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# Model Reduction by Moment Matching for Linear and Nonlinear Time-Delay Systems

Giordano Scarciotti

**This is a joint work with Alessandro Astolfi**

**Workshop on Model Order Reduction of Transport-dominated Phenomena,
Berlin, Germany**

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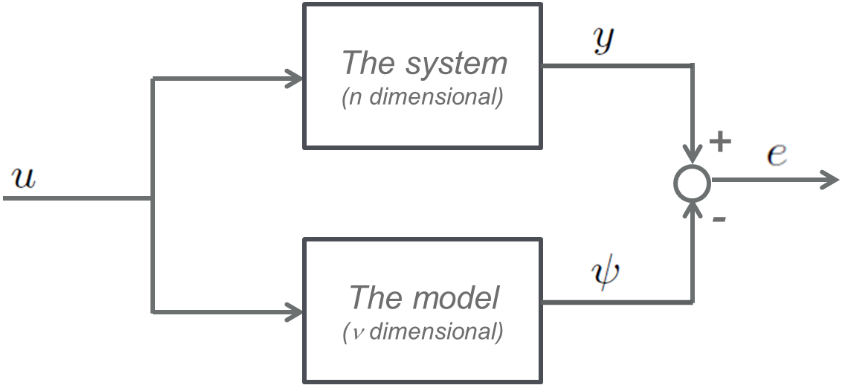
G. Scarciotti is with CAP EEE, **Imperial College London**.

# Contents

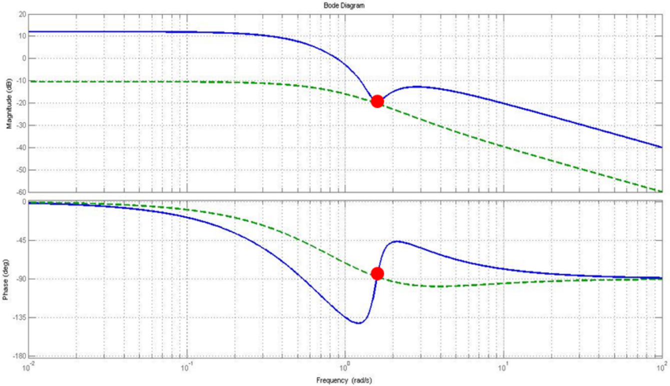
# Contents

# Moments - Interpolation approach




$$||e|| \leq \beta(\nu)||u|| \quad \text{with} \quad \lim_{\nu \to n} \beta(\nu) = 0$$

# Moments - Interpolation approach




$$W(s^*) = W_r(s^*) \quad \ldots \quad \left.\frac{d^k W(s)}{ds^k}\right|_{s=s^*} = \left.\frac{d^k W_r(s)}{ds^k}\right|_{s=s^*}$$

# Moments - Interpolation approach

Let

$$V = \left[(s^* I - A)^{-1} B,\ (s^* I - A)^{-2} B,\ \ldots,\ (s^* I - A)^{-k} B\right],$$

be the generalized reachability matrix and $W$ any matrix such that

$$W^* V = I$$

Then a reduced order model which matched the moments of the system at $s^*$ is described by the equations

$$\dot{\xi} = W^* A V \xi + W^* B u$$
$$y = C V \xi + D u$$

# Contents

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# Moments - Time domain approach

Consider a linear, single-input, single-output, continuous-time, system described by the equations

$$\dot{x}(t) = Ax(t) + Bu(t), \qquad y(t) = Cx(t) \tag{1}$$

and let

$$W(s) = C(sI - A)^{-1}B$$

be the associated transfer function.

## Definition

The 0-*moment of system (1) at* $s_i \in \mathbb{C}$ is the complex number $\eta_0(s_i) = C(s_i I - A)^{-1}B$. The *k-moment of system (1) at* $s_i \in \mathbb{C}$ is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!}\left[\frac{d^k}{ds^k}(C(sI - A)^{-1}B)\right]_{s=s_i}$$

with $k \geq 1$ and integer.

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# Moments - Time domain approach

## Lemma

*Suppose $s_i \notin \sigma(A)$. Then there exists a one-to-one relation between the moments $\eta_0(s_1), \ldots, \eta_{k_1}(s_1), \ldots, \eta_0(s_\eta), \ldots, \eta_{k_\eta}(s_\eta)$ and the matrix $C\Pi$, where $\Pi$ is the unique solution of the Sylvester equation*

$$A\Pi + BL = \Pi S,$$

*with $S \in \mathbb{R}^{\nu\times\nu}$ any non-derogatory matrix with characteristic polynomial $p(s) = \prod_{i=1}^{\eta}(s - s_i)^{k_i}$, where $\nu = \sum_{i=1}^{\eta} k_i$, and $L$ such that the pair $(L, S)$ is observable.*

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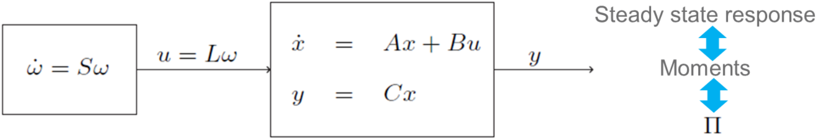


The interconnected system has a globally invariant manifold given by

$$\mathcal{M} = \left\{(x, \omega) \in \mathbb{R}^{n+\nu} : x = \Pi\omega\right\}$$

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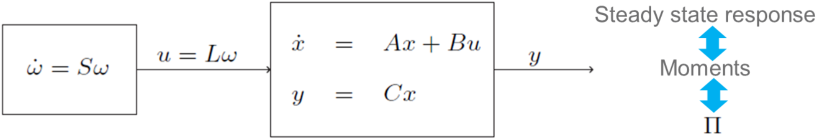


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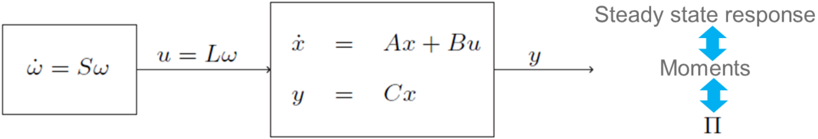


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$$y(t) = C\Pi\omega(t) + Ce^{At}(x(0) - \Pi\omega(0))$$

where the first term on the right-hand side describes the steady-state response of the system, and the second term on the right-hand side the transient response.

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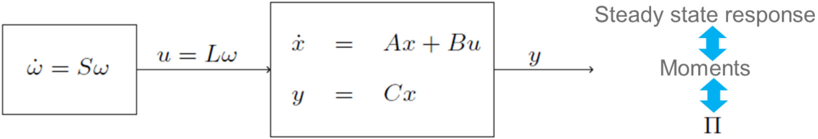


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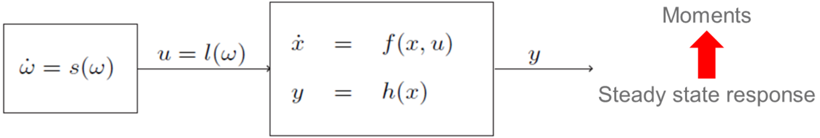
# Moments for nonlinear systems




The interconnected system has a local invariant manifold

$$\mathcal{M} = \left\{ (x, \omega) \in \mathbb{R}^{n+\nu} : x = \pi(\omega) \right\}$$

if $\pi(\omega)$ solves $f(\pi(\omega), l(\omega)) = \dfrac{\partial \pi}{\partial \omega} s(\omega)$.

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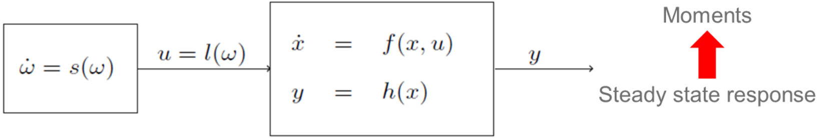


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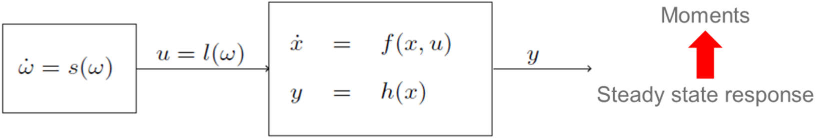


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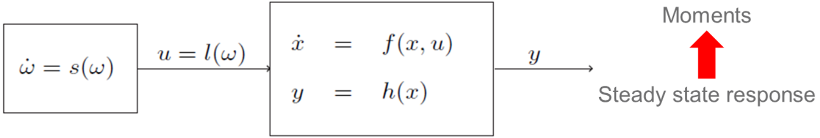
$$y(t) = h(\pi(\omega)) + \varepsilon(t, x(0) - \pi(\omega(0)))$$

and the steady-state response is **by definition** the moment of the nonlinear system at $s(\omega)$.

# Moments for nonlinear systems

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# The key message

In this talk I will try to convey the message that **the one-to-one relation between moments and steady-state response is a flexible and powerful tool** to extend the moment matching approach to general class of systems.

Our toolbox is constituted by the steady-state equations

$$x(t) = \Pi\omega(t)$$

$$x(t) = \pi(\omega(t))$$

$$x(t) = \Pi(t)\omega(t)$$



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# Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research



# Contents

# Motivations

- Time-delay systems are ubiquitous

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- Time-delay systems are ubiquitous
- Delays generate unexpected behavior
- Model reduction for linear and nonlinear systems
- What is the role of the delay in the reduced order model?

# Definition of moment for LTD systems

Consider a linear, single-input, single-output, continuous-time, time-delay system described by the equations

$$\dot{x} = \sum_{j=0}^{\varsigma} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j u_{\tau_j}, \qquad y = \sum_{j=0}^{\varsigma} C_j x_{\tau_j}, \tag{2}$$

and let $W(s) = \sum_{j=0}^{\varsigma} C_j e^{-s\tau_j} \left( sI - \sum_{j=0}^{\varsigma} A_j e^{-s\tau_j} \right)^{-1} \sum_{j=\varsigma+1}^{\mu} B_j e^{-s\tau_j}.$

## Definition

The $k$-moment of system (2) at $s_i \in \mathbb{C}$ is the complex number

$$\eta_k(s_i) = \frac{(-1)^k}{k!} \left[ \frac{d^k}{ds^k} \left( \sum_{j=0}^{\varsigma} C_j e^{-s\tau_j} \left( sI - \sum_{j=0}^{\varsigma} A_j e^{-s\tau_j} \right)^{-1} \sum_{j=\varsigma+1}^{\mu} B_j e^{-s\tau_j} \right) \right]_{s=s_i}$$

with $k \geq 0$ integer.

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*Let* $\bar{A}(s) = \sum_{j=0}^{\varsigma} A_j e^{-s\tau_j}$ *and suppose* $s_i \notin \sigma(\bar{A}(s_i))$ *for all* $i = 1, \ldots, \eta$. *Then there exists a one-to-one relation between the moments* $\eta_0(s_1)$, ..., $\eta_{k_1}(s_1)$, ..., $\eta_0(s_\eta)$, ..., $\eta_{k_\eta}(s_\eta)$ *and the matrix* $\sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j}$, *where* $\Pi$ *is the unique solution of the Sylvester-like equation*

$$\sum_{j=0}^{\varsigma} A_j \Pi e^{-S\tau_j} - \Pi S = -\sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j}$$

*with* $S \in \mathbb{R}^{\nu \times \nu}$ *any non-derogatory matrix with characteristic polynomial* $p(s) = \prod_{i=1}^{\eta} (s - s_i)^{k_i}$, *where* $\nu = \sum_{i=1}^{\eta} k_i$ *and* $L$ *such that the pair* $(L, S)$ *is observable.*

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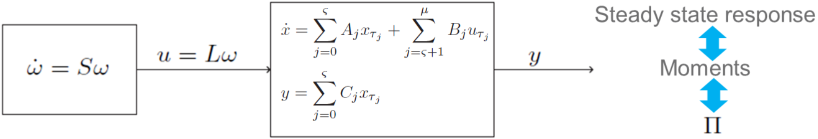


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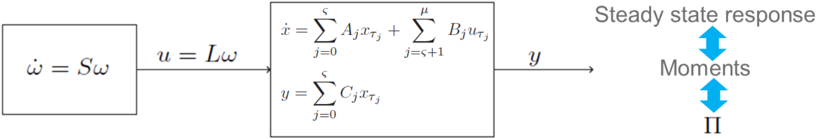


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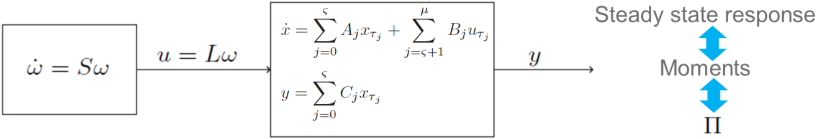


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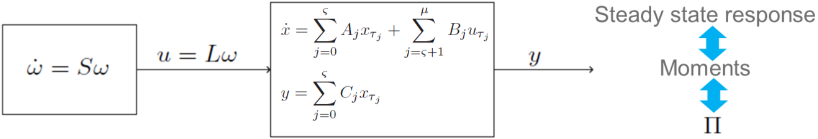


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# Neutral type - Distributed delays

$$\dot{x} = \sum_{j=1}^{q} D_j \dot{x}_{c_j} + \sum_{j=0}^{\varsigma} A_j x_{\tau_j} + \sum_{j=\varsigma+1}^{\mu} B_j u_{\tau_j} + \sum_{j=1}^{r} \int_{t-h_j}^{t} (G_j x(\theta) + H_j u(\theta)) d\theta$$

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The relation between moments and steady-state response is a powerful tool!

$$x(t) = \Pi\omega(t) \qquad \omega_\tau = e^{S\tau}\omega(t) \qquad \int_{t-h}^{t} \omega(\theta) d\theta = S^{-1}(I - e^{-Sh})\omega(t)$$

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$$\sum_{j=0}^{\varsigma} A_j \Pi e^{-S\tau_j} + \sum_{j=1}^{r} G_j \Pi S^{-1}(I - e^{-Sh_j}) + \sum_{j=1}^{q} D_j \Pi S e^{-Sc_j} - \Pi S =$$
$$= -\sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j} - \sum_{j=1}^{r} H_j L S^{-1}(I - e^{-Sh_j}).$$

$\Pi$ unique if $s_i \notin \sigma\left(\sum_{j=1}^{q} D_j s e^{-sc_j} + \sum_{j=0}^{\varsigma} A_j e^{-s\tau_j} + \sum_{j=1}^{r} G_j \frac{1 - e^{-sh_j}}{s}\right)$ and $s_i \neq 0$.

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$$= - \sum_{j=\varsigma+1}^{\mu} B_j L e^{-S\tau_j} - \sum_{j=1}^{r} H_j L S^{-1}(I - e^{-Sh_j}).$$

$\Pi$ unique if $s_i \notin \sigma \left( \sum_{j=1}^{q} D_j s e^{-sc_j} + \sum_{j=0}^{\varsigma} A_j e^{-s\tau_j} + \sum_{j=1}^{r} G_j \frac{1 - e^{-sh_j}}{s} \right)$ and $s_i \neq 0$.

# Reduced model with free parameters

The system

$$\dot{\xi} = \sum_{j=0}^{\varrho} F_j \xi_{\chi_j} + \sum_{j=\varrho+1}^{\rho} G_j u_{\chi_j}, \qquad \psi = \sum_{j=0}^{d} H_j \xi_{\chi_j},$$

is a model of the original system at $S$, if $s_l \notin \sigma\left(\sum_{j=0}^{\varrho} F_j e^{-s_l \chi_j}\right)$ for all $l = 1, \ldots, \eta$, and there exists a unique solution $P$ of the equation

$$\sum_{j=0}^{\varrho} F_j P e^{-S\chi_j} - PS = -\sum_{j=\varrho+1}^{\rho} G_j L e^{-S\chi_j},$$

such that

$$\sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} = \sum_{j=0}^{d} H_j P e^{-S\chi_j}$$

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# Reduced model with free parameters

To construct a family of models that achieves moment matching at $\nu$ points select $P = I$. This yields the family of reduced order models

$$
\begin{aligned}
\dot{\xi} &= \left( S - \sum_{j=\varrho+1}^{\rho} G_j L e^{-S\chi_j} - \sum_{j=1}^{\varrho} F_j e^{-S\chi_j} \right) \xi + \sum_{j=1}^{\varrho} F_j \xi_{\chi_j} + \sum_{j=\varrho+1}^{\rho} G_j u_{\chi_j}, \\
\psi &= \left( \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} - \sum_{j=1}^{d} H_j e^{-S\chi_j} \right) \xi + \sum_{j=1}^{d} H_j \xi_{\chi_j},
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with $G_j$, $F_j$ and $H_j$ any matrices.

The delay-free model is in this family

$$\dot{\xi} = (S - G_1 L)\xi + Gu,$$

$$\psi = \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} \xi$$

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# Example - Exploiting $F_j$

Consider the model of a LC transmission line described by the equations

$$\dot{x}_1 = -\frac{1}{C_1}\left(\frac{1}{R_1} + \sqrt{\frac{C_0}{L}}\right)x_1 - \frac{2}{C_1}\sqrt{\frac{C_0}{L}}\frac{1 - R_0\sqrt{\frac{C_0}{L}}}{1 + R_0\sqrt{\frac{C_0}{L}}}x_{2\tau} + b_1 u,$$

$$\dot{x}_2 = x_1 + \frac{1 - R_0\sqrt{\frac{C_0}{L}}}{1 + R_0\sqrt{\frac{C_0}{L}}}x_{2\tau} + b_1 u,$$

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$$
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$$F_1 = (S_b - S_a - G_3(e^{-S_b\chi_3} - e^{-S_a\chi_3}))(e^{-S_b\chi_1} - e^{-S_a\chi_1})^{-1},$$

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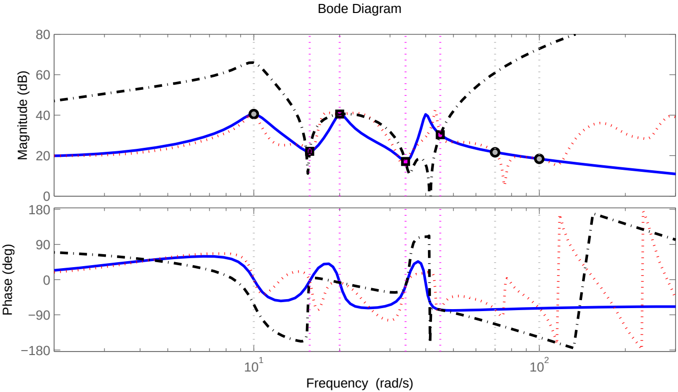
# From delay-free to time-delay




Bode plot of a $n = 1006$ delay-free system (blue/solid line), of a $\nu = 8$ delay-free reduced order model (black/dash-dotted line) and a $\nu = 8$ time-delay reduced order model (red/dotted line). The squares indicate the first set of interpolation points, whereas the circles indicate the second set.

# Contents

# Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research

# Moment for NLTD systems

Consider a nonlinear, single-input, single-output, continuous-time, time-delay system described by the equations

$$\dot{x} = f(x_{\tau_0}, \ldots, x_{\tau_\varsigma}, u_{\tau_\mu}), \qquad y = h(x)$$

Consider a signal generator described by the equations

$$\dot{\omega} = s(\omega), \qquad \theta = l(\omega),$$

and the interconnected system

$$\dot{\omega} = s(\omega), \qquad \dot{x} = f(x_{\tau_0}, \ldots, x_{\tau_\varsigma}, l(\omega_{\tau_\mu})), \qquad y = h(x).$$

## Assumption

*There exists a unique mapping $\pi(\omega)$, locally defined in a neighborhood of $\omega = 0$, which solves the partial differential equation*

$$\frac{\partial \pi}{\partial \omega} s(\omega) = f(\pi(\bar{\omega}_{\tau_0}), \ldots, \pi(\bar{\omega}_{\tau_\varsigma}), l(\bar{\omega}_{\tau_\mu})) \qquad (3)$$

*where $\bar{\omega}_{\tau_i} = \Phi^s_{\tau_i}(\omega)$ is the flow of the vector field $s$ at $\tau_i$.*

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# Moment for NLTD systems

## Assumption

*The signal generator is observable.*

## Definition

The function $h(\pi(\omega))$, with $\pi$ solution of equation (3), is the *moment of the system at* $(s(\omega), l(\omega))$.

## Theorem

*Assume the zero equilibrium of the system* $\dot{x} = f(x_{\tau_0}, \ldots, x_{\tau_\varsigma}, 0)$ *is locally exponentially stable and* $s(\omega)$ *is Poisson stable. Then there exists a unique* $\pi(\omega)$ *and the moment of the system at* $(s(\omega), l(\omega))$ *coincides with the steady-state response of the output of the interconnected system.*

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# Reduced order model for NLTD systems

A family of models that achieves moment matching at $(s(\omega), l(\omega))$ is described by the equations

$$\begin{aligned}\dot{\xi} &= s(\xi) - \delta(\xi) l(\bar{\xi}_{\chi_u}) - \gamma(\bar{\xi}_{\chi_1}, \dots, \bar{\xi}_{\chi_\varrho}) + \gamma(\xi_{\chi_1}, \dots, \xi_{\chi_\varrho}) + \delta(\xi) u_{\chi_u} \\ \psi &= h(\pi(\xi))\end{aligned}$$

where $\bar{\omega}_{\chi_i} = \Phi^s_{\chi_i}(\omega)$ and $\delta$ and $\gamma$ are arbitrary mappings such that

$$\begin{aligned}\frac{\partial p}{\partial \omega} s(\omega) &= s(p(\omega)) - \delta(p(\omega)) l(p(\bar{\omega}_{\chi_u})) + \delta(p(\omega)) l(\omega_{\chi_u}) - \\ &- \gamma(p(\bar{\omega}_{\chi_1}), \dots, p(\bar{\omega}_{\chi_\varrho})) + \gamma(p(\omega_{\chi_1}), \dots, p(\omega_{\chi_\varrho}))\end{aligned}$$

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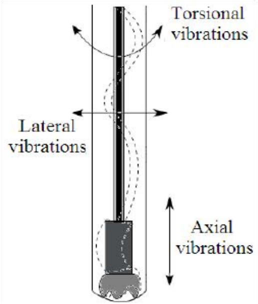
# Dynamics of an oilwell drillstring

$$\frac{\partial^2\theta}{\partial z^2}(z,t) = \frac{I}{GJ}\frac{\partial^2\theta}{\partial t^2}(z,t), \qquad z \in (0,L), t > 0$$

coupled to the mixed boundary conditions

$$GJ\frac{\partial\theta}{\partial z}(0,t) = c_a\left(\frac{\partial\theta}{\partial z}(0,t) - \Omega(t)\right), \quad GJ\frac{\partial\theta}{\partial z}(L,t) + I_B\frac{\partial^2\theta}{\partial t^2}(L,t) = -T\left(\frac{\partial\theta}{\partial t}(L,t)\right)$$



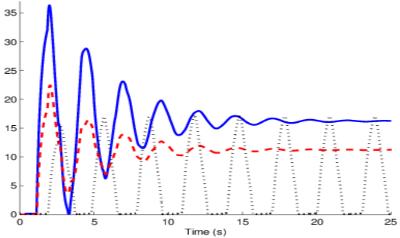




Fig. Angular speed of the system (17), with (20), for different values of $r$: 20 $rad/s$ (solid line), 15 $rad/s$ (dashed line) and 10 $rad/s$ (dotted line). Note the stick-slip phenomenon for $r(t) = 10\ rad/s$.

# Dynamics of an oilwell drillstring

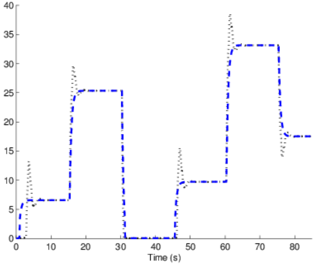



Fig. Time histories of the output of system (17), with (20) and (21), (dotted line) and the output $\psi$ of the reduced order model (22), with $\delta(\cdot) = 2$, (dashed line) for various desired velocities.

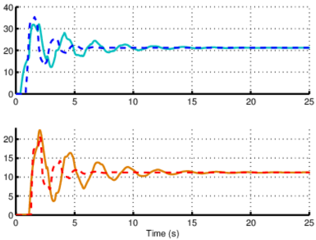



Fig. Time histories of the output of system (17), with (20), (solid line) and of system (23) (dashed line) with $\delta(z) = qz^2 + \varepsilon$ for $r = 25$, $q = 0.0333$, $\varepsilon = 0.3$ (top) and $r = 15$, $q = 0.125$, $\varepsilon = 0.01$ (bottom).

Reduced order model

$$\dot{\xi} = -\delta(\xi)\,[\xi - r_{\tau_2}]$$

$$\psi = \pi(\xi)$$

Open-loop reduced order model

$$\dot{\xi} = -\delta(\xi)\,[\xi - \mu_{\tau_2}]$$

$$\mu = -k_1\pi(\dot{\xi}_{\tau_2}) - k_2\pi(\xi_{\tau_2}) + r$$

$$\psi = \pi(\xi)$$

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# Recap and further directions

The problem of model reduction by moment matching has been changed from a problem of interpolation of points to a problem of interpolation of signals. The output of the reduced order model has to behave as the output of the original system for a class of input signals, a concept which can be translated to nonlinear systems, time-delay systems and...

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The results described are based on the availability of a differential representation of the signal generator, namely $\dot{\omega} = S\omega$. However, there are notable applications in which this may not be the case. For instance, the input of a dynamical system describing a power electronic device can often be a PWM wave (e.g. a square or sawtooth wave) which cannot be represented as the output of a system described by smooth differential equations.

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# Recap and further directions

The problem of model reduction by moment matching has been changed from a problem of interpolation of points to a problem of interpolation of signals. The output of the reduced order model has to behave as the output of the original system for a class of input signals, a concept which can be translated to nonlinear systems, time-delay systems and...

The results described are based on the availability of a differential representation of the signal generator, namely $\dot{\omega} = S\omega$. However, there are notable applications in which this may not be the case. For instance, the input of a dynamical system describing a power electronic device can often be a PWM wave (e.g. a square or sawtooth wave) which cannot be represented as the output of a system described by smooth differential equations.

# Analysis of a square wave

Consider a square wave $\sqcap(t)$ defined as

$$\sqcap(t) = \text{sign}(\sin(t)) = \begin{cases} 1, & (k-1)\pi < t < k\pi, \\ 0, & t = k\pi \text{ or } t = (k+1)\pi, \\ -1, & k\pi < t < (k+1)\pi, \end{cases}$$

*i.e.* with $\text{sign}(0) = 0$, and $k = 1, 3, 5, \ldots, +\infty$.

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*i.e.* with $\text{sign}(0) = 0$, and $k = 1, 3, 5, \dots, +\infty$.

The Laplace transform of this function is

$$\mathcal{L}(\sqcap(t)) = \frac{1 - e^{-s\pi}}{s(1 + e^{-s\pi})},$$

and this has the poles

$$s_1 = 0, \qquad s_i = (2j+1)\iota,$$

with $j = -\infty, \dots, -1, 0, 1, \dots, +\infty$.

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Since the function $\sqcap(t)$ is periodic, it admits a Fourier series, namely

$$\sqcap(t) = \frac{4}{\pi} \sum_{i=1,3,5,\ldots,+\infty}^{\infty} \frac{1}{i} \sin(it).$$

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The Laplace and Fourirer transform of the square wave suggest that we could describe $\sqcap(t)$ by means of the infinite dimensional system

$$\dot{\omega} = \begin{bmatrix} \ddots & \ddots & & & & & \\ \ddots & +2\iota & 0 & & & & \\ & 0 & +\iota & 0 & & & \\ & & 0 & 0 & 0 & & \\ & & & 0 & -\iota & 0 & \\ & & & & 0 & -2\iota & \ddots \\ & & & & & \ddots & \ddots \end{bmatrix} \omega$$

with output $\sqcap = P\omega$ for some "matrix" $P$.

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# Generator in explicit form

To overcome these issues we consider signal generators in explicit form. Thus, consider

$$\omega(t) = \Lambda(t, t_0)\omega_0, \qquad u = L\omega,$$

Note that for linear systems in implicit form

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But it describes a very large class of signals: noncontinuos periodic signals, time-varying systems, a subclass of hybrid systems, a subclass of nonlinear systems,...

We want to characterize the "moments" of the following interconnection

$$\omega(t) = \Lambda(t, t_0)\omega_0$$

$$\dot{x} = Ax + BL\omega$$

$$y = Cx$$

# Characterization of the moments

## Theorem

*Let* $\Pi(t) = \left(e^{A(t-t_0)}\Pi(t_0) + \int_{t_0}^{t} e^{A(t-\tau)} BL\Lambda(\tau, t_0) d\tau\right)\Lambda(t, t_0)^{-1}$

*be a family of matrix valued functions parametrized in* $\Pi(t_0) \in \mathbb{R}^{n\times\nu}$*. Given "mild" assumptions there exists a unique* $\Pi_\infty(t_0)$ *such that, for any* $\Pi(t_0)$*,* $\lim_{t\to+\infty} \Pi(t) - \Pi_\infty(t) = 0$*. Moreover, if* $x(t_0) = \Pi_\infty(t_0)\omega(t_0)$ *then* $x(t) - \Pi_\infty(t)\omega(t) = 0$ *for all* $t \geq t_0$*, and the set* $\{(x, \omega) \,|\, x(t) = \Pi_\infty(t)\omega(t)\}$ *is attractive.*



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## Remark

$\Pi_\infty(t)$ *is also the unique solution of*

$$\dot{\Pi}(t) = A\Pi(t) + BL - \Pi(t)\dot{\Lambda}(t, t_0)\Lambda(t, t_0)^{-1}$$

*with the initial condition* $\Pi(t_0) = \Pi_\infty(t_0)$. *From a practical point of view, it is necessary to know the initial condition* $\Pi_\infty(t_0)$. *However, since the motion* $\Pi_\infty(t)$ *is attractive, any solution of the two equations converges to* $\Pi_\infty(t)$.

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# The periodic case

Consider the signal generator

$$\begin{aligned} \omega(t) &= \omega(t-T), \\ \omega(t) &= h(t,t_0)\omega_0, \qquad\qquad t_0 - T \leq t < t_0, \\ u &= L\omega, \end{aligned}$$

then $\Pi_\infty(t)$ becomes

$$\Pi_\infty(t) = (I - e^{AT})^{-1}\left[\int_{t-T}^{t} e^{A(t-\tau)}BL\Lambda(\tau,t_0)d\tau\right]\Lambda(t,t_0)^{-1}$$

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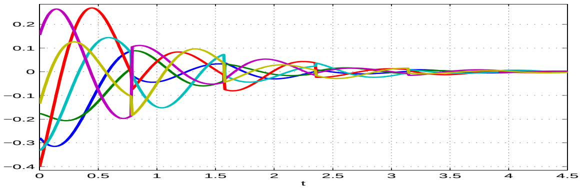
# A numerical example

Consider the matrix of square waves

$$\Lambda_{\sqcap}(t,0) = \begin{bmatrix} \sqcap\left(\frac{2\pi}{T}t+\frac{\pi}{2}\right) & -\sqcap\left(\frac{2\pi}{T}t\right) \\ \sqcap\left(\frac{2\pi}{T}t\right) & \sqcap\left(\frac{2\pi}{T}t+\frac{\pi}{2}\right) \end{bmatrix}.$$

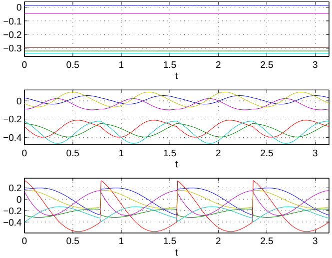
The previous equation computed for $t = 0$

$$\Pi_{\infty}(0) = -A^{-1}(I-e^{AT})^{-1}\left[\left(e^{\frac{3}{4}AT}-e^{AT}+e^{\frac{1}{2}AT}-e^{\frac{1}{4}AT}\right)BL + \right.$$
$$\left. + \left(e^{\frac{1}{2}AT}-e^{\frac{3}{4}AT}+e^{\frac{1}{4}AT}-I\right)BL\Lambda_{\sqcap}\left(\frac{T}{4},0\right)\right]$$

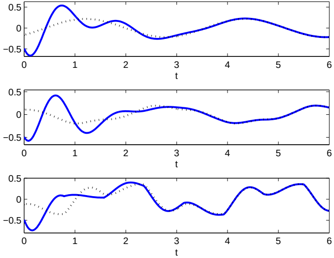


# Looking at new $\Pi$'s

$$\Lambda_{\sim}(t,0) = \begin{bmatrix} \cos\left(\frac{2\pi}{T}t\right) & -\sin\left(\frac{2\pi}{T}t\right) \\ \sin\left(\frac{2\pi}{T}t\right) & \cos\left(\frac{2\pi}{T}t\right) \end{bmatrix} \qquad \Lambda_{\wedge}(t,0) = \begin{bmatrix} \wedge\left(\frac{2\pi}{T}t+\frac{\pi}{2}\right) & -\wedge\left(\frac{2\pi}{T}t\right) \\ \wedge\left(\frac{2\pi}{T}t\right) & \wedge\left(\frac{2\pi}{T}t+\frac{\pi}{2}\right) \end{bmatrix}$$




Time history of the entries of the matrices $\Pi_{\sim}$ (top), $\Pi_{\wedge}$ (middle) and $\Pi_{\sqcap}$ (bottom).




Time history of the output (solid lines) $y_{\sim}$ (top), $y_{\wedge}$ (middle) and $y_{\sqcap}$ (bottom). Time histories of the steady-state of the output (dotted lines) computed as $C\Pi_{\sim}\omega$, $C\Pi_{\wedge}\omega$ and $C\Pi_{\sqcap}\omega$.

# A new family of reduced order models

## Definition

The system described by the equations

$$\xi(t) = F(t, t_0)\xi_0 + \int_{t_0}^{t} G(t-\tau)u(\tau)d\tau,$$
$$\psi(t) = H(t)\xi(t),$$

is a *model of the system*, if there exists a unique solution $P_\infty(t)$ of the equation

$$P(t) = \left( F(t, t_0)P(t_0) + \int_{t_0}^{t} G(t-\tau)L\Lambda(\tau, t_0)d\tau \right) \Lambda^{-1}(t, t_0)$$

with $P(t_0) = P_\infty(t_0)$ such that for any $P(t_0)$, $\lim\limits_{t\to+\infty} P(t) - P_\infty(t) = 0$ and

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# The periodic family

## Definition

The system

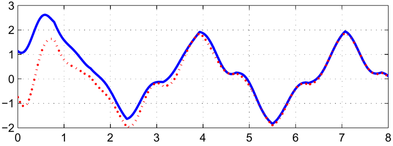
$$\dot{\xi} = \widetilde{F}\xi + \widetilde{G}u,$$

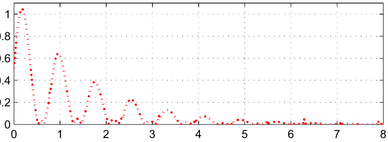
$$\psi(t) = C\Pi_\infty(t)P_\infty(t)^{-1}\xi(t),$$

is a *model of the system*, if $\sigma(\widetilde{F}) \in \mathbb{C}_{<0}$ and

$$P_\infty(t) = (I - e^{\widetilde{F}T})^{-1}\left[\int_{t-T}^{t} e^{\widetilde{F}(t-\tau)}\widetilde{G}L\Lambda(\tau,t_0)d\tau\right]\Lambda(t,t_0)^{-1},$$

is non-singular for all $t \in \mathbb{R}_{\geq 0}$.



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# Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research

# Contents

# Systems with unknown description

If we have the steady-state response $C\Pi\omega(t)$, how do we recover the moments $\Pi$?

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How do we obtain a reduced order model if we do not have the matrices $A$, $B$, $C$, but we have measurements of the input and output of the system?

# Let's manipulate the response

Recall that the output of a linear system can be written as

$$y(t) = C\Pi\omega(t) + Ce^{At}(x(0) - \Pi\omega(0))$$

This can be rewritten as

$$\mathrm{vec}(C\Pi\omega(t)) - \mathrm{vec}(Ce^{At}\Pi\omega(0)) = \mathrm{vec}(y(t) - Ce^{At}x(0)),$$

and

$$(\omega(t)^\top \otimes C - \omega(0)^\top \otimes Ce^{At})\,\mathrm{vec}(\Pi) = \mathrm{vec}(y(t) - Ce^{At}x(0)).$$

Finally

$$(\omega(0)^\top \otimes C)(e^{S^\top t} \otimes I - I \otimes e^{At})\,\mathrm{vec}(\Pi) = \mathrm{vec}(y(t) - Ce^{At}x(0))$$

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# Let's manipulate the response

Define the time-snapshots $R_k \in \mathbb{R}^{w \times n\nu}$ and $\Upsilon_k \in \mathbb{R}^{w}$ as

$$R_k = \begin{bmatrix} (\omega(0)^\top \otimes C)(e^{S^\top t_{k-w+1}} \otimes I - I \otimes e^{At_{k-w+1}}) \\ \vdots \\ (\omega(0)^\top \otimes C)(e^{S^\top t_{k-1}} \otimes I - I \otimes e^{At_{k-1}}) \\ (\omega(0)^\top \otimes C)(e^{S^\top t_{k}} \otimes I - I \otimes e^{At_{k}}) \end{bmatrix},$$

$$\Upsilon_k = \begin{bmatrix} y(t_{k-w+1}) - Ce^{At_{k-w+1}}x(0) \\ \vdots \\ y(t_{k-1}) - Ce^{At_{k-1}}x(0) \\ y(t_k) - Ce^{At_k}x(0) \end{bmatrix}.$$

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Note that the equation can be written as

$$y(t) = C\Pi\omega(t) + \varepsilon(t),$$

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Thus, let $\widetilde{C\Pi}$ be such that

$$y(t) = \widetilde{C\Pi}\omega(t),$$

and define the time-snapshots $\widetilde{R}_k \in \mathbb{R}^{w\times\nu}$ and $\widetilde{\Upsilon}_k \in \mathbb{R}^{w}$ as

$$\widetilde{R}_k = \left[\begin{array}{cccc} \omega(t_{k-w+1}) & \dots & \omega(t_{k-1}) & \omega(t_k) \end{array}\right]^\top$$

and

$$\widetilde{\Upsilon}_k = \left[\begin{array}{cccc} y(t_{k-w+1}) & \dots & y(t_{k-1}) & y(t_k) \end{array}\right]^\top.$$

Then

$$\mathrm{vec}(\widetilde{C\Pi}_k) = (\widetilde{R}_k^\top \widetilde{R}_k)^{-1}\widetilde{R}_k^\top \widetilde{\Upsilon}_k,$$

is an approximation of the on-line estimate $C\Pi_k$.

# A recursive implementation

It is easy to derive a recursive least-squares estimation of $\widetilde{C\Pi}_k$. To this end, let

$$\Phi_k = (\widetilde{R}_k^\top \widetilde{R}_k)^{-1},$$

$$\Psi_k = (\widetilde{R}_{k-1}^\top \widetilde{R}_{k-1} + \omega(t_k)\omega(t_k)^\top)^{-1}.$$

Then

$$\begin{aligned}\widetilde{C\Pi}_k &= \widetilde{C\Pi}_{k-1} + \Phi_k\omega(t_k)(y(t_k) - \omega(t_k)^\top \widetilde{C\Pi}_{k-1}) \\ &\quad -\Phi_k\omega(t_{k-w})(y(t_{k-w}) - \omega(t_{k-w})^\top \widetilde{C\Pi}_{k-1}),\end{aligned}$$

with

$$\begin{aligned}\Phi_k &= \Psi_k - \Psi_k\omega(t_{k-w})\times \\ &\quad \times(I + \omega(t_{k-w})^\top \Psi_k\omega(t_{k-w}))^{-1}\omega(t_{k-w})^\top \Psi_k\end{aligned}$$

and

$$\begin{aligned}\Psi_k &= \Phi_{k-1} - \Phi_{k-1}\omega(t_k)\times \\ &\quad \times(I + \omega(t_k)^\top \Phi_{k-1}\omega(t_k))^{-1}\omega(t_k)^\top \Phi_{k-1}.\end{aligned}$$

For SISO systems the two matrix inversions are two divisions. The computation complexity of updating the estimate is $\mathcal{O}(1)$.

# A family of reduced order models

## Definition

The system described by the equations

$$\dot{\xi} = F_k \xi + G_k u, \qquad \phi = H_k \xi,$$

is a *model of the system at (S,L) at time $t_k$*, if there exists a unique solution $P_k$ of the equation

$$F_k P_k + G_k L = P_k S,$$

such that

$$\widetilde{C\Pi}_k = H_k P_k,$$

## Remark

*Select $P_k = I$, for all $k \geq 0$. If $\sigma(F_k) \cap \sigma(S) = \emptyset$ for all $k \geq 0$, then the model*

$$\dot{\xi} = (S - G_k L)\xi + G_k u,$$

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# Linear time-delay systems

These resuls can be easily extended to linear time-delay systems. In fact, we have already seen that for linear time-delay systems the following holds

$$y(t) = \sum_{j=0}^{\varsigma} C_j \Pi e^{-S\tau_j} \omega(t) + \varepsilon(t),$$

Then

$$\operatorname{vec}\left( \widetilde{\sum_{j=0}^{\varsigma} C_j \Pi e_k^{-S\tau_j}} \right) = (\widetilde{R}_k^\top \widetilde{R}_k)^{-1} \widetilde{R}_k^\top \widetilde{\Upsilon}_k,$$

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# Matching with prescribed eigenvalues

Determining at every $k$ the matrix $G_k$ such that $\sigma(F_k) = \{\lambda_{1,k}, \ldots, \lambda_{\nu,k}\}$ for some prescribed values $\lambda_{i,k}$. The solution of this problem is well-known and consists in selecting $G_k$ such that

$$\sigma(S - G_k L) = \sigma(F_k).$$

This is possible for every $k$ and for all $\lambda_{i,k} \notin \sigma(S)$ and note that $G_k$ is independent from the estimate $\widetilde{C\Pi}_k$. Note also that by observability of $(L, S)$, $G_k$ is unique at every $k$.

# Prescribed relative degree, zeros, compartmental constraints

These problems can be solved at each $k$ if and only if

$$\text{rank}\begin{bmatrix} sI - S \\ \widetilde{C\Pi_k} \end{bmatrix} = n,$$

for all $s \in \sigma(S)$ at $k$. Even though the asymptotic value of $\widetilde{C\Pi_k}$ satisfies this condition there is no guarantee that the condition holds for all $k$. However, if the condition holds for the asymptotic value, there exists $\bar{k} \gg 0$ such that for all $k \geq \bar{k}$ the equation has a solution.

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# System with $n = 1006$

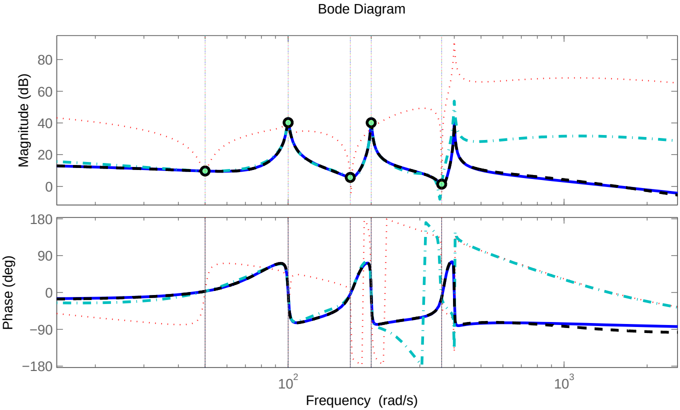



Figure: Bode plot of the system (solid line), of the reduced order model at $t_k = 90s$ (dotted line), of the reduced order model at $t_k = 110s$ (dash-dotted line) and of the reduced order model at $t_k = 140s$ (dashed line). The circles indicate the interpolation points.

# A nonlinear example

The averaged model of the DC-to-DC Ćuk converter is given by the equations

$$L_1 \frac{d}{dt} i_1 = -(1-u)v_2 + E, \qquad L_3 \frac{d}{dt} i_3 = -uv_2 - v_4,$$
$$C_2 \frac{d}{dt} v_2 = (1-u)i_1 + ui_3, \qquad C_4 \frac{d}{dt} v_4 = \ i_3 - Gv_4,$$
$$y = v_4,$$

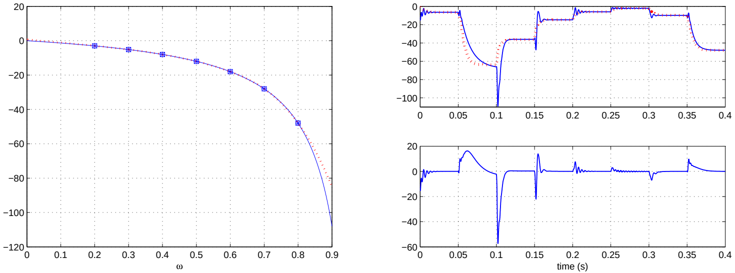
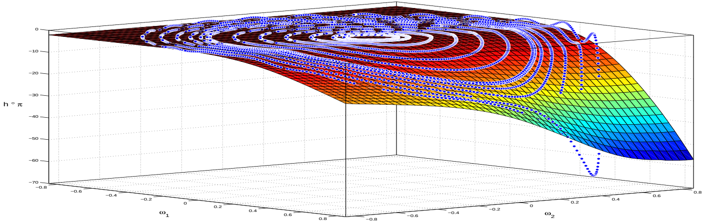
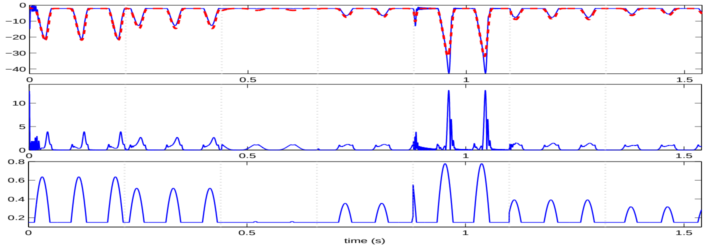



Figure: $h(\pi(\omega)) = E\dfrac{\omega}{\omega - 1}$

# A nonlinear example



# Contents

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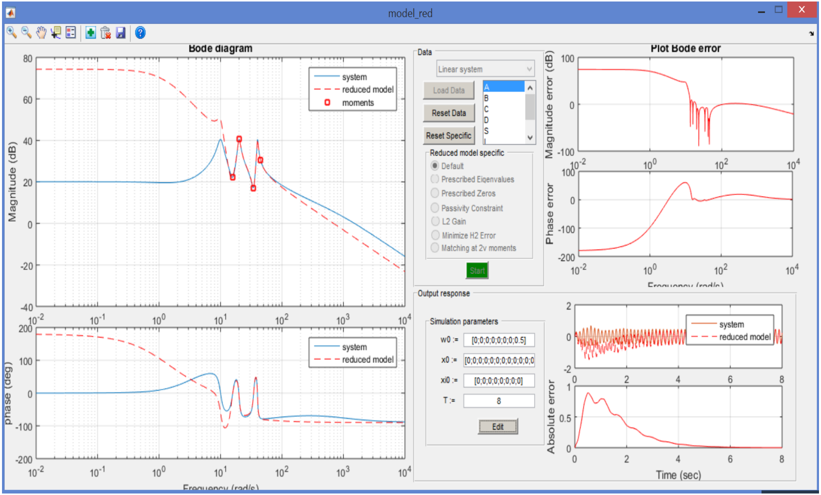
# Contents

# A matlab toolbox for moment matching



# A matlab toolbox for moment matching

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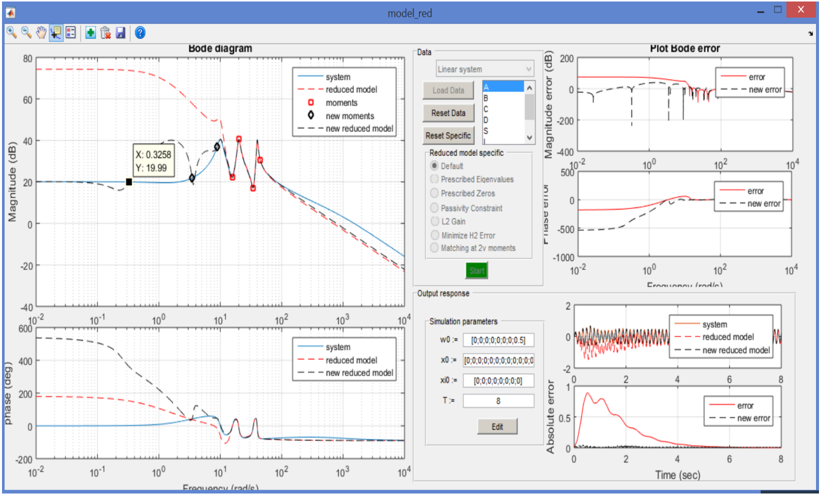

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# Contents

# Contents

- Introduction to moment matching
- The time domain approach to moment matching
- Model reduction for linear time-delay systems
- Model reduction for nonlinear time-delay systems
- Interpolation at infinitely many points
- Model reduction from input/output data
- A toolbox for the model reduction by moment matching
- Remarks and further research

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# Thank you for your attention!