously problematic. Most mathematics, especially deep work, is slow and methodical rather than quick and clever. Further, the everyday power of mathematics is that persistence, appropriate techniques, and good work habits succeed with long, hard problems where quick cleverness is powerless. Competitions miss, and often discourage the deep slow thinkers. I was one of these, but unlike so many others I managed to recover from it. Conversely, quick clever people often find the long-haul tenacity required for real accomplishment either unattractive or impossible, and most high-scoring competitors do not become mathematicians.

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