A SCIENCE-OF-LEARNING APPROACH TO MATHEMATICS EDUCATION

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Abstract. A scientifically disciplined, learning-oriented approach to mathematics education is described and illustrated through examples. Mathematicians tend to focus on teacher education, with an implicit presumption that the current teacher corps is not sufficiently competent. The analysis here suggests that the teachers are competent enough, but the methodology they have been taught to use is incompetent. In other words, significantly better outcomes are unlikely without profound changes in educational philosophy and teacher-education programs.

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1. Introduction

The focus in this approach is directly on individual, mathematics-specific learning. It is 'science' in the sense that it developed bottom-up from micro-scale observations. It addresses long-term needs for abilities and skills because these impose strong constraints on elementary education. Our eventual goal is to produce competent citizens and capable scientists and engineers. To do this, college faculty

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need certain skills from high school; to develop these skills, high-school teachers need certain outcomes from middle school; and so on down. Finally the account draws on, and is consistent with, modern professional practice. These features are described a bit more in the next section and in detail from [4] from which this material was drawn.

Much of the output from this approach is also micro-scale: highly-effective ways to treat fractions, polynomial multiplication, word problems, etc., but with little indication as to how these might be put together in a course or curriculum. In other words a toolkit without much assembly instruction, in contrast to educational-philosophy approaches that provide assembly instructions (pedagogy and content-independent methodology) without much toolkit. There are, however, strategies to make the tools work, and we invert the logical progression and begin with these in §3. Section 4 provides examples, mostly from [4]. Topics in the first four subsections are fractions, multiplication of polynomials, and partial fractions in both integers and polynomials. The goal is to indicate how topics might develop over time, and to suggest that a cognitively-adapted and logically clear development might be more ambitious. Word problems are the last topic in the section. The final section concerns teacher preparation relevant to this approach. Among other things it throws light on why study of advanced topics does not seem to improve K-12 teaching.

This article was developed as a response to articles in the AMS *Notices* special issue on education in March 2011 [1]. The supplement [5] includes comparisons to these articles.

2. Background

The most significant background for this material is extensive (over a thousand hours) one-on-one diagnostic work with students[1]. The procedure is that I go to students when they need help. I review their work, diagnose the specific problem, and show them how to fix it to avoid a recurrence. This is not tutoring. The students do most of the talking and most of the work, and I leave them to finish on their own as soon as they are past the specific difficulty. The focus on diagnosis also helps students see how to diagnose and correct their own errors.

The context for the diagnostic work is a computer-tested engineering calculus course that I developed. Diagnosis of problems provides feedback for the courseware: it is designed to emphasize structure and encourage work habits that avoid these problems. The material is also designed so that pre-existing learning errors will cause serious difficulty. The goal is to expose these errors so they can be diagnosed and repaired.

A final ingredient is an extensive bottom-up analysis of contemporary mathematical practice, [3]. This draws on my own research experience, editorial work, work on publication policy, cognitive neuroscience, history of mathematics, the work with students mentioned above, and many other sources. It is well-known that modern professional practice is quite different from that in the nineteenth century, and from models used in education and most philosophical accounts. It was no surprise to find that contemporary practice is better adapted to the subject. It was a surprise to find that, within the subject constraints, it is much better adapted

[1]In the Math Emporium at Virginia Tech.

to human cognitive facilities. It seems that contemporary practice is not just for freaks: carefully understood, it can be a powerful resource for education.

These experiences have given a fine-grained perspective on mathematical learning and its difficulties. Moreover the perspective directly concerns individual learning: not teaching, and not learning mediated by teachers. It has become uncomfortably clear that teaching is not nearly as closely connected with learning as educators like to think.

3. Mathematics and teaching

Mathematics offers insights into the world and its own realms to explore, but it is not a spectator sport. Current educational approaches do not provide enough functionality to bring it to life. Current introductions to numbers and fractions, for instance, have rather passive 'understanding' as goals rather than *doing* things with them. The difference for fractions is illustrated in the examples.

The goal here is to suggest that a more active approach could be successful in real-life settings, but there are very strong constraints. First, weak goals permit relaxed attitudes toward educational philosophy and individual practice. Achieving strong goals is delicately dependent on approach, and provides a criterion that identifies many approaches as 'wrong'. To achieve success one must first avoid failure. Second, there are guidelines for 'right' approaches, but no easy formula. Successful practice is a delicate negotiation between mathematical structure and the complex oddities of human cognition. This has to be done on a case-by-case (micro-scale) basis, but in ways that fit into the long-term development.

This section provides an overview of structure and practice of mathematics, extracted from a detailed account in [3] and formulated for use in elementary education. However it is an after-the-fact description of the methodological constraints found in the micro-scale material, *not* a philosophical foundation from which methodology can be deduced. Again these are—at best—guidelines that help avoid failure, not formulas that guarantee success.

3.1. Precision and functionality.

Mathematics has evolved a set of explicit rules of reasoning with the following astounding property: arguments without rule violations have completely reliable conclusions[2]! People often have trouble learning to use these rules effectively because rules in most other systems are sufficiently vague and ineffective that precision is a waste of time, while in mathematics anything less than *full* precision (*no* rule violations) is a waste of time. However mathematical rules are simple compared to those of the tax code, religions, politics, physical science, etc. Most people can master the basics.

3.1.1. *Precision.*

Precision is particularly important for functionality in elementary mathematics: sloppy thinking that is harmless to experienced users can seriously confuse beginners. Confusion about fractions, for instance, results from failing to distinguish between things and names for them; between representatives of an equivalence class and the equivalence class; and from common ambiguity in the use of '='. This is acceptable if the goal is passive 'understanding', but ambitious goals require much more clarity. The following subsections suggest ways to avoid these and other problems.

[2]For all practical purposes; see [3] §2 for nuances.

3.1.2. *Potential proofs.* For users the key enabling concept ([3] §4, [4](c)) is *potential proof*: a record of reasoning, using methods of established reliability, and detailed enough to check for errors. Potential proofs in this sense are very common. The scratch work done by a child multiplying multi-digit integers by hand provides a record of reasoning that can be checked for errors. When teachers tell students to "show your work" they mean "provide a record of reasoning...". The basic method is already in wide use, and the potential-proof description mainly clarifies the features and activities needed to make it fully effective.

In these terms a "real" proof is a potential proof that has been checked (and repaired if necessary) and found to be error-free. However the benefits come from reasoning and checking. Errors happen. A flawed first attempt with sufficient detail can provide a framework for diagnosis and improvement. A flawed first attempt with insufficient detail is useless and the user must start again from scratch. Focusing on the formal and error-free aspects of completed proofs actually distracts from the features that provide power to users.

3.1.3. *Work templates.* To exploit the power of potential proofs, students should be taught to record their reasoning in a way that can be checked for errors. Criteria for good work formats are:

(1) record enough detail so reasoning can be reconstructed and checked for errors;
(2) be compact and straightforward; and
(3) help organize the work in ways consistent with human cognitive constraints.

Designing such a format is hard for professionals and cannot be expected of students, so they must be provided with *templates*. These can range from literal fill-in-the-blank forms (c.f. §4.3.2) to routinely recording crucial intermediate steps (c.f. §4.5).

The standard formats for multi-digit multiplication and long division are well-known examples of templates, though they are not fully satisfactory. They satisfy the first two conditions, but are less successful with (3) because they have been optimized for production arithemetic by experienced users rather than to provide support for beginners. A less-efficient but potentially more supportive template for multiplication is explored in [4](a).

3.1.4. *Accuracy and diagnosis.* Mathematical methods enable near 100% accuracy and extended arguments depend on this. If a single operation can be done with 70% accuracy then the likelihood of getting the right answer in problems with ten operations is under 3%. If one wants 70% accuracy in the ten-operation problem then individual operations must be done with 97% accuracy. Elementary teachers who accept 70% success rates are not only wasting the unique potential of mathematics, but setting their students up for failure in later courses. The proposal, then, is that the goal should be quality, not quantity, and not speed. Fewer problems, but expected to be 100% right. Is this reasonable in education? Maybe not, but I sketch strategies that should make it possible to get close. One is described above: carefully-designed work templates for students to emulate.

The most important strategy is teacher diagnosis of errors that students can't find themselves. Ideally *every* wrong answer should corrected. For diagnosis, the student should explain his reasoning following his written work record. If the record is illegible, steps were skipped, or appropriate templates were not used, then the work should be redone before being diagnosed for specific errors: it is remarkable how often this resolves the problem, and it is valuable for students to see this. If work record is appropriate then it can be reviewed efficiently and mistakes quickly pinpointed. Premature guesses about the difficulty are often wrong and will make confusion worse, so the teacher should wait until the student has come to the error before doing anything. Further, the student learns more if he spots the error himself; wait and see if he says, "oh, now I see what went wrong". In such cases the teacher can ask what happened, to see if the insight is right, but if so then the teacher should leave well enough alone and *not* explain further, even if a clearer description can be given. If this approach is used systematically then errors will become remarkably uncommon. Confusions don't accumulate and students learn to diagnose their own work.

3.2. **Definitions and key statements.** Precise reasoning requires precise descriptions: vague or intuitive things are inaccessible to mathematics. I discuss four aspects: when explicit descriptions are necessary; how things are described precisely; where good descriptions come from; and how they support the intuitive processes that usually precede precise reasoning.

3.2.1. *When statements are needed.* There are no hard-and-fast rules about when things should be formalized because it depends more on how our brains work than on mathematical structure. Professional practice gives clues: mathematicians have essentially the same neural equipment as everyone else, and are very ambitious about exploiting it to the fullest.

For instance, addition seems to be internalized as a kind of behavior, and work habits learned in the integers transfer relatively easily to much more general contexts that behave in the same way. This is formalized as: any binary operation that is commutative, associate, and has a neutral element and inverses, is entitled to be called 'addition' and denoted by '+'. Similarly, a second operation that distributes over + is entitled to be denoted as multiplication. This enables transparent transfer of work habits. That this is a cognitive strategy rather than just mathematical structure becomes clear when people move out of their comfort zone. Technical examples: homology is written additively, and the bar construction multiplicatively, and it is rather awkward to translate these to the other notation. It can be amusing to watch mathematicians encountering tropical arithmetic in the integers for the first time. This has two operations, and one distributes over the other just as multiplication does over addition. However the operation that distributes like multiplication is ordinary addition. This means it cannot be rewritten as multiplication, so distributivity looks backwards. The operation analogous to addition is the max function, which doesn't have inverses and also doesn't follow familiar patterns.

The point is that while it is a good idea to have the associative and commutative axioms available for reference, it is unnecessary and distracting to require students to work with them explicitly. See [4](b) for an elaboration. In contrast, making explicit the definition of fractions and the quadratic formula has significant cognitive benefits.

3.2.2. *Formats.* Contemporary practice uses axiomatic definitions to provide precision but the format is not forced by the job to be done. In fact the definition format seems to have won out over the alternatives because it is more effective for human use; see [3] §5. In other words, precise definitions can be an enabling technology for people. Similarly for concise theorems.

Definitions and theorems provide precision but, as with any format, they do not turn junk into gold. *Effective* definitions usually result from long and painful searches. Mathematicians will have some intuitive goal, make a precise description that they hope will realize their intuition, work with it for a while, and most of the time it doesn't do what they hope for. They junk the attempt and try again. In some areas these false starts probably account for 50% of all mathematical effort. A really good definition is therefore a treasure, and because it can efficiently guide newcomers around years of confusion, a powerful legacy. Really sharp and effective theorem statements are similarly precious.

3.2.3. *Suggestion.* Important objects, terminology, and properties (fraction, prime, relatively prime, proper fraction, quadratic formula, . . .) should be given brief and genuinely precise formulations.

- Formulations should be constructed primarily by professional mathematicians, with education feedback. Professionals habitually hone and fine-tune them to be brief and effective, and can ensure that they are compatible with later material.
- Ideally, students should memorize them so they can be reproduced exactly. Definitions provide anchor points, and genuinely functional understanding nucleates and deepens around the definition. This is particularly true for weaker students.
- Explanations of what a definition 'means' is best given after the definition, not before. Putting too much explanation first almost guarantees confusion.

Compared to current educational practice this is very rigid, but it is a baby version of the way mathematicians approach unfamiliar material, and it works. More precisely, less rigid approaches take longer, are less effective, and were abandoned when this approach became available about a century ago.

Is this approach reasonable in education? Well-crafted definitions evoke mathematical objects with the economy and grace of poetry. Asking students to memorize them is like asking them to memorize unusually powerful poems. There are not so many that this is burdensome, and if it is done consistently it will become routine. Students will also see quick payoffs because good definitions are immediately functional.

3.2.4. *Intuition.* The final point concerns the role of intuition. A great deal of mathematical activity is at least guided by intuition, but this makes precise definitions *more* important. Intuitions developed by working with a good definition are often effective. Intuitions based on sloppy 'definitions', or naive intuitions 'refined' by exposure to definitions, tend to be unreliable. Intuitions based on 'meaning' or physical analogs are almost never satisfactory. This is acceptable if passive 'understanding' is the goal, but it strongly limits long-term outcomes.

3.3. **Limits of mathematics.** The need for complete precision provides power but also limits the scope of mathematics. In particular nothing in the physical world can be described with mathematical precision, so mathematical methods do not apply directly. People applying mathematics accommodate this with an intermediate step: a symbolic *mathematical model* is developed to represent the physical situation, and it is the model that is analyzed mathematically. Mathematical conclusions about the model are reliable in a way that the connection between the physical situation and the model cannot be (see §4.5.1 and the supplement for examples) and working without a model confuses this structural difference.

Historically, systematic use of modeling developed in the seventeenth century as an essential part of the 'scientific revolution'.

There are additional reasons mixing modeling and analysis has been obsolete for four centuries: they are quite different cognitively and mixing causes cognitive interference. In fact cognitive interference in word problems may be the most serious single difficulty I have seen in students. For weaker students "crippling" may not be too strong a word. Examples are given here in §4.5, and there are many more in [4].

Most contemporary educators are philosophically committed to word problems as a different *format* rather than a different activity. They encourage a holistic approach with reasoning 'in context', and discourage modeling. In other words they follow sixteenth-century practice and, naturally, get sixteenth-century outcomes.

3.4. **Technology.** A great deal of [4] is concerned with educational use (or misuse) of technology. The issue is too complex to be addressed here, but I make one comment.

The traditional classroom was a tightly bound package, and technology is making it come apart. In particular it is loosening the link between teaching and learning. For instance calculators appear to make teaching easier, and improve student performance on traditional test problem types. However these problem types no longer have the learning correlates they once did. Easier teaching and better test performance are, in some cases, masking learning declines. A genuinely learning-oriented approach reveals this.

4. Examples

Explicit problem-oriented methods for student use are the main products of the science-of-learning approach, and are the bases for generalities described in the previous section. Space constraints do not permit enough to really illuminate the generalities, but we give enough to illustrate how they fit together. The first section gives a precise but unconventional approach to fractions. It does not immediately give the usual passive 'parts-of-a-whole' picture, though that could be given as an application of *integer* fractions. However it does immediately give effective contexts for partial fractions in both the integers (§4.2) and polynomials (§4.2). An unconventional procedure for multiplying polynomials in §4.3 carefully exploits structure to give access to much more complex problems than usual, and is adapted to find coefficients in polynomial partial fractions in §4.2. Many more examples and interconnections are given in [4]. The polynomial multiplication procedure, for instance, is adapted to give an algorithm for multiplying multi-digit integers in [4](a, c).

Section §4.5 illustrates the discussion of word problems in §3.3.

4.1. **Fractions.** These are a perennial source of trouble, and there seem to be two reasons. First, a fraction is a name for, or description of something, not the thing itself. The expression $\frac{9}{4} = 2.25$ really means "2.25 can be written as (is a name for) $\frac{9}{4}$". The different names encode different properties, and we want to work with both because we want to exploit the different information they encode. Of course all our symbols are names, not things. This point is too subtle for explicit use in elementary education, but students learn a great deal subliminally from the way things are presented so confusion can be avoided if teachers are aware of it and accommodate it in their phrasing (see the Caution below).

The second point about fractions is that they specify things implicitly. The name 2.25 encodes an explicit procedure for assembling a number from single-digit integers and powers of 10. Fractions and names such as $\sqrt{2}$ encode properties that determine the thing, but do not encode a procedure. See the supplement for a completely analogous description of square roots.

4.1.1. *Definition.* $\square$ can be expressed as $\frac{a}{b}$ means $b\square = a$.

4.1.2. *Notes.*

- Examples in the rest of this section demonstrate that this is an effective approach for high- and middle school. It is not yet clear it can be used directly in elementary grades, but it should be clear that whatever approach *is* used should be carefully upward-compatible with this.
- The structure of the definition implies that the default way to work with fractions is to clear denominators. The usual rules are shortcuts that sometimes avoid this, but these shortcuts are verified by clearing denominators, and if you can't see how to use shortcuts then clear denominators.
- Note that $0\square = 0$ so if $a \neq 0$ then *nothing* can be expressed as $\frac{a}{0}$. It is a useless notation and we avoid it by requiring denominators to be nonzero.

4.1.3. *Examples.*

(1) The decimal 2.25 can be expressed as $\frac{9}{4}$ because it satisfies $4 \times 2.25 = 9$.

(2) Show that if $c \neq 0$ then $\frac{ca}{cb}$ can be expressed as $\frac{a}{b}$. To check, set $\frac{ca}{cb} = \square$ and clear denominators. This gives $ca = cb\square$. If $c \neq 0$ then c can be cancelled to give $a = b\square$, and this means $\square = \frac{a}{b}$.

(3) Is $a - 2 \stackrel{?}{=} \frac{a^2-4}{a+2}$? To find out, clear denominators:

$$(a+2) \times (a-2) = a^2 - a2 + 2a - 4 = a^2 - 4,$$

this is true so the answer is 'yes' (but see the caution below).

(4) Is $\frac{a}{x} + \frac{b}{y} \stackrel{?}{=} \frac{a+b}{x+y}$? Clear denominators (multiply by $xy(x+y)$) to get $a(y(x+y)) + b(x(x+y)) \stackrel{?}{=} (a+b)(xy)$. Expanding and canceling gives $ay^2 + bx^2 \stackrel{?}{=} 0$. This hardly ever works, so the answer in general is 'no'.

(5) Express $\frac{a}{x} + \frac{b}{y}$ as a fraction. First give it a neutral name, $\square = \frac{a}{x} + \frac{b}{y}$, and clear denominators by multiplying by x and y: $xy\square = ya + xb$. But this means $\square$ can be written as $\square = \frac{ya+xb}{xy}$.

4.1.4. *Comments and cautions.* First, I explain the notations $\square$ and $\stackrel{?}{=}$.

$\square$. If we are trying to express something as a fraction then we need another name for it to use during the process. The thing is usually given with a compound name (e.g. $\frac{2}{a} + 7 \times \frac{2}{b}$) but it is hard to work with this and still think of it as a single object. It helps to use a temporary name, e.g "let $A = \frac{2}{a} + 7 \times \frac{2}{b}$". But using a symbol means we have *three* symbolic names in play, and recycling a symbol (for instance using 'A' in both Examples (2) and (5)) invites confusion. The neutral placeholder, $\square$, encodes no structure, and has no hidden or lasting significance.

$\stackrel{?}{=}$. This addresses problems with sloppy use of '='. In common use this can mean "define x by…"; "it *is always* true that…"; "*is it* always true that…?", or "determine *when* it is true that…". The equation alone is incomplete and the intended meaning is to be inferred from context. This ambiguity becomes invisible to experienced users, but students—and many teachers—do not understand it and it is a constant source of confusion (see question 2, p. 390, in [2]). Computers don't understand it either, and the programming language of *Mathematica* provides at least six different symbols for different meanings of '='. In the example here we want to begin with an equality posed as a question, and end with the same expression as an assertion. Confusion is inevitable if the notation does not display the difference between the two.

Caution. There is an additional ambiguity in the use of $\square = \frac{a}{b}$ as short for "$\square$ can be written as $\frac{a}{b}$". The difficulty can be seen by rephrasing Example (2) as "show that $\frac{ca}{cb} = \frac{a}{b}$". The typical response is "clear denominators by multiplying by cb and b to get $b(ca) \stackrel{?}{=} cb(a)$. This is true so the original expression is true." But this logic must be flawed because it does not require c to be nonzero.

The source of the flaw is that fractions (as names) encode a property as well as an underlying object. The symbol '=' only concerns underlying objects, while "can be expressed as" includes the additional information. Technically this leads people to overlook one of the two implications in the "if and only if" aspect of the definition. This error is essentially harmless, especially after "don't divide by 0" is firmly internalized, and I think trying to fix or explain it would cause more problems than it would solve. I suggest teachers should be aware of it and use the more-precise phrasing to evoke correct reasoning when appropriate—especially when verifying rules—and to answer questions about meaning. In ordinary practice the flawed version should be accepted.

More about names. Care about the difference between things and names resolves a lot of confusion. Again this is too subtle for explicit use in elementary education but teachers should be aware of it and terminology should be consistent with it. Examples:

- "Express $1.23 + 45.6$ as a decimal" rather than "find $1.23 + 45.6$". The expression specifies a number so it is already "found". The decimal name encodes a particular construction of the number in addition to the number itself, and the objective is to find the construction-describing name.
- "Express $\frac{1}{2} + \frac{2}{3}$ as a fraction" rather than either "find" or "simplify". As above the expression already specifies a number so it is not "lost". We could convert the fractions to decimals and "find" it by adding these, but this does not give the desired form. Finally, "simplify" is ambiguous: the sum is the partial-fraction form of the rational number so is already "simplified" in this sense.

4.2. **Integer partial fractions.** One objective is to foreshadow a standard rational-function topic in §4.4. This might also illustrate that there are lots of interesting problems within mathematics.

First, a theorem. Theorems are great labor-savers: they explain what you can do with complete confidence, and what the limits are. In this case, for instance, you should expect that attempts to separate non-relatively-prime factors will fail in some way.

Theorem 1. *If b and c are relatively prime, then fractions with denominator bc can be expanded as $\frac{a}{bc} = \frac{x}{b} + \frac{y}{c}$, for some x, y.*

This is true essentially whenever the terms make sense (any commutative ring) and is an immediate consequence of the definition of relatively prime: a, b are *relatively prime* if there are m, n so that $am + bn = 1$. This is the correct general definition but it is cumbersome to use, and in practice (integers here, and real polynomials in §4.4) one uses refinements of both 'relatively prime' and the expansion.

Theorem 2. (Refinements for integers):

(1) *For integers, 'relatively prime' is the same as 'have no common factor' (except 1, of course).*

(2) *if $b_1, b_2 \dots b_n$ are pairwise relatively prime then there is a unique expansion*

$$\frac{a}{b_1 \times \cdots \times b_n} = \frac{r_1}{b_1} + \frac{r_2}{b_2} + \cdots + \frac{r_n}{b_n} + q$$

with all the $\frac{r_i}{b_i}$ proper fractions and q an integer.

For integer fractions, $\frac{a}{b}$ *proper* means that $b > a \geq 0$; see §4.4 for the polynomial version. Note that the version for a single factor, $\frac{a}{b} = \frac{r}{b} + q$ with $\frac{r}{b}$ proper, is the usual quotient-with-remainder expression for $a \div b$.

This refinement is true because the Euclidean algorithm works in the integers. In fact there is an algorithm for the numerators in the expansion, based on the extended Euclidean algorithm (see the Wikipedia entry). The algorithm is cumbersome and obscure, so modular arithmetic is used here.

4.2.1. *Problem.* Express 47/180 as a sum of an integer and proper fractions with prime-power denominators.

4.2.2. *Solution.* $180 = 2^2 3^2 5$ so the maximal partial fraction expansion is of the form $\frac{a}{4} + \frac{b}{9} + \frac{c}{5} + d$.

The first step must be to clear denominators: fractions are defined implicitly, and we must convert to an explicit form to work with them (see §4.1). This gives

$$47 = (4 \cdot 9 \cdot 5)(\frac{a}{4} + \frac{b}{9} + \frac{c}{5} + d) = 9 \cdot 5a + 4 \cdot 5b + 4 \cdot 9c + 4 \cdot 9 \cdot 5d.$$

Reduce mod 4 to get $47 \overset{4}{\equiv} 3 \overset{4}{\equiv} 45a \overset{4}{\equiv} a$. This means $a = 3 + 4r$, and the numerator that gives a proper fraction is 3. Next reduce mod 9 to get $2 \overset{9}{\equiv} 2b$, so $b \overset{9}{\equiv} 1$. Finally reduce mod 5 to get $2 \overset{5}{\equiv} c$.

We now know that $47/180 = \frac{3}{4} + \frac{1}{9} + \frac{2}{5} + d$, with d an integer. We can find d by converting the partial fractions back to a common denominator, but we could also use a calculator. The left side is approximately 0.2611, while the fractions on the right add up to about 1.2611. Since d is an integer it must be -1, and

$$47/180 = \frac{3}{4} + \frac{1}{9} + \frac{2}{5} - 1$$

4.3. **Multiplying polynomials.** This section illustrates how careful attention to cognitive issues and mathematical structure can significantly extend the problem types students can do. The main cognitive concern is that different activities (here organization, addition, and multiplication) should be separated as much as possible. Structure is used to make the procedure flexible; see the next section for a variation. See [4](a) for detailed discussion.

4.3.1. *Problem.* Write the product $(2y^3 + y^2 - 9)(-y^2 + 5y - 1)$ as a polynomial in y, in standard form.

"Standard form" here means coefficients times powers of the variable, with exponents in descending order. The factors are given in this standard form. Sometimes ascending order is used.

4.3.2. *Solution.* The first step is purely organizational. We see that the outcome will be a polynomial of degree 5 so set up a template for this:

$$y^5(\qquad\qquad\qquad) + y^4(\qquad\qquad\qquad) + y^3(\qquad\qquad\qquad) + \cdots$$

Next scan through the factors and put coefficient products in the right places. To get the y^3 coefficient, for example, begin with the leftmost term in the first factor. Record the coefficient, (2). The exponent is 3, so the complementary exponent is 0. Beginning at the right in the second factor, we see that the coefficient on y^0 is -1, so we record this as a product $(2)(-1)$. Now move one place to the right in the first factor and record coefficient (1). The exponent is 2 so look for complementary exponent 1 in the second factor. The coefficient on this is 5, so record this as $(1)(5)$. Again move one place to the right in the first factor and record coefficient (-9). The exponent is 0 so we look for complementary exponent 3 in the second factor. There is none, so we record coefficient 0 as $(-9)(0)$. The template will now look like

$$\cdots + y^3\big((2)(-1) + (1)(5) + (-9)(0)\big)\cdots$$

It is quite important that *no arithmetic* be done in the organizational phase, not even $(1)(5) = 5$. Also, we usually quit scanning the first factor when complementary exponents are above the degree of the second, but it is better to record overruns such as $(-9)(0)$ than to worry about this. The reason is that even trivial on-the-fly arithmetic requires a momentary break in focus that significantly increases error rates in both organization and arithmetic. For the same reason, every coefficient is automatically enclosed in parentheses whether they are needed or not.

The first arithmetic step is to do the multiplications:

$$y^5\big(\underbrace{(2)(-1)}_{-2}\big) + y^4\big(\underbrace{(2)(5)}_{10} + \underbrace{(1)(-1)}_{-1}\big) + y^3\big(\underbrace{(2)(-1)}_{-2} + \underbrace{(1)(5)}_{5} + \underbrace{(-9)(0)}_{0}\big) + \cdots$$

The second arithmetic step is to do additions:

$$y^5\big(\underbrace{(2)(-1)}_{-2}\big) + y^4\big(\underbrace{\underbrace{(2)(5)}_{10} + \underbrace{(1)(-1)}_{-1}}_{9}\big) + y^3\big(\underbrace{\underbrace{(2)(-1)}_{-2} + \underbrace{(1)(5)}_{5} + \underbrace{(-9)(0)}_{0}}_{3}\big) + \cdots$$

Additions and multiplications are separated because, again, switching back and forth invites errors. Note also that this pattern requires essentially no organizational activity during the arithmetic steps: everything is in standard places, and outcomes are written just below inputs.

The final result is

$$y^5(-2) + y^4(9) + y^3(3) + y^2(8) + y^1(-45) + 9$$

4.3.3. *Discussion.* Standard high-school practice is to restrict to product of binomials, and use the intelligence-free algorithm with acronym "FOIL". This is so ingrained that many students say "FOIL it" rather than "expand", even when the factors are not binomials. However this algorithm mixes organization and arithmetic enough that some students have trouble even with this simple case, and since the expansion step is not adapted to the eventual goal, a second sorting step is needed. Finally, the near-exclusive focus on binomials makes larger expressions foreign territory. Some students are at a loss about how to deal with them, others generalize incorrectly from the pattern, and almost all are anxious and hesitant. A better algorithm fixes all this, and opens up a much wider world.

4.4. **Polynomial partial fractions.** The final example brings together fractions, partial fractions, and polynomial multiplication. The analog of modular arithmetic is included in the supplement [5]. Theorem (1) in §4.2 applies, but as in that section we need a refinement.

Theorem 3. (Refinement for polynomials):

(1) *For polynomials 'relatively prime' is the same as 'have no common factor', and the same as 'have no common roots' (including complex roots).*
(2) *If $b_1, b_2 \dots b_n$ are pairwise relatively prime real-coefficient polynomials then there is a unique expansion*
$$\frac{a}{b_1 \times \cdots \times b_n} = \frac{r_1}{b_1} + \frac{r_2}{b_2} + \cdots + \frac{r_n}{b_n} + q$$
with r_i, q polynomials and each $\frac{r_i}{b_i}$ a proper fraction.

A *polynomial* fraction is called proper if the degree of the numerator is strictly smaller than the degree of the denominator. The term has the same significance as the integer version even though the definitions are different. As with integers, q is the result of division (the "quotient") and the r_i are remainders.

This is statement almost identical to the integer statement. This emphasizes commonality of the underlying structure, but the main reason is efficiency: since the underlying structures *are* the same, a maximally efficient statement for one context will also be maximally efficient for the other.

4.4.1. *Problem.* Find the real partial-fraction expansion of the polynomial fraction

$$\frac{3x^3 + 2x + 9}{(4x^2 - 4x + 1)(x^2 - 2x + 3)}$$

4.4.2. *Solution, setup.* The first quadratic in the denominator factors as $(2x-1)^2$, but these factors cannot be separated because they are not relatively prime. The second quadratic has complex roots so it can be factored over the complexes as a product of linear terms. These could be separated in a partial-fraction expansion over $\mathbb{C}$, but not over $\mathbb{R}$. The denominators in the partial fractions are therefore the two quadratics. Finally, the input fraction is proper so the expansion cannot have a (non-fraction) polynomial term (note this conclusion is special to polynomials: it does not work for integers).

Both pieces in the expansion have denominators of degree 2, and we know that the numerators have smaller degree. The expansion is therefore of the form

$$\frac{3x^3+2x+9}{(4x^2-4x+1)(x^2-2x+3)} = \frac{ax+b}{4x^2-4x+1} + \frac{cx+d}{x^2-2x+3}$$

As usual, to work with fractions we have to clear denominators. This gives

$$3x^3+2x+9 = (ax+b)(x^2-2x+3) + (cx+d)(4x^2-4x+1) \tag{1}$$

4.4.3. *Solution, linear-algebra approach.* Here we illustrate the use of linear algebra to find the coefficients in the expansion above. The supplement [5] also includes a modular-arithmetic approach analogous to the integer procedure. Equation (1) is an equality of polynomials, so the coefficients must be equal. The coefficient equations give the linear system that determines a–d.

The plan is as follows: for each exponent n scan through the products above and pick out coefficients on x^n, exactly as in the section on polynomial products. Recording coefficients gives

$$\begin{array}{ll} x^3: & 3 = a(1) + c(4) \\ x^2: & 0 = a(-2) + b(1) + c(-4) + d(4) \\ x^1: & 2 = a(3) + b(-2) + c(1) + d(-4) \\ x^0: & 9 = b(3) + d(1) \end{array}$$

Writing this linear system in matrix form gives

$$\begin{pmatrix} 1 & 0 & 4 & 0 \\ -2 & 1 & -4 & 4 \\ 3 & -2 & 1 & -4 \\ 0 & 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 2 \\ 9 \end{pmatrix}$$

Problem this involved are necessary to illustrate some of the phenomena but full solutions are not always necessary. A possibility is to ask "find a linear system that determines the coefficients…", to avoid the linear algebra. Setting up the system is an important goal in any case because in the long run they would enter

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LinearSolve[{{1,0,4,0},{-2,1,-4,4}, {3,-2,1,-4},{0,3,0,1}},{3,0,2,9}]
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in a computer-algebra system[3], and obtain $(\frac{319}{81}, \frac{214}{81}, -\frac{19}{81}, \frac{29}{27})$. Even if a full solution is required then the problem statement should be "find a linear system that determines the coefficients, and then solve to find the expansion". Requiring explicit display of the intermediate step has two functions: first, if the final answer is wrong then the intermediate step can be used to locate errors. The second reason is to ensure that students carefully formulate the intermediate step, especially if the system is to be solved by hand. Many will try to save writing (and thinking) by solving on the fly as coefficients are found (e.g. for the x^3 coefficient writing $a = 3 - 4c$ and using this to eliminate a). They could use this strategy to solve the system after it is set up, but mixing the steps increases the error rate and in the long term will limit the problems they can handle.

As a final note, if the input fraction were not proper then the q term in Theorem 3 would be nonzero. It can be included in the cleared-denominator form (1) and handled the same way. Most approaches recommend first using long division to get q (the quotient) and a proper fraction (with remainder as numerator), and

[3] This is *Mathematica* syntax.

then expanding the proper fraction. The division algorithm gives the coefficients in the quotient, so the remaining coefficients give a smaller linear system. But division takes a lot more work. The extra part of the larger linear system has lower triangular coefficient matrix so does not add much to the difficulty.

4.5. **Word problems.** This is a crucial topic because contemporary educators put great emphasis on word problems, but use a problematic approach. There is an abstract discussion in §3.3; here I illustrate the use of modeling to avoid the problem.

A *mathematical model* is a translation of a real-world or word problem to a symbolic form suitable for mathematical analysis. The key feature is that little or no analysis is done during modeling, and no modeling is done during analysis of the model. Three examples are given. The first and second are typical number-and-word problem with little or no mathematical content. The third illustrates the use of algebra to explore a problem about statistics. The supplement to the essay [5] has additional examples and illustrates difficulties with 'trick problems'.

4.5.1. *Example 1: Buying Pencils.* This was suggested to me as representative of a common and important problem type. Educational goals are discussed below.

Frank bought 8 pencils at 32 cents apiece. How much did he pay?

4.5.2. *Mathematical model.* This is a rate problem, with model:

$$(\text{ number of pencils, items }) \times (\text{ cost rate, cents/item }) = (\text{ total cost, cents })$$

Plugging in values gives $8 \times 32 = 256$ cents, or \$2.56.

Rate equations, and the format that includes units (here items and cents) are discussed in the next problem.

4.5.3. *Problems with the model.* As a reality check, Frank actually did buy some pencils.

- Pencils are now sold in blister packs. I got one pack of 8 for \$2.57. Math was unnecessary.
- I paid \$2.70, including \$0.13 Virginia sales tax.

No-one really expects problems like this to have much relation to reality, but this illustrates a point made in §3.3. If this is a math problem then, evidently, math does not connect well with reality. Separating modeling and analysis gives a better picture: it is the model that is faulty and the mathematical analysis of the model is perfectly correct.

4.5.4. *Problems with educational goals.* There are two points of view on such problems. One is that they bring math to life. Teachers reading them can almost smell the pencils. The other is that they are a "thinly disguised invitation to multiply", with some unpackaging activity that does not qualify as modeling.

The students I have asked about this (dozens of them) almost unanimously despise these problems and ignore the charming context. They parse it as "...8...32 ...apiece": two numbers, multiply. Some educators even teach a 'keyword' approach to determine the operation. So the bring-to-life idea fails, but pencils no longer smell good anyway.

I have serious concerns with the second view. First, why are large numbers of thinly-disguised trivial arithmetic problems a good thing? They improve grades and test scores, but do they contribute in any real way to learning? Second, it is not a good idea to encourage 'unpackaging' without modeling. The carefully-formulated model uses a rate equation and making it explicit connects with more complicated work (see the next problem). Conversely, hiding rate equations will cause confusion later. More generally, 'reasoning in context' with intuitive 'unpackaging' is a hard-to-break habit that seriously limits outcomes in later courses.

4.5.5. *Example 2: Leaky fuel tank.*

> A car begins a trip with 40 liters of fuel and drives at a constant speed 70 km/hr. At this speed the car gets 12 km/liter, but it runs out of fuel after 360 km. Apparently the fuel tank is leaking. If it leaks at a constant rate, what is the rate (liter/hr)[4]?

4.5.6. *Model.* The rate of loss is (fuel Lost)/(Time); denote this by L/T. There are two rate equations, and the relation of (fuel Lost) to (fuel Used):

(1) (speed, km/hr)×(Time, hr) = (distance, km)
(2) (distance rate, km/liter)×(Used, liter) = (distance, km)
(3) (Lost, liter) = (total, liter) - (Used, liter)

Now put in the numbers from the problem statement:

(1) $70\,T = 360$
(2) $12\,U = 360$
(3) $L = 40 - U$

Solve to get $T = 36/7$, $U = 30$, $L = 10$, and therefore $L/T = 70/36 = 35/18 \simeq 1.94$ liters/hour.

4.5.7. *Discussion.* The first step was to write the rate equations in a symbolic form designed to be easily and reliably remembered. Neither numerical values nor *any* analytic reasoning are done in this step. Trying to look ahead, for instance "I need to know the time, so write the speed rate equation as (time)=(distance)/(speed)…" invites errors. This problem is significantly more difficult without modeling because there are so many opportunities for such errors. Note also that the rate equations are written with units, so dimensional analysis can be used as a check that they are correct. In particular, the rate in the fuel use equation is given as a *distance* rate (km/liter) rather than a *fuel* rate (liter/km). The latter seems more logical (see the next problem), and confusion on this point is likely to be a source of error.

The last step is to substitute numbers into the symbolic forms, and then solve. This is routine, and easy because the analysis is protected from cognitive interference with modeling.

As an aside, it is a common criticism of such problems that "constant rate" cannot be made precise without derivatives (c.f. [6]). This is not correct: 'constant rate' means a rate equation applies, and this makes perfect sense if it is made explicit. Variable rates need deriviatives.

[4]Adapted from a Massachusetts high-school problem via a 'puzzler' from the National Public Radio program *Car Talk*.

4.5.8. *Example 3: Fuel efficiency.* This illustrates a common way to misuse statistics. Comparison with the degraded version illustrates the benefits of symbols rather than the usual inert numbers.

> A regulatory agency wants to promote development of fuel-efficient cars. The regulations require a certain *average* efficiency to provide automakers flexibility and incentive: they can offer inefficient models if these are balanced by super-efficient ones. However an environmental group claims that if the regulators use the traditional km/liter measure of automotive efficiency then more-efficient models will actually lead to *increased* fuel use. Are they right? Which measure should be used to avoid this?

4.5.9. *Symbolic specific problem.* Suppose there is a target efficiency T km/liter. A manufacturer sells two models: an efficient one that gets rT km/liter for some efficiency multiplier $r > 1$, and an inefficient one that gets b km/liter so that the average of rT and b is T.

(1) Find the average of the fuel required by the two models to go distance d. Express this as the product of the fuel required by a car with average efficiency T, and a function $h(r)$.
(2) Evaluate $h(r)$ for $r = 4/5$ and $5/3$. Plot $h(r)$ (by calculator or computer) on the interval $[1, 1.8]$. For which efficiency multipliers r does the two-model fleet use more fuel than average-efficiency cars?
(3) Find r so that the two-model fleet uses twice as much fuel as a fleet of average-efficiency cars.
(4) Explain what could theoretically happen if the automakers developed a car with efficiency *twice* the target (i.e. $r = 2$).

4.5.10. *Discussion.* The previous problem provides an appropriate template for the modeling step, so an explicit solution is not given here.

Is there a better measure? Here we are interested in total fuel use. The km/liter measure has rate equation

$$(\text{fuel, liter}) = \frac{(\text{distance, km})}{(\text{distance rate, km/liter})}$$

so it is *inversely* related to fuel use. But standard statistical methods are additive: they work well if the data has a linear structure, poorly otherwise. This explains why the km/liter measure works poorly. The *linearly* related efficiency measure is the inverse, the fuel rate in liter/km with rate equation

$$(\text{fuel, liter}) = (\text{distance, km})(\text{fuel rate, liter/km})$$

It would therefore be more honest to use the fuel rate in the statistical analysis.

4.5.11. *Degraded specific problem.* The regulatory agency sets 12 km/liter as the target efficiency. A manufacturer sells two models: an efficient one that gets 18 km/liter and an inefficient one that gets b km/liter so that the average of 18 and b is 12.

(1) Find the average of the fuel required by the two models to go 100 km. Compare to the fuel required by a car with average efficiency 12.
(2) Repeat for high-efficiency rate 20, and 22. Describe the pattern you see as the better efficiency increases.

The use of specific numbers is typical of contemporary practice. This makes it easier to do without modeling, and more accessible to direct evaluation on a calculator. On the other hand a great deal of mathematical structure has been lost. The average fuel use as a function of the upper efficiency is now represented by a few numerical values. These suggest part of the pattern, but do not make obvious the infinite limit at upper efficiency 24. Further, the use of a specific numerical target hides the fact that the controlling parameter is the quotient (max efficiency)/(target efficiency), called r in the first version. All this structure emerges from essentially the same work done symbolically.

The use of numerical values is also a favorite tactic of test designers. The first version is a single problem. Plugging in numbers gives a large number of seemingly different problems. But don't we *want* students to see structure and patterns? Destroying it for the convenience of test design and calculator evaluation seems inappropriate.

5. Teacher preparation

Outcomes from K-12 education are unsatisfactory and not improving. Angst about this has mainly focused on teacher preparation, implicitly assuming that educational methodology is effective and the problem is incompetent teachers. The discussion here suggests an alternative: it is the methodology that is incompetent, and worse outcomes might actually reflect *better* teaching of the methodology. If this is the case then the methodology has to be straightened out before there is much point in discussing teacher preparation.

The issue is approached through analysis of two common but conflicting viewpoints about advanced study.

5.1. **Advanced study *not* necessary?** The school-of-education argument is essentially "our students have trouble with advanced courses and don't get much from them. Pedagogy is more important than content anyway."

Wu [6] tries to give this point of view more substance by citing an article of Begle that found no correlation between advanced coursework and better student outcomes. Wu concludes that advanced viewpoints are irrelevant to elementary education, and uses fractions to illustrate the reasons. However his description of fractions is inaccurate (see the supplement [5] for discussion) and what this actually shows is that imprecision and sloppy notation that are harmless at advanced levels can render the presentation irrelevant to elementary teaching. More care can give more relevance: see [4](b) for a relatively painless extension of the fraction material in §4.1 to commutative rings. This includes dealing with zero-divisors; not K-12 classroom material but it certainly clarifies why having 0 in the denominator is a bad idea. It also connects with modern themes by observing that Grothendieck groups are just a fancy name for fractions when they are approached this way. On the other hand I agree that courses such as real and complex analysis are unnecessary: they have little to do with K-12 topics and benefits to survivors is unlikely to justify the severe attrition among potential teachers.

Examples like the above suggest that modern advanced courses with appropriate topics and precision could connect nicely with K-12 *topics*, but the next discussion reveals a problem with *methodology*.

5.2. **Advanced study *is* necessary?** Most people with technical backgrounds feel teachers need more mathematics, but cannot clearly articulate why. It seems to me the essence is "my background leads me to feel that educators are doing bad things. If they had stronger backgrounds they would feel the same way, and do better." Roughly, the need is for more sophistication and mathematically disciplined thinking. The standard way to get these is subliminally from the material during extensive study, so more study seems to be the solution. According to Begle, however, the teachers who did have enough advanced study to internalize the mindset don't do any better as teachers.

There is a disturbing explanation for this. Advanced study leads to internalization of *modern* methodology. Education is modeled on a philosophical description of mathematical practice in the nineteenth century and before[5], centered around romanticized ideas about the power of intuition. In professional practice this was abandoned as ineffective in the early twentieth century. However educators reject contemporary professional methods as inappropriate for normal humans. This bias makes internalization from advanced courses useless. In fact the conviction that the nineteenth century was a better time for children is so strong that mathematicians working in education tend to buy into it, and suspend their professional skills and sensibilities.

5.3. **Conclusions.** Genuinely productive discussion of teacher preparation must wait on resolution of deeper problems:

(1) Core methodology for mathematics education is stuck in the nineteenth century.
(2) The approach to applications (word problems) is stuck in the sixteenth century, and much of geometry is stuck in the third century BCE.
(3) Until this changes, we should not expect outcomes better than those of antiquity.
(4) Until this changes, advanced coursework with twentieth-century methodology will be irrelevant.

On a brighter note, the science-of-learning methodology described here is compatible with modern mathematics, and advanced coursework would be relevant to teaching based on this approach. Moreover, students trained this way would not find modern mathematics so totally alien, and the calculus/proof transition would not be the killer that it is today.

References

[1] Special Issue on Education, *Notices of the American Mathematical Society* **58** (March 2011).

[2] Papick, Ira; *Strengthening the mathematical content knowledge of middle and secondary mathematics teachers*, pp. 389–393 in [1].

[3] Quinn, Frank; *Contributions to a science of contemporary mathematics*, preprint; current draft at http://www.math.vt.edu/people/quinn/.

[4] ______ *Contributions to a science of mathematical learning*, preprint; current draft at http://www.math.vt.edu/people/quinn/.

- **a:** *Neuroscience experiments for mathematics education*
- **b:** *Proof projects for teachers*
- **c:** *Contemporary proofs for mathematics education* (to appear in the proceedings of ICMI Study 19, held in Taipei May 2009).

[5] For extensive historical and technical background on this see [3].

[5] ______ *Supplement to 'A science-of-learning approach to mathematics education'*, at the web site above.

[6] Wu, H., *The mis-education of mathematics teachers*, pp. 372–384 in [1].