Spatially-discretised Distributed Port-Hamiltonian Systems

Lumping, model reduction & control synthesis

Workshop on
Model Reduction and Transport-dominated Phenomena

Berlin-Brandenburg Academy of Sciences — June 19-20, 2015

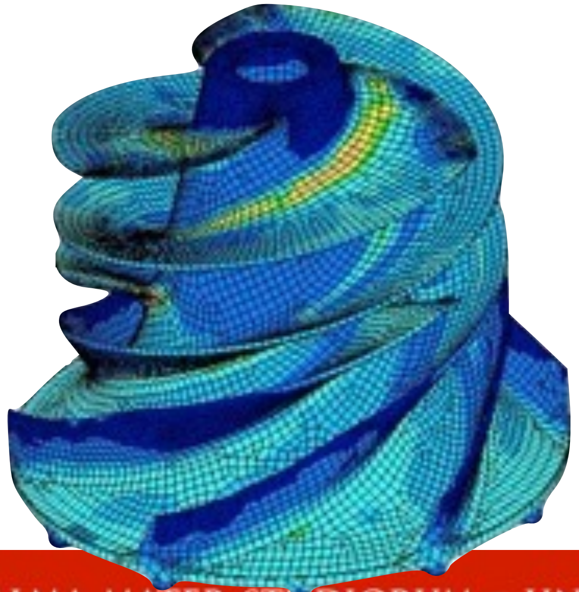
Outline

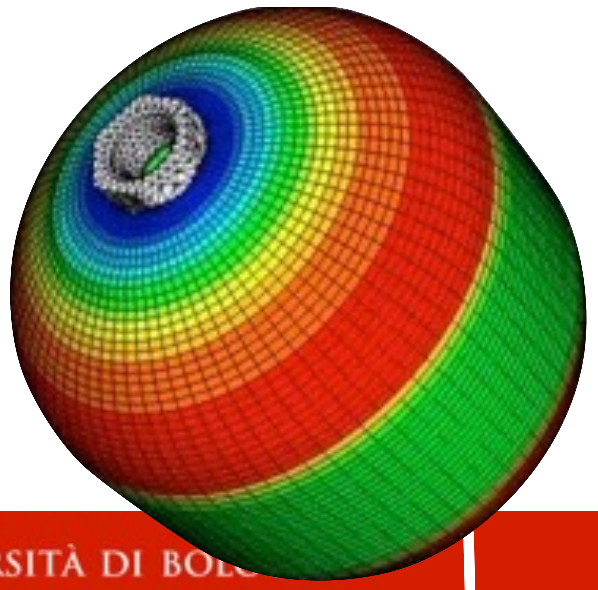
- This talk is organised into three main parts
  - Structure-preserving *Spatial Discretisation* of Distributed Port-Hamiltonian Systems
  - Basic Results on *Model Reduction* for Port-Hamiltonian Systems
  - *Control Synthesis* for Implicit Port-Hamiltonian Systems
- These topics are strongly related, and they show how it is possible to treat different problems within the same *methodological framework*
- Some academic examples are presented, and also one real-world industrial application

Structure-preserving Spatial Discretisation of Distributed Port-Hamiltonian Systems

Spatial discretisation of distributed pH systems

- A fundamental problem in the *simulation* and *control* of systems containing *distributed-parameter components* concerns *finite-dimensional approximation*
  - *Numerical methods usually assume the boundary conditions to be given;*
  - Finite dimensional approximation methods are not easily relatable to numerical techniques for solving PDEs,
  - and are mainly confined to *linear PDEs*
- We propose a method for spatial discretisation of boundary control systems based on a particular type of *finite elements*
- The spatially discretised system is **again** a *port-Hamiltonian system*
- *Examples:*
  - Ideal transmission line;
  - Two dimensional wave equation;
  - Nonlinear flexible link





Spatial discretisation of distributed pH systems

G. Golo, *et al.*, Automatica, 2004

- Let us start from a system of *two conservation laws:*

  - *Bond space:*

$Z \subset \mathbb{R}^n$

$$f_x = \begin{pmatrix} f_q \\ f_p \end{pmatrix} \in \mathcal{F}_x = \Omega^q(Z) \times \Omega^p(Z)$$

$$x = \begin{pmatrix} q \\ p \end{pmatrix} \in \mathcal{X} = \Omega^q(Z) \times \Omega^p(Z)$$

$$e_x = \begin{pmatrix} e_q \\ e_p \end{pmatrix} \in \mathcal{E}_x = \Omega^{n-q}(Z) \times \Omega^{n-p}(Z)$$

$p + q = n + 1$

$$\begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} \in \mathcal{F}_\partial \times \mathcal{E}_\partial = \Omega^{n-p}(\partial Z) \times \Omega^{n-q}(\partial Z)$$

  - *Stokes-Dirac structure:*

$$\begin{pmatrix} f_q \\ f_p \end{pmatrix} = \begin{pmatrix} 0 & -\mathrm{d} \\ (-1)^{pq}\mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} e_q \\ e_p \end{pmatrix} \qquad \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -(-1)^q & 0 \end{pmatrix} \begin{pmatrix} e_q\,|_{\partial Z} \\ e_p\,|_{\partial Z} \end{pmatrix}$$

with respect to the *scalar pairing*

$$\int_Z (e_q \wedge f_q + e_p \wedge f_p) + \int_{\partial Z} e_\partial \wedge f_\partial$$

$$f_q = \frac{\partial q}{\partial t} \quad f_p = \frac{\partial p}{\partial t} \qquad e_q = \frac{\delta H}{\delta q} \quad e_p = \frac{\delta H}{\delta p}$$

  - *Energy balance:*

$$\frac{\mathrm{d}H}{\mathrm{d}t}(t) + \int_{\partial Z} e_\partial \wedge f_\partial = 0$$

Spatial discretisation of distributed pH systems

$Z = [0,\ L]$

- For a *transmission line*, ***n = q = p = 1*** and

$p \equiv \phi$

$$H(q,\phi) = \frac{1}{2}\int_Z \left[\frac{\star q(z)}{C(z)}q(z) + \frac{\star\phi(z)}{L(z)}\phi(z)\right]$$

- The spatial discretisation procedure consists of *two steps:*
  - First, the *interconnection structure* is spatially discretised;
  - Secondly, the *constitutive relations* are approximated
- **STEP#1:** spatial discretisation of the interconnection structure
  - Consider *a part of the transmission line* between two points *a* and *b*: the spatial manifold corresponding to this part of line is $Z_{ab} = [a,\ b]$

Assumption 1. *Approximation of f_q and f_ϕ on Z_{ab}*

$$f_q(t,z) = f_{q,ab}(t)\omega_{q,ab}(z)$$
$$f_\phi(t,z) = f_{\phi,ab}(t)\omega_{\phi,ab}(z)$$

where

$$\int_{Z_{ab}} \omega_{q,ab} = \int_{Z_{ab}} \omega_{\phi,ab} = 1$$

Assumption 2. *Approximation of e_q and e_ϕ on Z_{ab}*

$$e_q(t,z) = e_{q,a}(t)\omega_{q,a}(z) + e_{q,b}(t)\omega_{q,b}(z)$$
$$e_\phi(t,z) = e_{\phi,a}(t)\omega_{\phi,a}(z) + e_{\phi,b}(t)\omega_{\phi,b}(z)$$

where

$$\omega_{q,a}(a) = \omega_{q,b}(b) = \omega_{\phi,a}(a) = \omega_{\phi,b}(b) = 1$$
$$\omega_{q,a}(b) = \omega_{q,b}(a) = \omega_{\phi,a}(b) = \omega_{\phi,a}(b) = 0$$

Spatial discretisation of distributed pH systems

Assumption 1

Assumption 2

$$f_{q,ab}(t)\omega_{q,ab}(z) = -e_{\phi,a}(t)\mathrm{d}\omega_{\phi,a}(z) - e_{\phi,b}(t)\mathrm{d}\omega_{\phi,b}(z) \quad (1)$$

$$f_{\phi,ab}(t)\omega_{\phi,ab}(z) = -e_{q,a}(t)\mathrm{d}\omega_{q,a}(z) - e_{q,b}(t)\mathrm{d}\omega_{q,b}(z) \quad (2)$$

Assumption 3 (Compatibility of forms). *(1) The 1-form $\omega_{q,ab}$ and functions $\omega_{\phi,a}$ and $\omega_{\phi,b}$ should be chosen in such way that for every $e_{\phi,a}$ and $e_{\phi,b}$ we can find $f_{q,ab}$ such that **1** is satisfied.*
*(2) The 1-form $\omega_{\phi,ab}$ and functions $\omega_{q,a}$ and $\omega_{q,b}$ should be chosen in such way that for every $e_{q,a}$ and $e_{q,b}$ we can find $f_{\phi,ab}$ such that **2** is satisfied.*

$$\mathrm{d}\omega_{\phi,a} = -\omega_{q,ab}$$
$$\mathrm{d}\omega_{\phi,b} = \omega_{q,ab}$$
$$\mathrm{d}\omega_{q,a} = -\omega_{\phi,ab}$$
$$\mathrm{d}\omega_{q,b} = \omega_{\phi,ab}$$

Proposition. *$\omega_{q,a}$, $\omega_{q,b}$, $\omega_{\phi,a}$, $\omega_{\phi,b}$, $\omega_{q,ab}$ and $\omega_{\phi,ab}$ satisfy*

$$(i) \quad \omega_{q,a}(z) + \omega_{q,b}(z) = 1$$
$$(ii) \quad \omega_{\phi,a}(z) + \omega_{\phi,b}(z) = 1$$
$$(iii) \quad \int_{Z_{ab}} [\omega_{q,a}(z) + \omega_{q,b}(z)]\,\omega_{q,ab}(z) = 1$$
$$(iv) \quad \int_{Z_{ab}} [\omega_{\phi,a}(z) + \omega_{\phi,b}(z)]\,\omega_{\phi,ab}(z) = 1$$
$$(v) \quad \int_{Z_{ab}} [\omega_{q,a}(z)\omega_{q,ab}(z) + \omega_{\phi,a}(z)\omega_{\phi,ab}(z)] = 1$$

e.g., linear spline

Spatial discretisation of distributed pH systems

- Integration over Z_{ab} leads to

$$f_{q,ab}(t) = e_{\phi,a}(t) - e_{\phi,b}(t)$$
$$f_{\phi,ab}(t) = e_{q,a}(t) - e_{q,b}(t)$$

$$\Rightarrow \begin{pmatrix} e_{\partial,a} \\ e_{\partial,b} \\ f_{\partial,a} \\ f_{\partial,b} \\ f_{q,ab} \\ f_{\phi,ab} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 0 \end{pmatrix} \begin{pmatrix} e_{q,a} \\ e_{q,b} \\ e_{\phi,a} \\ e_{\phi,b} \end{pmatrix}$$

- *Net power* of the considered part of the transmission line:

$$P_{ab}^{\text{net}} = \int_{Z_{ab}} e_q(z) f_q(z) + \int_{Z_{ab}} e_\phi(z) f_\phi(z) - e_{\partial,a} f_{\partial,a} + e_{\partial,b} f_{\partial,b}$$

$$\alpha_{ab} = \int_{Z_{ab}} \omega_{q,a}(z)\omega_{q,ab}(z) = 1 - \alpha_{ba}$$

$$\Downarrow$$

$$P_{ab}^{\text{net}} = \left[\alpha_{ab} e_{q,a} + (1-\alpha_{ab}) e_{q,b}\right] f_{q,ab} + \\ + \left[(1-\alpha_{ab}) e_{\phi,a} + \alpha_{ab} e_{\phi,b}\right] f_{\phi,ab} - \\ - e_{\partial,a} f_{\partial,a} + e_{\partial,b} f_{\partial,b}$$

$$0 < \alpha_{ab} < 1$$

$$e_{q,ab} = \alpha_{ab} e_{q,a} + (1-\alpha_{ab}) e_{q,b}$$
$$e_{\phi,ab} = (1-\alpha_{ab}) e_{\phi,a} + \alpha_{ab} e_{\phi,b}$$

$$\Downarrow$$

$$P_{ab}^{\text{net}} = e_{q,ab} f_{q,ab} + e_{\phi,ab} f_{\phi,ab} - e_{\partial,a} f_{\partial,a} + e_{\partial,b} f_{\partial,b}$$

Spatial discretisation of distributed pH systems

$$f_{ab} = \begin{pmatrix} f_{q,ab} & f_{\phi,ab} & f_{\partial,a} & f_{\partial,b} \end{pmatrix}^{\mathrm{T}}$$
$$e_{ab} = \begin{pmatrix} e_{q,ab} & e_{\phi,ab} & e_{\partial,a} & e_{\partial,b} \end{pmatrix}^{\mathrm{T}}$$

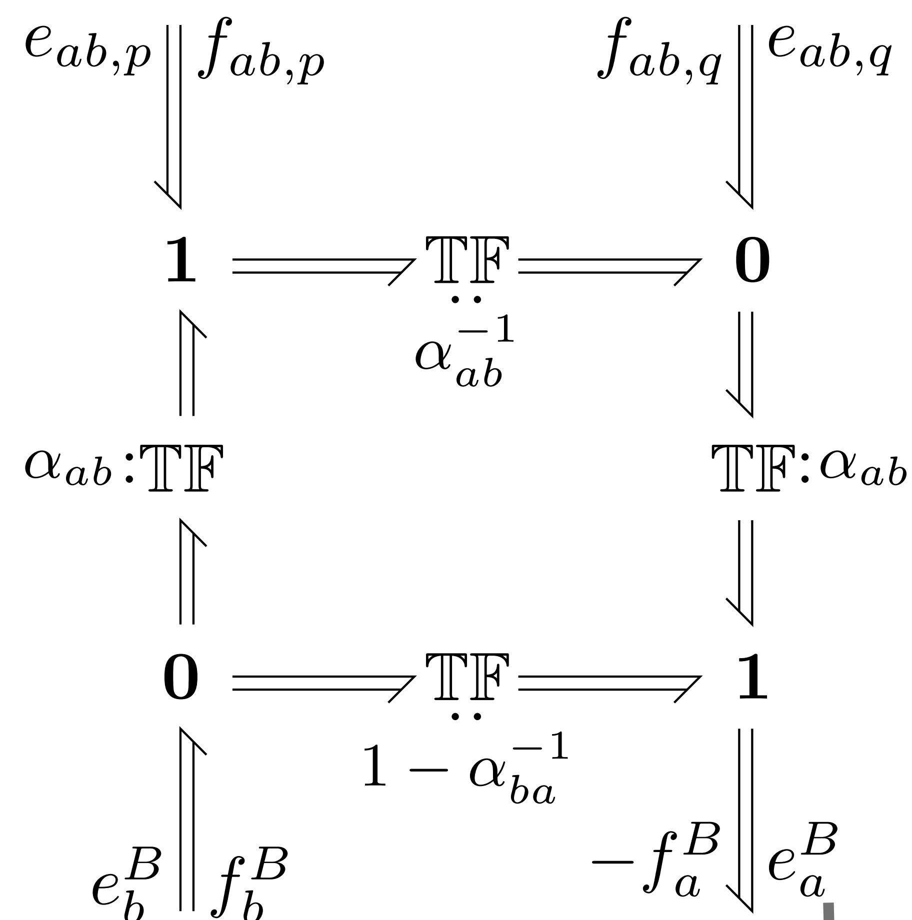
- After simple computations:

$$\underbrace{\begin{pmatrix} -1 & 0 & \alpha_{ab} & \alpha_{ba} \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix}}_{E_{ab}} \begin{pmatrix} e_{q,ab} \\ e_{\phi,ab} \\ e_{\partial,a} \\ e_{\partial,b} \end{pmatrix} + \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha_{ba} & \alpha_{ab} \\ 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}}_{F_{ab}} \begin{pmatrix} f_{q,ab} \\ f_{\phi,ab} \\ f_{\partial,a} \\ f_{\partial,b} \end{pmatrix} = 0$$

Proposition. *The subspace*

$$\mathcal{D}_{ab} = \left\{ (f_{ab}, e_{ab}) \in \mathbb{R}^8 \;\middle|\; E_{ab} e_{ab} + F_{ab} f_{ab} = 0 \right\}$$

is a Dirac structure



bond graph

Spatial discretisation of distributed pH systems

STEP#2: approximation of the energy part

- Since *f_q and f_ϕ are related to q and ϕ,* it is consistent to impose

$$q(t,z) = Q_{ab}(t)\omega_{q,ab}(z) \qquad \phi(t,z) = \Phi_{ab}(t)\omega_{\phi,ab}(z)$$

$$\frac{\mathrm{d}Q_{ab}(t)}{\mathrm{d}t} = f_{q,ab}(t) \qquad \frac{\mathrm{d}\Phi_{ab}(t)}{\mathrm{d}t} = f_{\phi,ab}(t)$$

$$\int_{Z_{ab}} \frac{\star q(t,z)}{2C(z)} q(t,z)$$

- The approximation of the *electric energy* of the part of the line is

$$H_{q,ab}(Q_{ab}(t)) = \frac{1}{2}\frac{Q_{ab}^2}{C_{ab}} \qquad C_{ab}^{-1} = \int_{Z_{ab}} \star\left(\frac{\omega_{q,ab}(z)}{C(z)}\right)\omega_{q,ab}(z)$$

- Similarly, the *magnetic energy* is approximated by:

$$H_{\phi,ab}(\Phi_{ab}(t)) = \frac{1}{2}\frac{\Phi_{ab}^2}{L_{ab}} \qquad L_{ab}^{-1} = \int_{Z_{ab}} \star\left(\frac{\omega_{\phi,ab}(z)}{L(z)}\right)\omega_{\phi,ab}(z)$$

- *Total energy:*

$$H_{ab}(Q_{ab},\Phi_{ab}) = H_{q,ab}(Q_{ab}) + H_{\phi,ab}(\Phi_{ab})$$

$$e_{q,ab}(t) = \frac{\partial H_{ab}(Q_{ab},\Phi_{ab})}{\partial Q_{ab}}(t) = \frac{Q_{ab}(t)}{C_{ab}} \qquad e_{\phi,ab}(t) = \frac{\partial H_{ab}(Q_{ab},\Phi_{ab})}{\partial \Phi_{ab}}(t) = \frac{\Phi_{ab}(t)}{L_{ab}}$$

Spatial discretisation of distributed pH systems

$$Q = \begin{pmatrix} Q_{S_0S_1} & Q_{S_1S_2} & \cdots & Q_{S_{n-1}S_n} \end{pmatrix}^{\mathrm{T}}$$
$$\Phi = \begin{pmatrix} \Phi_{S_0S_1} & \Phi_{S_1S_2} & \cdots & \Phi_{S_{n-1}S_n} \end{pmatrix}^{\mathrm{T}}$$

☑ Spatial discretisation of the transmission line:

- The transmission line is split into *n parts*
- The *i*-th part $(S_{i-1}; S_i)$ is discretised as explained in the previous two sub-sections, where $a = S_{i-1}$ and $b = S_i$
- Since the interconnection of port-Hamiltonian systems is a port-Hamiltonian system, *the total discretised system is also a port-Hamiltonian system*
- The *total Hamiltonian* is given by the sum of the individual Hamiltonians

$$H(Q,\Phi) = \sum_{i=1}^{n} \frac{Q^2_{S_{i-1}S_i}}{2C_{S_{i-1}S_i}} + \sum_{i=1}^{n} \frac{\Phi^2_{S_{i-1}S_i}}{2L_{S_{i-1}S_i}}$$

- *Energy balance:*

$$\frac{\mathrm{d}H(Q(t),\Phi(t))}{\mathrm{d}t} - e_{\partial,0} f_{\partial,0} + e_{\partial,L} f_{\partial,L} = 0$$

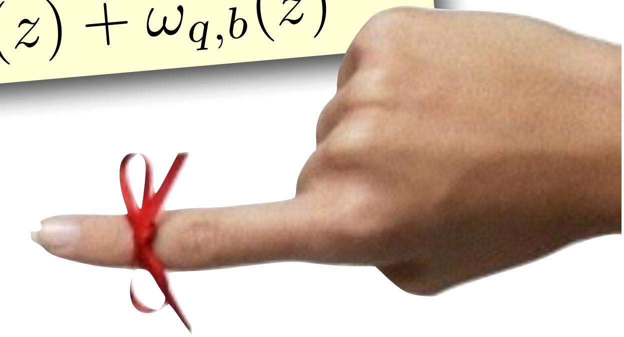
boundary port

$$(f_{\partial,S_0}, e_{\partial,S_0}) = (f_{\partial,0}, e_{\partial,0})$$
$$(f_{\partial,S_n}, e_{\partial,S_n}) = (f_{\partial,L}, e_{\partial,L})$$

Spatial discretisation of distributed pH systems

$$\alpha_{ab} = \int_{Z_{ab}} \omega_{q,a}(z)\omega_{q,ab}(z)$$

$$1 = \omega_{q,a}(z) + \omega_{q,b}(z)$$



☑ How to chose the approximating 1-forms?

$$Q_{ab}(t) = C_{ab}e_{q,ab} = C_{ab}\alpha_{ab}e_{q,a}(t) + C_{ab}(1-\alpha_{ab})e_{q,b}(t)$$

$$Q_{ab}(t) = \int_{Z_{ab}} q(t,z) = \int_{Z_{ab}} C(z)e_q(t,z)$$

$$= \left(\int_{Z_{ab}} C(z)\omega_{q,a}(z)\right)e_{q,a}(t) + \left(\int_{Z_{ab}} C(z)\omega_{q,b}(z)\right)e_{q,b}(t)$$

$$L_{ab}\alpha_{ab} = \int_{Z_{ab}} L(z)\omega_{\phi,a}$$

$$L_{ab}(1-\alpha_{ab}) = \int_{Z_{ab}} L(z)\omega_{\phi,b}$$

Proposition. *These conditions are satisfied if and only if:*

$$(i) \quad \alpha_{ab} = \int_{Z_{ab}} \frac{C(z)}{C_{ab}}\omega_{q,a}(z) \qquad (ii) \quad C_{ab} = \int_{Z_{ab}} C(z)$$

$$(iii) \quad \alpha_{ab} = \int_{Z_{ab}} \frac{L(z)}{L_{ab}}\omega_{\phi,a}(z) \qquad (iv) \quad L_{ab} = \int_{Z_{ab}} L(z)$$

additional physical characteristics of the system are now taken into account

$$\omega_{q,ab} = \frac{C(z)\mathrm{d}z}{\int_{Z_{ab}} C(z)\mathrm{d}z} \qquad \omega_{\phi,ab} = \frac{L(z)\mathrm{d}z}{\int_{Z_{ab}} L(z)\mathrm{d}z}$$

Spatial discretisation of distributed pH systems

- *Two-dimensional case:* the wave equation

$$\mu\ddot{u} + E\Delta u = 0, \qquad u(t,z) \in \mathbb{R},\ z = (z_1, z_2) \in Z$$

- Port-Hamiltonian description:
  - *State (energy) variables:*

$$p(t,z_1,z_2) \in \Omega^2(Z) \qquad \epsilon(t,z_1,z_2) = \frac{\partial u}{\partial z_1}\mathrm{d}z_1 + \frac{\partial u}{\partial z_2}\mathrm{d}z_2 \in \Omega^1(Z)$$

  - *Hamiltonian function:*

$$H(p,\epsilon) = \int_Z \mathcal{H}(p,\epsilon), \qquad \mathcal{H}(p,\epsilon) = \frac{1}{2}\big(p \wedge v + \epsilon \wedge \sigma\big)$$

$$v = \frac{1}{\mu} \star p$$

$$\sigma = E \star \epsilon$$

  - *Port-Hamiltonian formulation:*

$$\begin{pmatrix} \dot{\epsilon} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{d} \\ -\mathrm{d} & 0 \end{pmatrix} \begin{pmatrix} \frac{\delta H}{\delta \epsilon} \\ \frac{\delta H}{\delta p} \end{pmatrix} \qquad \begin{pmatrix} v_\partial \\ \sigma_\partial \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \left.\frac{\delta H}{\delta \epsilon}\right|_{\partial Z} \\ \left.\frac{\delta H}{\delta p}\right|_{\partial Z} \end{pmatrix}$$

Spatial discretisation of distributed pH systems

- Passing from the interval grid for the 1-D case, we move onto the simplest possible grid for the 2-D example, i.e. the *triangular grid*
  - Approximation of *flow variables:*

$$f_p(t,z) = f_{p,abc}(t)\omega_{p,abc}(z)$$

$$f_\epsilon(t,z) = f_{\epsilon,ab}(t)\omega_{\epsilon,ab}(z) + f_{\epsilon,bc}(t)\omega_{\epsilon,bc}(z) + f_{\epsilon,ca}(t)\omega_{\epsilon,ca}(z)$$

  - Approximation of *efforts:*

$$e_v(t,z) = e_{v,a}(t)\omega_{v,a}(z) + e_{v,b}(t)\omega_{v,b}(z) + e_{v,c}(t)\omega_{v,c}(z)$$

$$e_\sigma(t,z) = e_{\sigma,ab}(t)\omega_{\sigma,ab}(z) + e_{\sigma,bc}(t)\omega_{\sigma,bc}(z) + e_{\sigma,ca}(t)\omega_{\sigma,ca}(z)$$

  - *Boundary variables:*

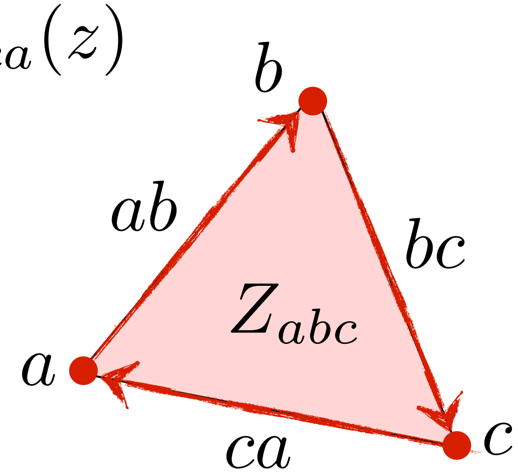
$$f_\partial(t,z) = f_{\partial,a}(t)\omega_{v,a}(z) + f_{\partial,b}(t)\omega_{v,b}(z) + f_{\partial,c}(t)\omega_{v,c}(z)$$

$$e_\partial(t,z) = e_{\partial,ab}(t)\omega_{\sigma,ab}(z) + e_{\partial,bc}(t)\omega_{\sigma,bc}(z) + e_{\partial,ca}(t)\omega_{\sigma,ca}(z)$$

$$\int_{Z_{abc}} \omega_{p,abc} = 1$$

$$\int_{l'} \omega_{\epsilon/\sigma,l} = \begin{cases} 0 & \text{if } l \neq l' \\ 1 & \text{if } l = l' \end{cases} \qquad l,\, l' \in \{ab, bc, ca\}$$

$$\omega_{v,x}(y) = \begin{cases} 0 & \text{if } x \neq y \\ 1 & \text{if } x = y \end{cases} \qquad x,\, y \in \{a, b, c\}$$



Spatial discretisation of distributed pH systems

- Similarly to the 1D case and after some time, you obtain the following relation, that defines a *Dirac structure:*

$$
\overbrace{\begin{pmatrix}
\alpha_{a,abc} & 0 & 0 & 0 & 0 & 0 \\
\alpha_{b,abc} & 0 & 0 & 0 & 0 & 0 \\
\alpha_{c,abc} & 0 & 0 & 0 & 0 & 0 \\
0 & \alpha_{b,abc} - \beta_{b,ab} & \beta_{c,ab} - \alpha_{c,abc} & 1 & 0 & 0 \\
0 & \alpha_{b,abc} - \beta_{b,bc} & \beta_{c,bc} - \alpha_{c,abc} & 0 & 1 & 0 \\
0 & \alpha_{b,abc} - \beta_{b,ca} & \beta_{c,ca} - \alpha_{c,abc} & 0 & 0 & 1
\end{pmatrix}}^{F_{abc}}
\overbrace{\begin{pmatrix}
f_{p,abc} \\ f_{\epsilon,ab} \\ f_{\epsilon,ca} \\ f_{\partial,ab} \\ f_{\partial,bc} \\ f_{\partial,ca}
\end{pmatrix}}^{f_{abc}}
+
$$

$$
+ \underbrace{\begin{pmatrix}
0 & -1 & 1 & \beta_{a,ab} & \beta_{a,bc} & \beta_{a,ca} \\
0 & 1 & 0 & \beta_{b,ab} & \beta_{b,bc} & \beta_{b,ca} \\
0 & 0 & -1 & \beta_{c,ab} & \beta_{c,bc} & \beta_{c,ca} \\
-1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}}_{E_{abc}}
\underbrace{\begin{pmatrix}
e_{p,abc} \\ e_{\epsilon,ab} \\ e_{\epsilon,ca} \\ e_{\partial,ab} \\ e_{\partial,bc} \\ e_{\partial,ca}
\end{pmatrix}}_{e_{abc}}
= 0
$$

Spatial discretisation of distributed pH systems

$$p(t,z) = p_{abc}(t)\omega_{p,abc}(z)$$
$$\epsilon(t,z) = \epsilon_{ab}(t)\omega_{\epsilon,ab}(z) + \epsilon_{bc}(t)\omega_{\epsilon,bc}(z) + \epsilon_{ca}(t)\omega_{\epsilon,ca}(z)$$

☑ Approximation of the constitutive relations:

- *Kinetic energy:*

$$H_{p,abc}(p) = \frac{1}{2}\int_{Z_{abc}} \star\frac{p(t,z)}{\mu(z_1,z_2)} \wedge p(t,z)$$
$$= \frac{p^2_{abc}(t)}{2} \underbrace{\int_{Z_{abc}} \frac{\star\omega_{p,abc}(z) \wedge \omega_{p,abc}(z)}{\mu(z)}}_{M^{-1}} = \frac{p^2_{abc}(t)}{2M}$$

- *Potential elastic energy:*

$$H_{\epsilon,abc}(\epsilon) = \frac{1}{2}\int_{Z_{abc}} \frac{\star\epsilon}{Y(z)} \wedge \epsilon = \frac{1}{2}\left[\frac{\epsilon^2_{ab}}{Y_1} + \frac{\epsilon^2_{ca}}{Y_2} - \frac{2\epsilon_{ab}\epsilon_{ca}}{Y_3}\right]$$

$$\epsilon(t,z) = \epsilon_{ab}(t)\mathrm{d}\omega_{v,b}(z) - \epsilon_{ca}\mathrm{d}\omega_{v,c}(z) + \left[\epsilon_{ab}(t) + \epsilon_{bc}(t) + \epsilon_{ca}(t)\right]\omega_{\epsilon,bc}$$

$$Y_1^{-1} = \int_{Z_{abc}} \frac{\star\mathrm{d}\omega_{v,b}(z) \wedge \mathrm{d}\omega_{v,b}(z)}{Y}$$
$$Y_2^{-1} = \int_{Z_{abc}} \frac{\star\mathrm{d}\omega_{v,c}(z) \wedge \mathrm{d}\omega_{v,c}(z)}{Y}$$
$$Y_3^{-1} = \int_{Z_{abc}} \frac{\star\mathrm{d}\omega_{v,b}(z) \wedge \mathrm{d}\omega_{v,c}(z) + \star\mathrm{d}\omega_{v,c}(z) \wedge \mathrm{d}\omega_{v,b}(z)}{2Y}$$

The port-behaviour is the same!!!

Spatial discretisation of distributed pH systems

☑ *A nonlinear flexible link*

Dirac structure

$$-\begin{pmatrix} f_q \\ f_p \end{pmatrix} = \boxed{\left\{ \begin{pmatrix} 0 & \mathrm{d} \\ \mathrm{d} & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathrm{ad}_{(q+\hat{n})} \\ -\mathrm{ad}^*_{(q+\hat{n})} & 0 \end{pmatrix} \right\}} \begin{pmatrix} e_q \\ e_p \end{pmatrix}, \qquad \begin{pmatrix} f_\partial \\ e_\partial \end{pmatrix} = \begin{pmatrix} e_p\,|\partial Z \\ e_q\,|\partial Z \end{pmatrix}$$

total energy

$$H(p,q) = \frac{1}{2}\int_Z \star\Big(\langle \star p \mid \star p \rangle_Y + \langle \star q \mid \star q \rangle_{C^{-1}} \Big)$$

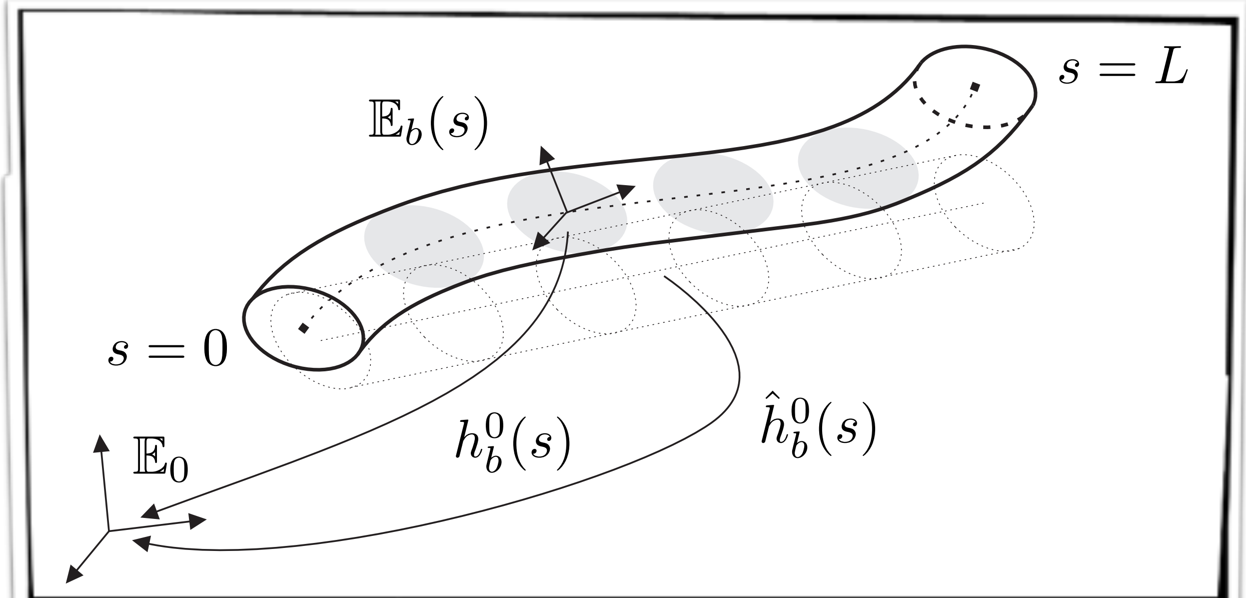
$$-f_q = \frac{\partial q}{\partial t}\ (= t_\delta^b) \qquad -f_p = \frac{\partial p}{\partial t} - p \wedge t_b^{b,0}\ (= w_\delta^b)$$

$$e_q = \frac{\delta H}{\delta q}\left(= w_b^{b,0}\right) \qquad e_p = \frac{\delta H}{\delta p}\left(= t_b^{b,0}\right)$$

port-behaviour

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \langle e_\partial(L), f_\partial(L)\rangle - \langle e_\partial(0), f_\partial(0)\rangle$$

energy balance

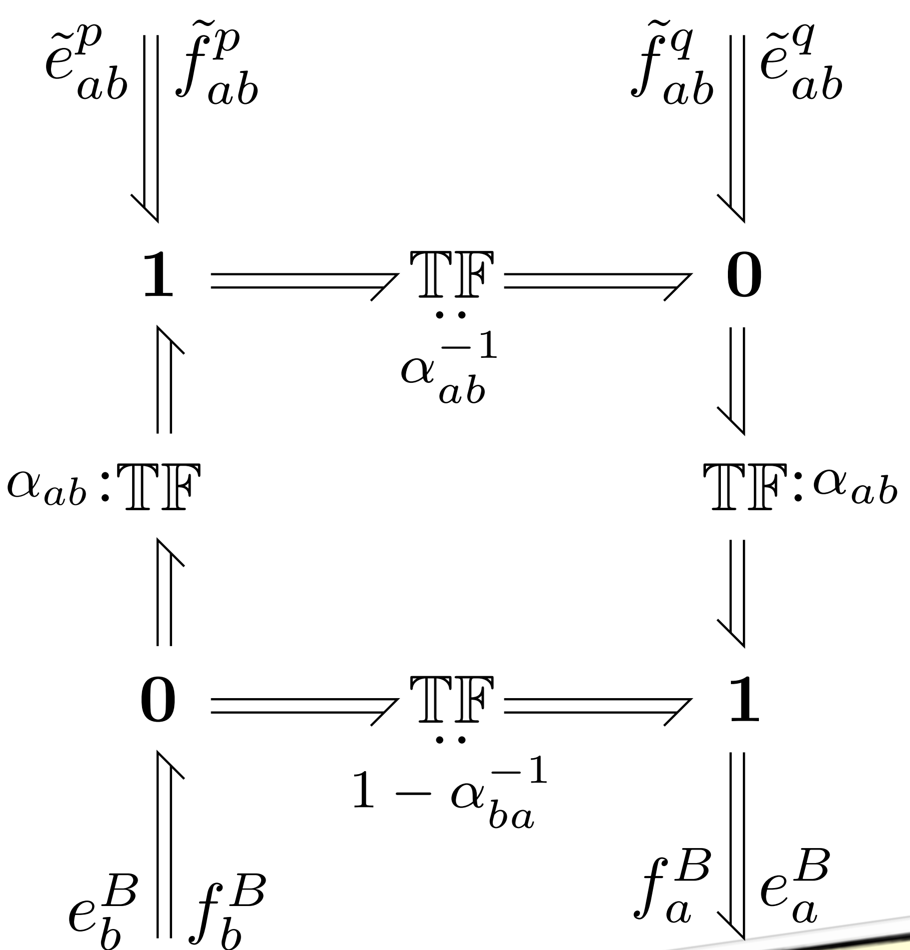


Spatial discretisation of distributed pH systems

- The *algebraic coupling* is easily tackled by defining

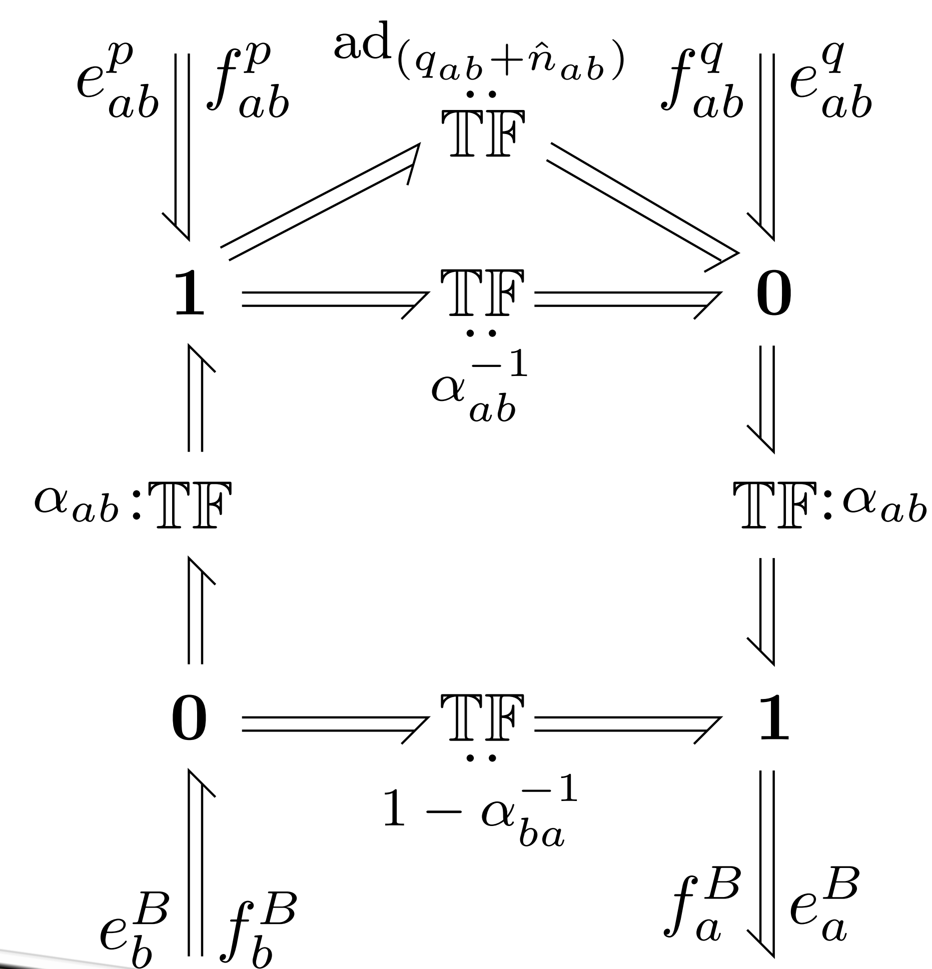
$$\tilde{f}_q = f_q + \mathrm{ad}_{q+\hat{n}} e_p \qquad \tilde{f}_p = f_p - \mathrm{ad}^*_{q+\hat{n}} e_q$$

- In this way, the *Stokes-Dirac structure of the transmission line* appears and the *boundary conditions* are not changed



transmission line

$$\tilde{f}_{q,ab} = f_{q,ab} + \mathrm{ad}_{q_{ab}+\hat{n}_{ab}} e_{p,ab}$$
$$\tilde{f}_{p,ab} = f_{p,ab} - \mathrm{ad}^*_{q_{ab}+\hat{n}_{ab}} e_{q,ab}$$



flexible link

Spatial discretisation of distributed pH systems

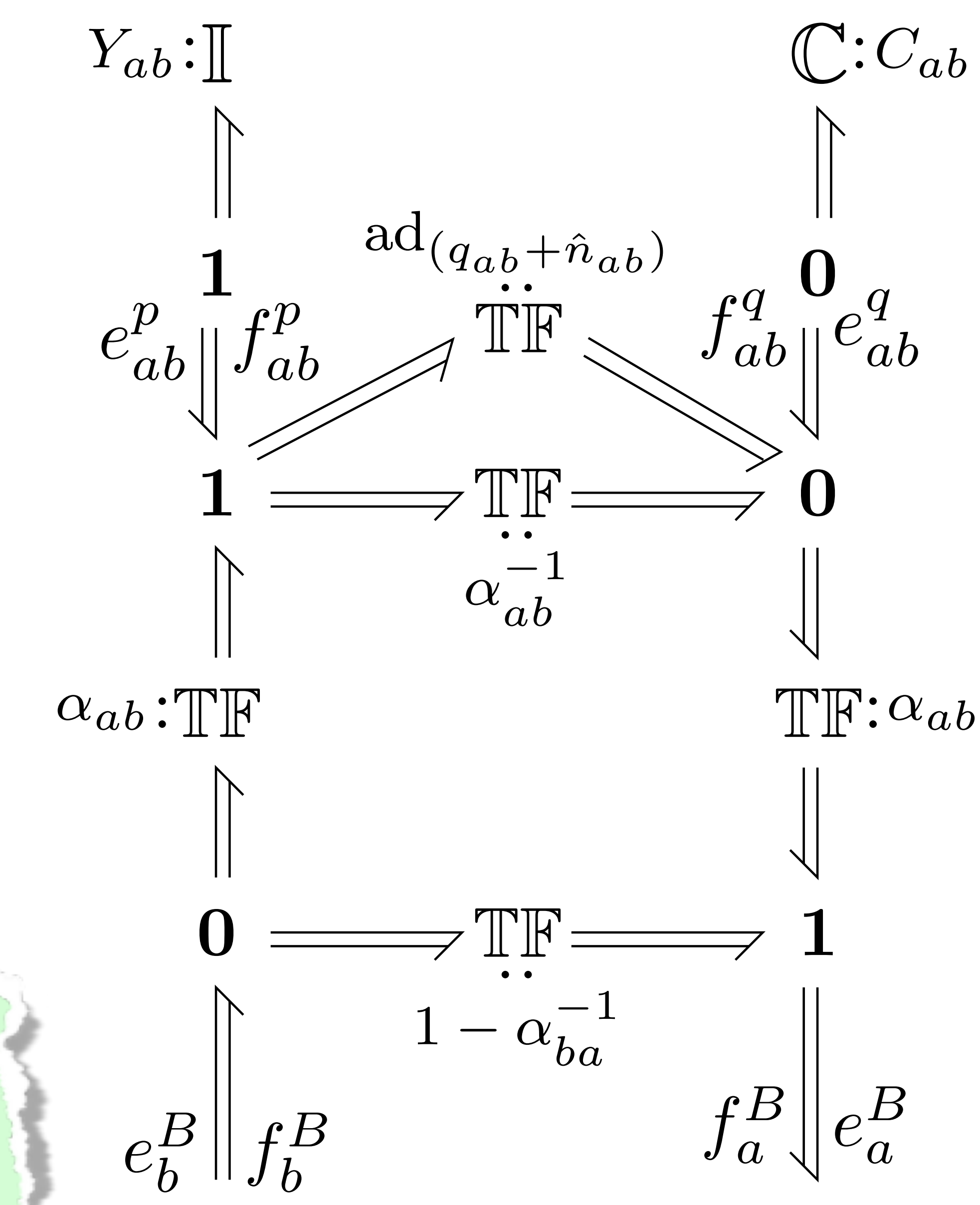
☑ The *constitutive equations* are treated in the usual way

- *Kinetic energy:*

$$\mathcal{K}(p_{ab}) = \frac{1}{2}\langle p_{ab} \mid p_{ab}\rangle_Y \int_{Z_{ab}} \star\omega_{p,ab}(z)\omega_{p,ab}(z) = \frac{1}{2}\langle p_{ab} \mid p_{ab}\rangle_{Y_{ab}}$$

- *Potential elastic energy:*

$$\mathcal{U}(q_{ab}) = \frac{1}{2}\langle q_{ab} \mid q_{ab}\rangle_{C^{-1}} \int_{Z_{ab}} \star\omega_{q,ab}(z)\omega_{q,ab}(z) = \frac{1}{2}\langle q_{ab} \mid q_{ab}\rangle_{C_{ab}^{-1}}$$



the complete model follows again via power conserving interconnection of such "simpler" elements

Spatial discretisation of distributed pH systems

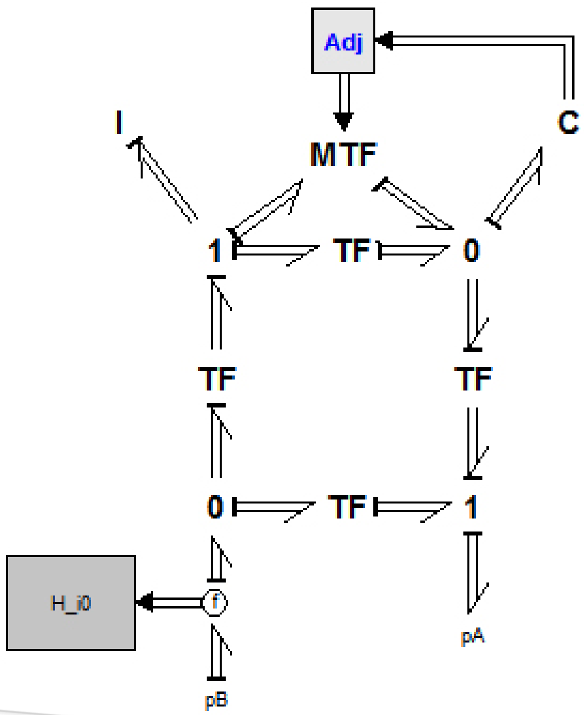
- How the discretisation procedure works? Is it accurate and does it assure a certain convergence as far as the number N of finite elements increases?
  - *The model is non-linear,* so it is not possible to perform a spectral analysis nor compute an exact solution
  - ...but for "small deformation" it is equivalent to the *Timoshenko beam*

M	**Eigenfrequencies** $\omega_1, \ldots, \omega_6$ [rad/s]					
6	8.4085	50.8229	141.7407	293.6579	532.5828	579.5340
9	8.3693	48.2608	124.5600	228.5422	365.4776	529.2416
12	8.3556	47.4113	119.3055	210.9628	321.2388	449.7060
15	8.3492	47.0260	116.9909	203.5459	303.5649	415.1332
18	8.3457	46.8187	115.7633	199.6910	294.6171	397.8813
21	8.3436	46.6945	115.0329	197.4233	289.4292	387.9998
24	8.3423	46.6141	114.5628	195.9736	286.1422	381.7907
∞ (exact)	8.3378	46.3530	113.0483	191.3562	275.8244	362.5837

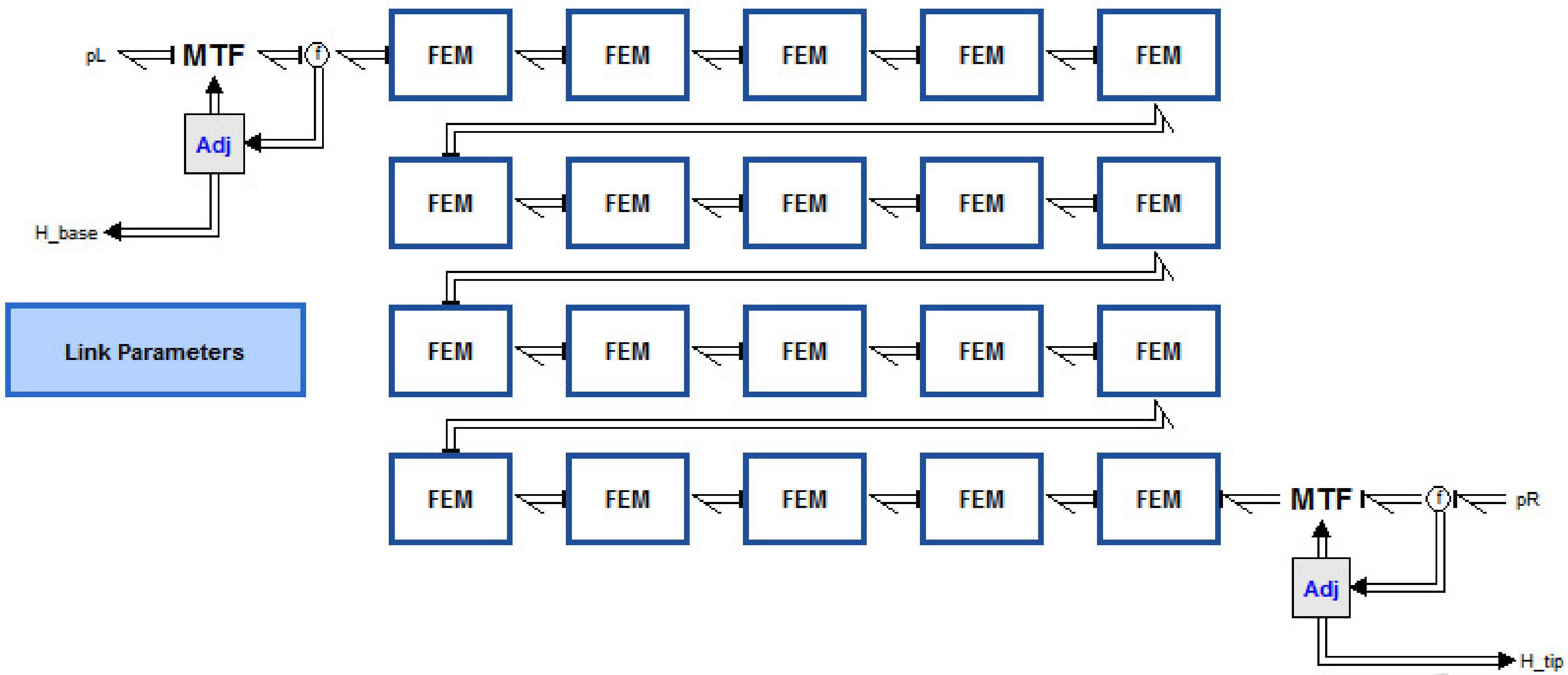
eigenfrequencies in the free-clamped case

Spatial discretisation of distributed pH systems

☑ *Simulation:* flexible link



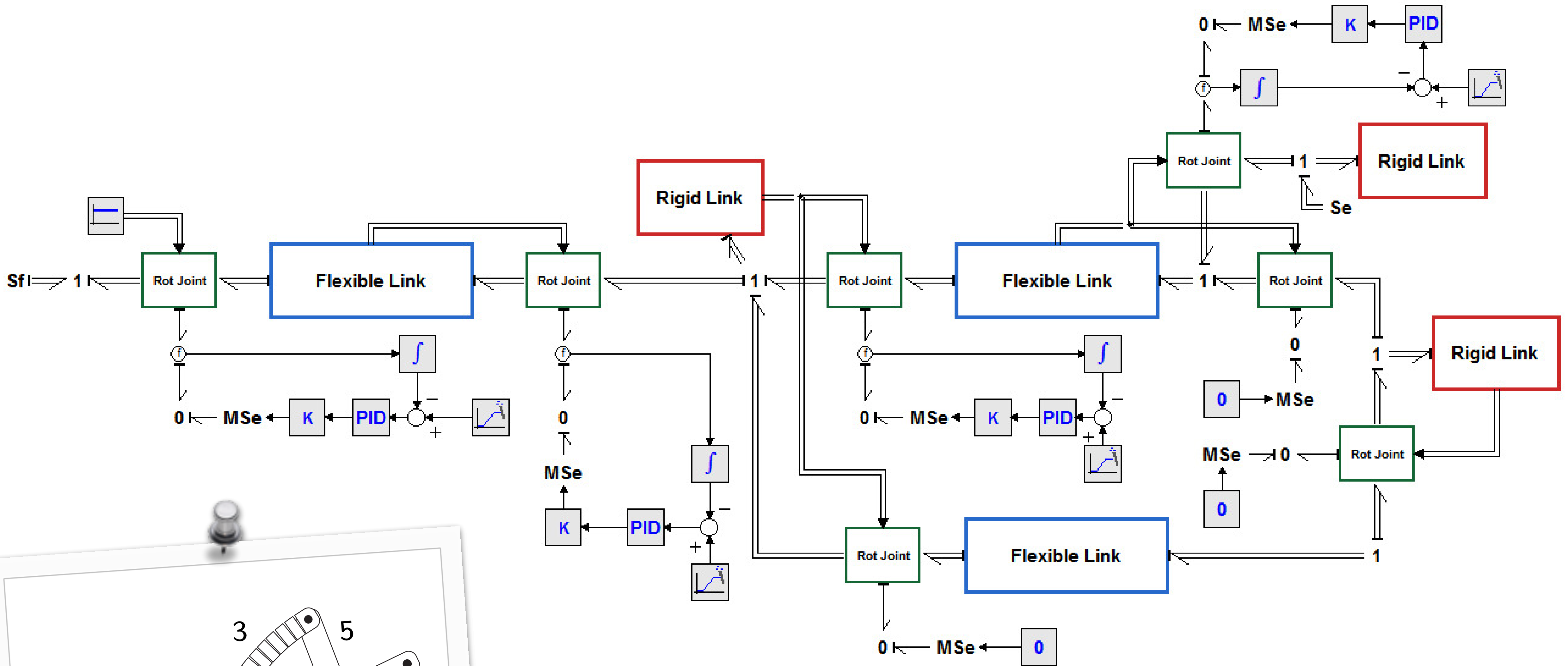
Finite element dynamics

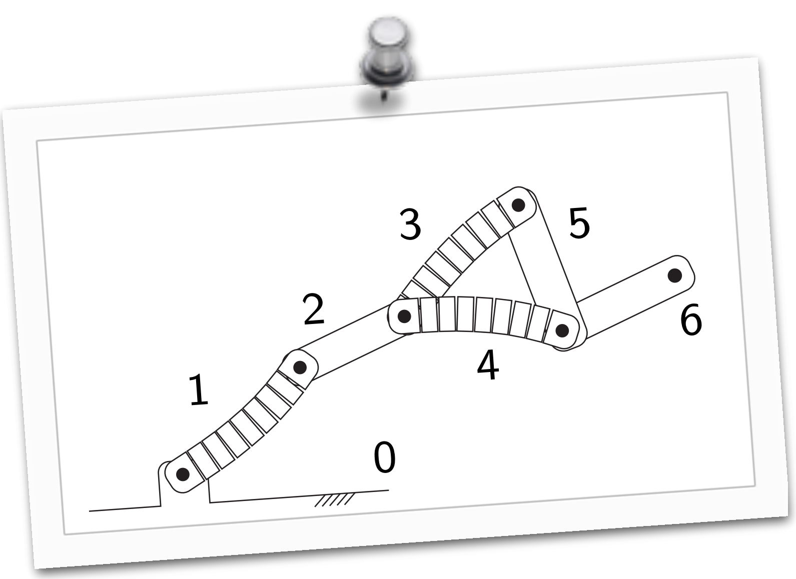


Finite element approximation

Spatial discretisation of distributed pH systems

☑ *Simulation:* complete system





Basic Results on Model Reduction for Port-Hamiltonian Systems

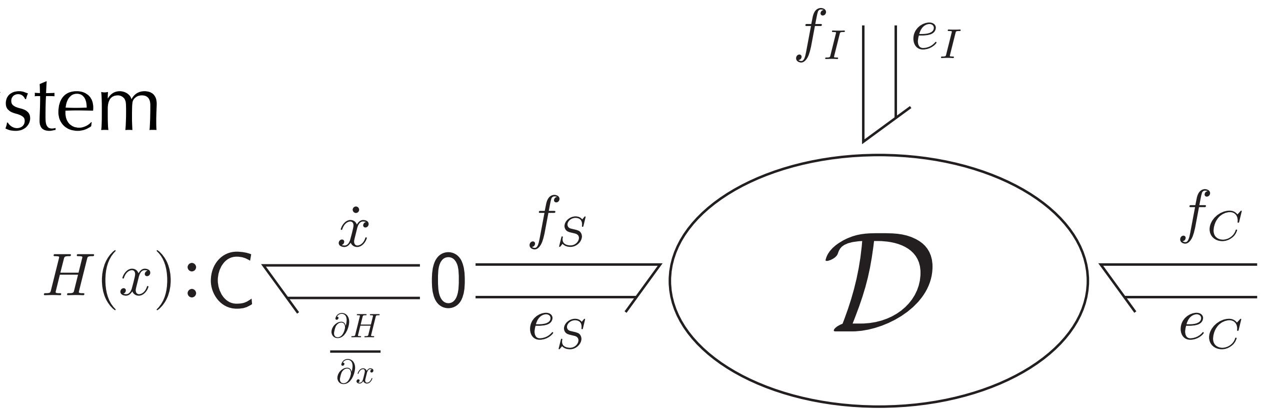
Model reduction of port-Hamiltonian systems

- The idea is to illustrate a novel procedure for the *model reduction of high-order port-Hamiltonian systems*
  - The method can be applied to port-Hamiltonian systems *not necessarily* in input-state-output form

- A typical application is an high-order systems obtained from the spatial discretisation of distributed port-Hamiltonian systems
  - The approximating system turns out to be completely *a-causal* and able to deal with *time varying boundary conditions*
  - "A-causality" implies that (boundary) inputs and outputs are determined by the causality of the interconnected subsystems
  - The plant dynamics are given as *a set of DAEs*
- The model-reduction technique is able to deal with such systems *without loosing the port-Hamiltonian structure* and related properties

Model reduction of port-Hamiltonian systems

- Let's start with a port-Hamiltonian system
  - With *constant* Dirac structure;
  - *Quadratic* Hamiltonian function



$$\mathcal{D} = \Big\{ (f_S, f_C, f_I, e_S, e_C, e_I) \in \mathcal{F} \times \mathcal{E} \ \Big| \ \Big| \ F_S f_S + F_C f_C + F_I f_I + + E_S e_S + E_C e_C + E_I e_I = 0 \Big\}$$

$$H(x) = \frac{1}{2} x^{\mathrm{T}} L x$$

- Based on "some" criteria, a reduced order model can be deduced by imposing a set of *(constant) constraints* at the energy-storage port:

$$A_f^{\mathrm{T}} f_S = 0 \qquad\qquad A_e^{\mathrm{T}} e_S = 0$$

- The reduced order model is obtained once the constraints have been *explicitly removed* from the system dynamics

$$S = \mathrm{Ker}\, A^{\mathrm{T}}$$

$$\mathrm{A} = \begin{pmatrix} \mathrm{A}_e & \mathrm{A}_f \end{pmatrix}$$

$$S^{\mathrm{T}} S = I_r$$

$$A_e^{\mathrm{T}} A_e = I$$
$$A_f^{\mathrm{T}} A_f = I$$
$$A_e^{\mathrm{T}} A_f = 0$$

Model reduction of port-Hamiltonian systems

- The reduction step is performed by using S as *projection matrix* on the Dirac structure and Hamiltonian function of the full-order system

$$\tilde{f}_S = \begin{pmatrix} \tilde{f}_{S,e} \\ \tilde{f}_{S,f} \\ \tilde{f}_{S,r} \end{pmatrix} = \begin{pmatrix} A_e^{\mathrm{T}} \\ A_f^{\mathrm{T}} \\ S^{\mathrm{T}} \end{pmatrix} f_S$$

$$\tilde{e}_S = \begin{pmatrix} \tilde{e}_{S,e} \\ \tilde{e}_{S,f} \\ \tilde{e}_{S,r} \end{pmatrix} = \begin{pmatrix} A_e^{\mathrm{T}} \\ A_f^{\mathrm{T}} \\ S^{\mathrm{T}} \end{pmatrix} e_S$$

coordinate change

$$T = \begin{pmatrix} A_e & A_f & S \end{pmatrix}$$

- In these new coordinates the constraints can be expressed as

$$\tilde{f}_{S,f} = 0 \qquad\qquad \tilde{e}_{S,e} = 0$$

- These conditions *fix the causality* on the corresponding power ports
  - Since these port variables do not play any role in the energy balance, *they can be removed* to obtain the reduced order Dirac structure

Model reduction of port-Hamiltonian systems

- Simple computations show that the *reduced Dirac structure* is

$$F_{S,r}\tilde{f}_{S,r} + F_{C,r}f_C + F_{I,r}f_I + \\ + E_{S,r}\tilde{e}_{S,r} + E_{C,r}e_C + E_{I,r}e_I = 0$$

with

$$F_{S,r} = G^{\perp}F_S S \quad F_{C,r} = G^{\perp}F_C \quad F_{I,r} = G^{\perp}F_I$$
$$E_{S,r} = G^{\perp}E_S S \quad E_{C,r} = G^{\perp}E_C \quad E_{I,r} = G^{\perp}E_I$$

$G = \begin{pmatrix} F_S A_e & E_S A_f \end{pmatrix}$

- The coordinate change on the storage flows induces a similar transformation on the energy variables and on the Hamiltonian:

$$\tilde{x} = \begin{pmatrix} \tilde{x}_e \\ \tilde{x}_f \\ \tilde{x}_r \end{pmatrix} = \begin{pmatrix} A_e^{\mathrm{T}} \\ A_f^{\mathrm{T}} \\ S^{\mathrm{T}} \end{pmatrix} x = T^{-1}x \qquad \tilde{H}(\tilde{x}) = H(\tilde{x}) = \frac{1}{2}\tilde{x}^{\mathrm{T}}\tilde{L}\tilde{x}$$

$$\tilde{x}_f = \kappa \qquad \frac{\partial \tilde{H}}{\partial \tilde{x}_e} = 0$$

Model reduction of port-Hamiltonian systems

- The condition on the *gradient of the Hamiltonian* implies that

$$S_e = \begin{pmatrix} A_f & S \end{pmatrix}$$

$$\bar{x} = \begin{pmatrix} \tilde{x}_f \\ \tilde{x}_r \end{pmatrix}$$

$$\frac{\partial \tilde{H}}{\partial \tilde{x}_e} = A_e^{\mathrm{T}} L A_e \tilde{x}_e + A_e^{\mathrm{T}} L S_e \bar{x} = 0$$

$$\tilde{H}(\tilde{x}_f, \tilde{x}_r) = \frac{1}{2} \begin{pmatrix} \tilde{x}_f^{\mathrm{T}} & \tilde{x}_r^{\mathrm{T}} \end{pmatrix} \tilde{L}' \begin{pmatrix} \tilde{x}_f \\ \tilde{x}_r \end{pmatrix}$$

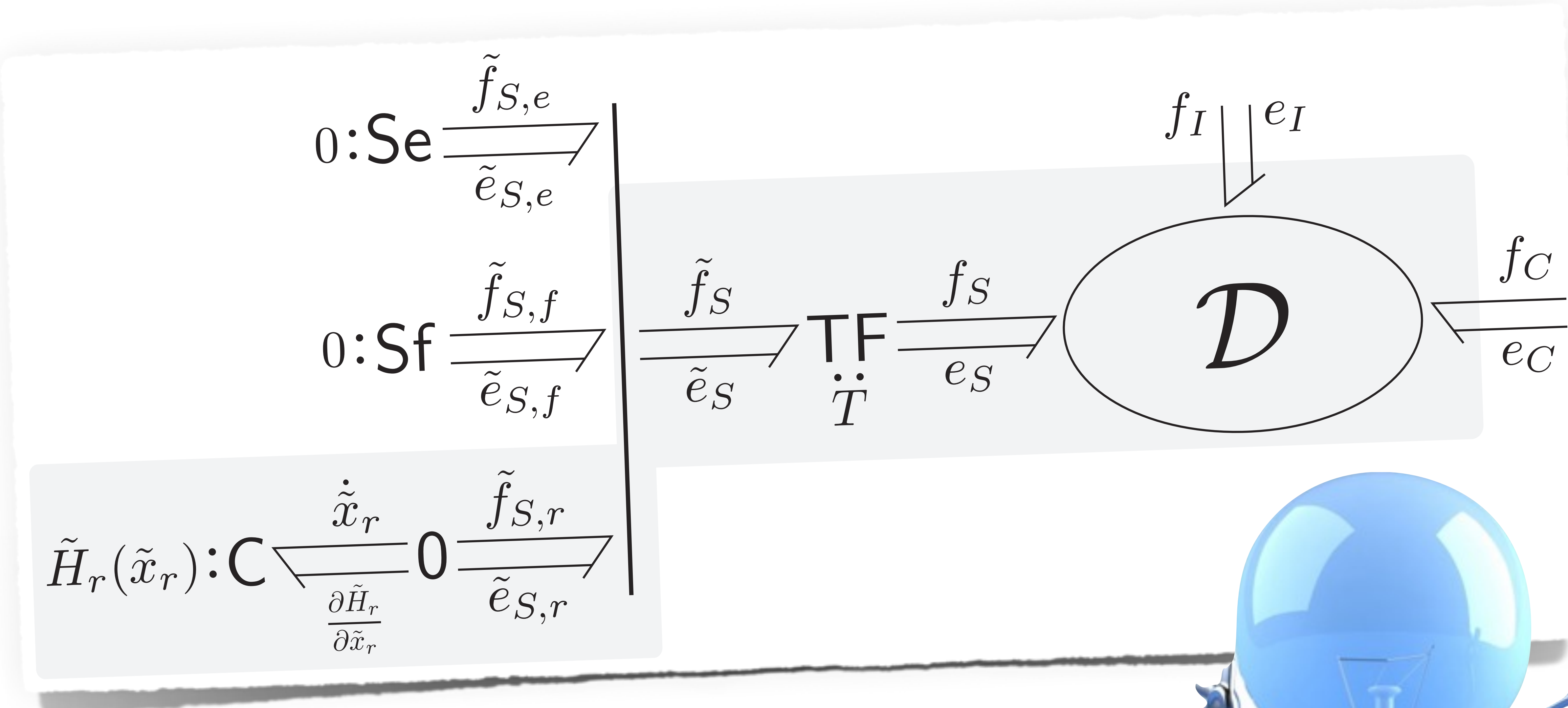
$$\tilde{L}' = \left(S_e^{\mathrm{T}} L^{-1} S_e\right)^{-1} = \begin{pmatrix} \tilde{L}_{ff} & \tilde{L}_{fr} \\ \tilde{L}_{fr}^{\mathrm{T}} & \tilde{L}_{rr} \end{pmatrix}$$

- The constraint on the *flow* is easily handled and leads to

$$\tilde{H}_r(\tilde{x}_r) = \frac{1}{2} \tilde{x}_r^{\mathrm{T}} \tilde{L}_{rr} \tilde{x}_r$$

$$\tilde{f}_{S,r} = -\dot{\tilde{x}}_r \qquad \tilde{e}_{S,r} = \frac{\partial \tilde{H}_r}{\partial \tilde{x}_r} = \tilde{L}_{rr} \tilde{x}_r$$

Model reduction of port-Hamiltonian systems

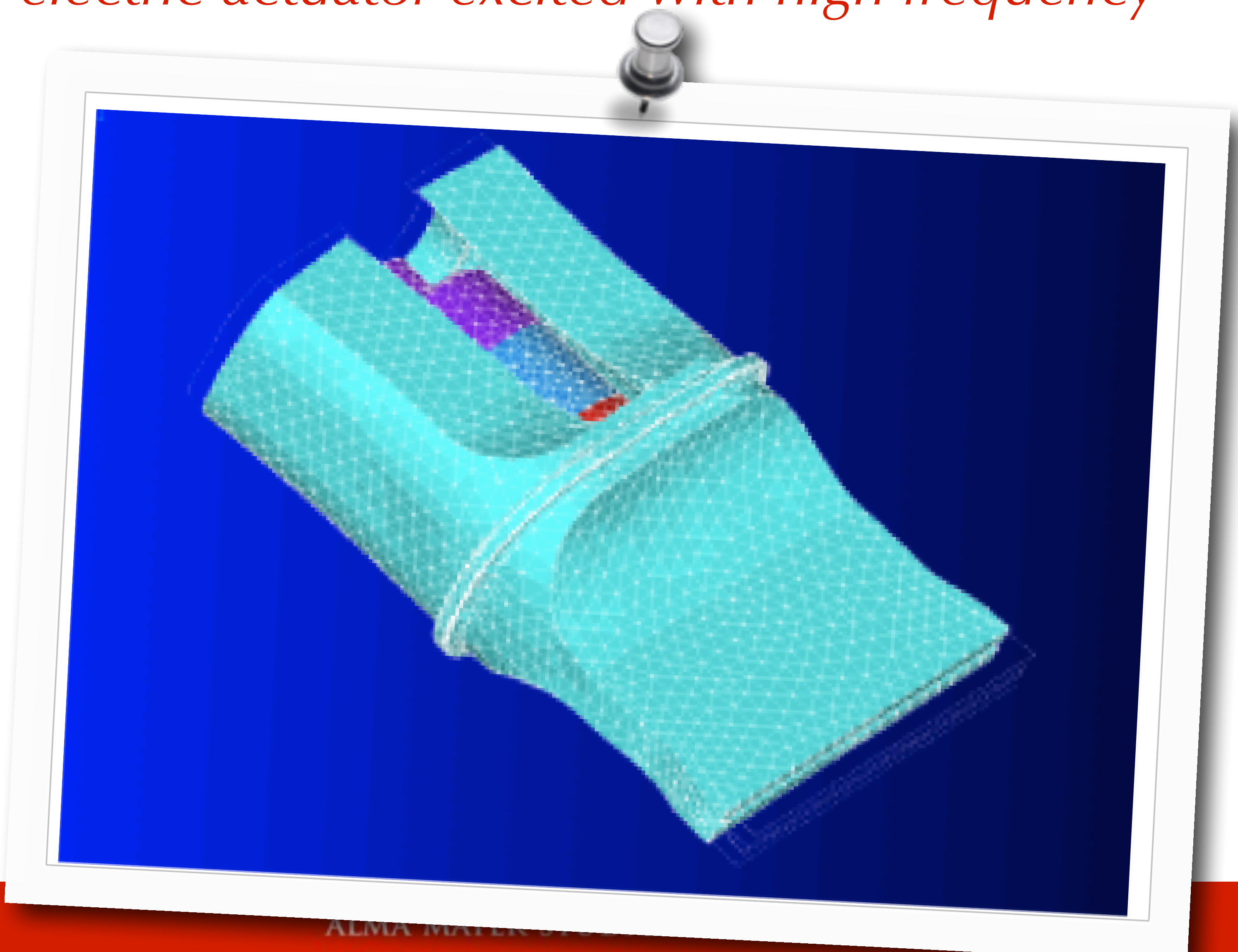


Model reduction of port-Hamiltonian systems

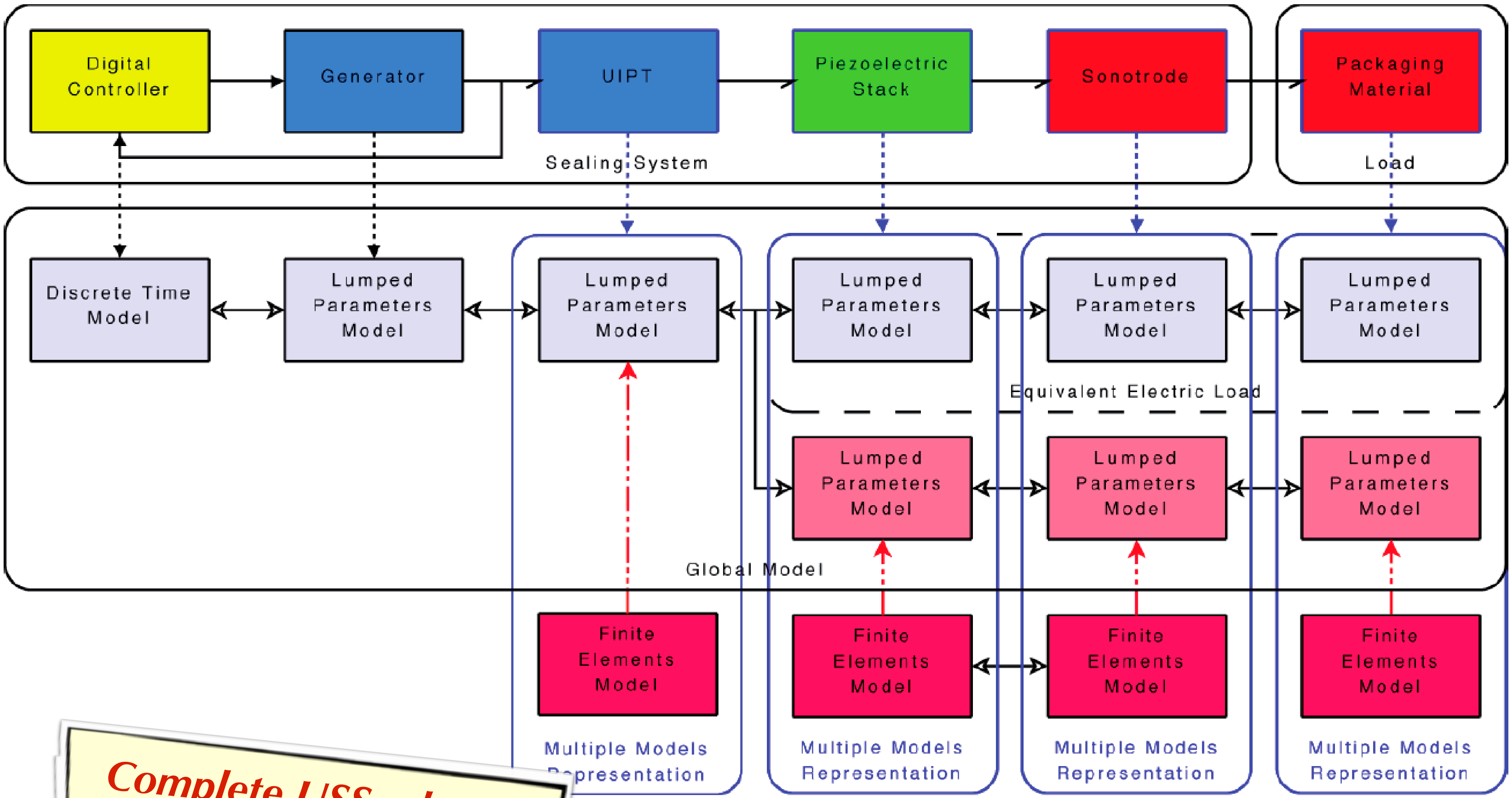
- *Application to a piezo-active structure:* the activity aimed at developing a model reduction procedure able to preserve the frequency behaviour of the original system in a neighbourhood of a predefined set of frequencies of interest
  - This research is carried out in collaboration with TetraPak and deals with modelling and simulation of the *Ultrasonic Sealing System (USS)*
- Due to the presence of a *Compact Transducer (CT),* it is not possible to perform a simulation of the complete system to
  - Test the validity of the controller;
  - *Perform the diagnosis of the sealing process in detail*
- The proposed procedure drastically reduces the simulation time without loosing the essential dynamical information

Model reduction of port-Hamiltonian systems

- The Ultrasonic Sealing System (USS) is a key technology used in the *sealing process* and based on *ultrasounds* for multi-layered packaging materials
  - The most complex part of the USS is the a Compact Transducer (CT) which is responsible of the sealing process
  - *Physically, it is a piezo-electric actuator excited with high frequency sinusoidal inputs*



Model reduction of port-Hamiltonian systems



Complete USS scheme

Model reduction of port-Hamiltonian systems

- The *f.e.m.* of a mechanical system embedding piezo-electric actuators and sensors is

$$\begin{pmatrix} M_{ww} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \ddot{w} \\ \ddot{\phi} \end{pmatrix} + \begin{pmatrix} K_{ww} & K_{\phi w}^{\mathrm{T}} \\ K_{\phi w} & K_{\phi\phi} \end{pmatrix} \begin{pmatrix} w \\ \phi \end{pmatrix} = \begin{pmatrix} F \\ G \end{pmatrix}$$

- The *inputs* of the system are *forces F* and *voltages* ϕ_C, which represents the effective input voltage over the electric d.o.f.

$$\phi = B_C \phi_C + B_F \phi_F$$

- Since there is *no electric charge* imposed on the free electrical d.o.f., it is possible to "eliminate" ϕ_F

$$B_F^{\mathrm{T}} G = 0$$

"mechanical" system

$$H(p,q) = \frac{1}{2} p^{\mathrm{T}} M^{-1} p + \underbrace{\frac{1}{2} q^{\mathrm{T}} K q}_{= V(q)}$$

$$u = \begin{pmatrix} F & \phi_C \end{pmatrix}^{\mathrm{T}}$$

$$K = K_{ww} - K_{\phi w}^{\mathrm{T}} B_F \left(B_F^{\mathrm{T}} K_{\phi\phi} B_F\right)^{-1} B_F^{\mathrm{T}} K_{\phi w}$$

Model reduction of port-Hamiltonian systems

- Due to the particular operative conditions of this device, the reduced order model preserves the behaviour of the original system in a *neighbourhood of a predefined set of frequencies* of interest
  - Solve an *eigenvalue/eigenvector problem* for *k* eigenvalues in a neighbourhood of the specified set

$$\bar{S} = \begin{pmatrix} \bar{S}_1 & \cdots & \bar{S}_k \end{pmatrix} \Rightarrow \bar{A} = \operatorname{Ker} \bar{S}^{\mathrm{T}} \Rightarrow \bar{A}^{\mathrm{T}} \dot{q} = 0$$

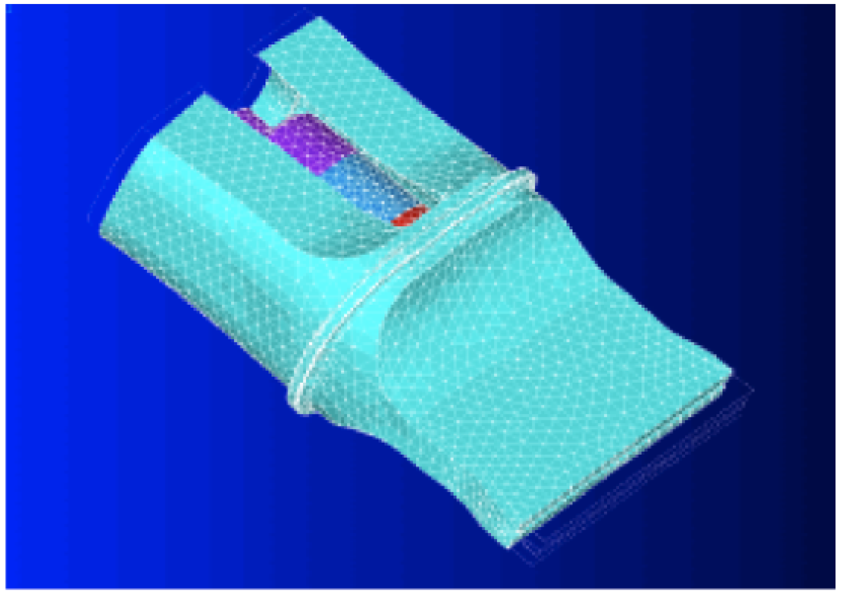
- The result is a set of independent constraints on *both the flows and the efforts* of the full-order dynamics

$$A_f = \begin{pmatrix} \bar{A} \\ 0 \end{pmatrix} \qquad A_e = \begin{pmatrix} 0 \\ \bar{A} \end{pmatrix}$$

- Simple computations show that

$$\tilde{L}_{rr} = \begin{pmatrix} K_r & 0 \\ 0 & M_r^{-1} \end{pmatrix} = \begin{pmatrix} \bar{S}^{\mathrm{T}} K \bar{S} & 0 \\ 0 & (\bar{S}^{\mathrm{T}} M \bar{S})^{-1} \end{pmatrix}$$

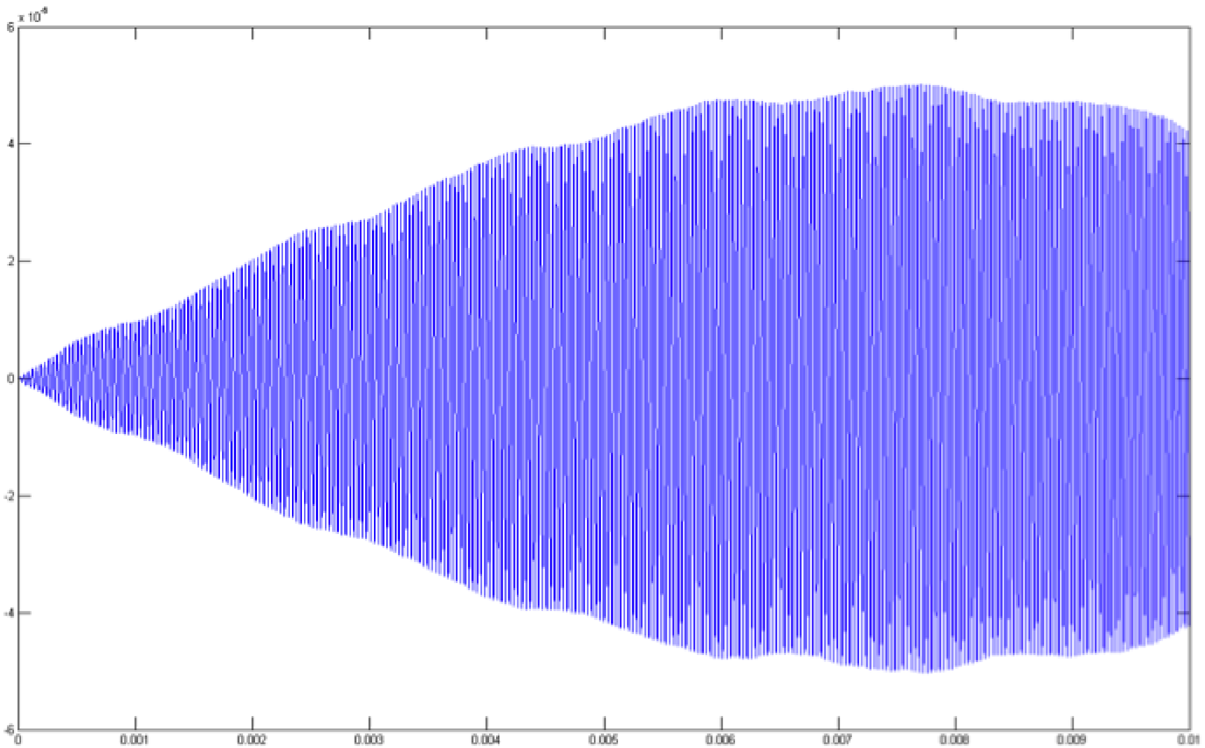
Model reduction of port-Hamiltonian systems



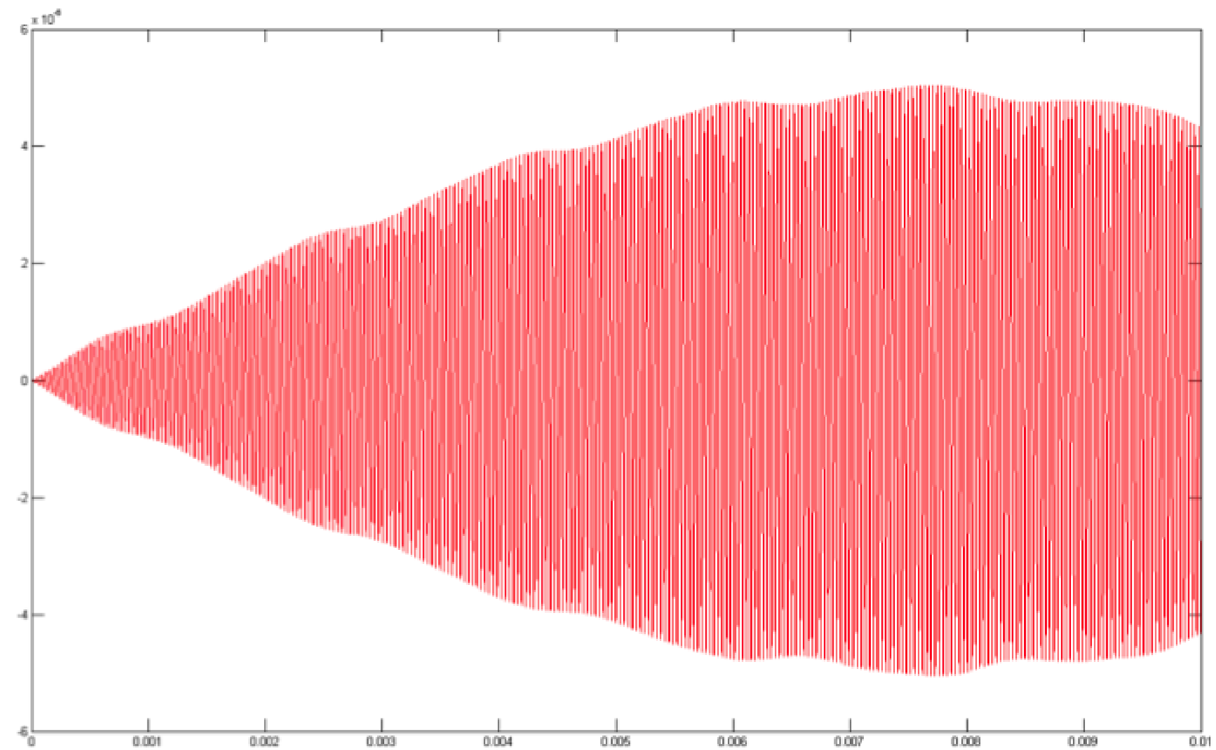
- *Abaqus FEM CAD* software has been used
- A sufficiently fine mesh have been designed:
  - 128322 nodes with **389286 d.o.f.**
- Reduction platform:
  - Core i7 940 (4 core + 4ht core) 2.93 Mhz
  - 12 Gb DDR3 Ram
  - Linux ubuntu 9.04 64bit
  - Matlab 2009a
- Reduction algorithm runs in 17 min
- Abaqus FEM CAD is able to simulate *0.5 ms in 24 hours*
  - Max step size 10^{-7} sec;
  - Complete sealing phase 150 ms; **1 year to perform a simulation!**
- Reduced order model considering 10 eigenvalues around 28 KHz is able to simulate **150 ms in 30 sec**

Simulation results

- Sinusoidal input: 100 V @28640 Hz, 0÷10 ms
- Head displacement



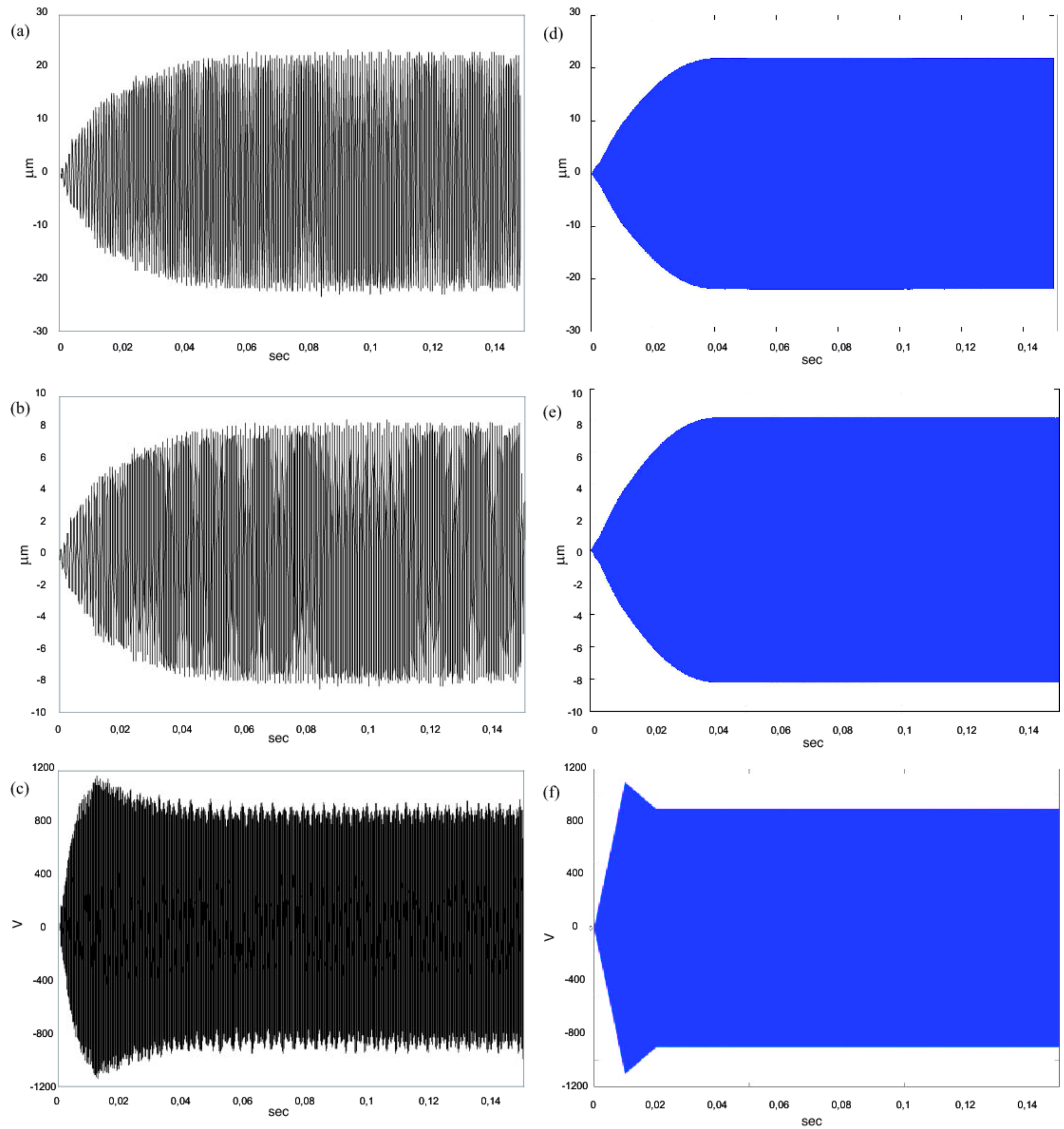
full order (Abaqus)



reduced order (Matlab)

Model reduction of port-Hamiltonian systems

Experimental results



Measured and simulated displacement during the sealing process of the center of the head

Measured and simulated displacement during the sealing process of the top of the screw the piezo-electric actuator

Measured and simulated input voltage

Part III

Control Synthesis for Implicit Port-Hamiltonian Systems

Dirac structures

- The idea is to develop a *general theory* for the energy-based control of implicit port-Hamiltonian systems, i.e. written as a set of *DAEs*
- Denote by $\mathcal{F} \times \mathcal{E}$ the space of *power variables,* and by $\langle e, f \rangle$ the *power* associated to the port $(f, e) \in \mathcal{F} \times \mathcal{E}$
- **Definition.** A (constant) Dirac structure on $\mathcal{F}$ is a *linear subspace* $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that

$$\dim \mathcal{D} = \dim \mathcal{F} \qquad \langle e, f \rangle = 0, \ \forall (f, e) \in \mathcal{D}$$

- Coordinate representations:

image $$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid f = E^{\mathrm{T}} \lambda, \ e = F^{\mathrm{T}} \lambda, \ \lambda \in \mathbb{R}^n \right\}$$

$$EF^{\mathrm{T}} + FE^{\mathrm{T}} = 0$$
$$\operatorname{rank} (F \mid E) = n$$

kernel $$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid Ff + Ee = 0 \right\}$$

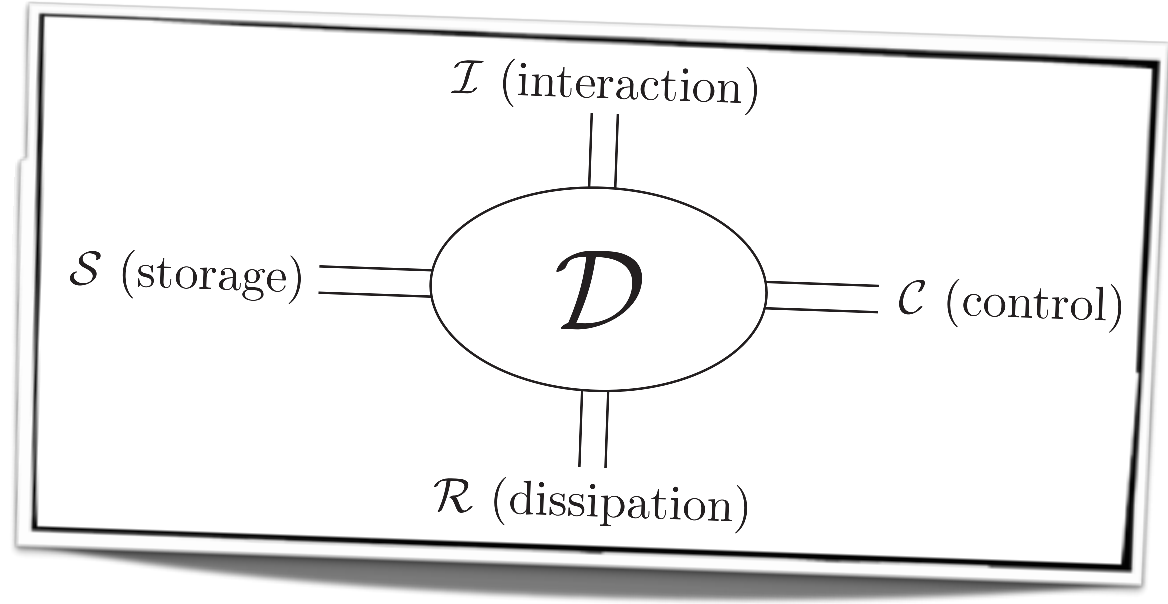
$$f = -\dot{x} \quad e = \frac{\partial H}{\partial x}$$

$$-F\dot{x} + E \frac{\partial H}{\partial x} = 0, \quad x(0) = x_0 \in \mathcal{X}$$

port-Hamiltonian system

Dirac structures & port-Hamiltonian systems

- In the general case:



$$
\begin{aligned}
\mathcal{D} = \Big\{ (f_S, f_R, f_C, f_I, e_S, e_R, e_C, e_I) \in \mathcal{F} \times \mathcal{E} \ \ \Big| \\
F_S f_S + F_R f_R + F_C f_C + F_I f_I + \\
+ E_S e_S + E_R e_R + E_C e_C + E_I e_I = 0 \Big\}
\end{aligned}
$$

$$
R_f f_R + R_e e_R = 0
$$

resistive relation

$$
\begin{aligned}
R_f R_e^{\mathrm{T}} &= R_e R_f^{\mathrm{T}} > 0 \\
\operatorname{rank}\left(R_f \ | \ R_e\right) &= \dim \mathcal{F}_R
\end{aligned}
$$

$$
\Rightarrow \quad
\begin{aligned}
-F_S \dot{x} + E_S \frac{\partial H}{\partial x} + F_R f_R + E_R e_R + \\
+ F_C f_C + E_C e_C + F_I f_I + E_I e_I = 0 \\
R_f f_R + R_e e_R = 0
\end{aligned}
$$

$$
\frac{\mathrm{d}}{\mathrm{d}t} H \leq e_C^{\mathrm{T}} f_C + e_I^{\mathrm{T}} f_I
$$

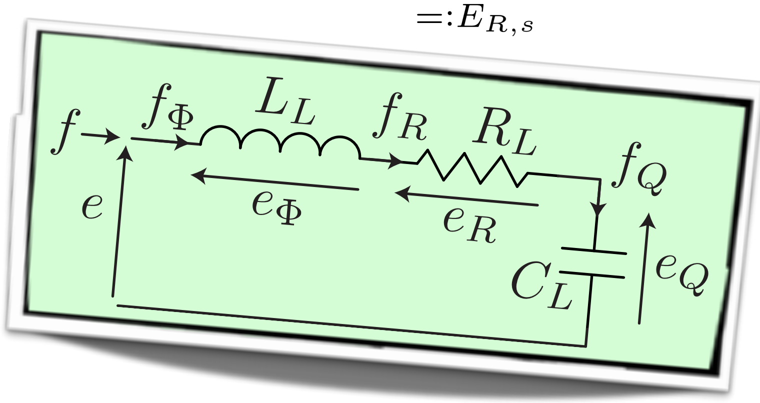
This is the *most general* formulation of a port-Hamiltonian system. Usually, the *spatial discretisation* of a distributed parameter systems is in this form

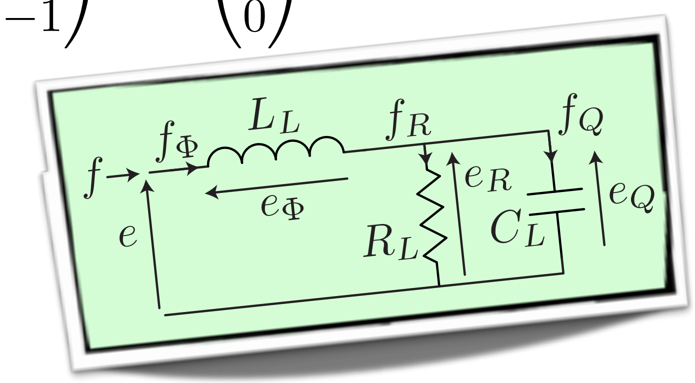
Dirac structures & port-Hamiltonian systems

☑ *Example:* the RLC circuit

$$\underbrace{\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}}_{=:F_{S,s}}\begin{pmatrix}f_Q\\f_\Phi\end{pmatrix}+\underbrace{\begin{pmatrix}0&1\\-1&0\\0&1\\0&1\end{pmatrix}}_{=:E_{S,s}}\begin{pmatrix}e_Q\\e_\Phi\end{pmatrix}+\underbrace{\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}}_{=:F_{R,s}}f_R+$$
$$+\underbrace{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{=:E_{R,s}}e_R+\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}f+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e=0$$

$$\underbrace{\begin{pmatrix}1&0\\0&1\\0&0\\0&0\end{pmatrix}}_{=:F_{S,p}}\begin{pmatrix}f_Q\\f_\Phi\end{pmatrix}+\underbrace{\begin{pmatrix}0&1\\-1&0\\1&0\\0&1\end{pmatrix}}_{=:E_{S,p}}\begin{pmatrix}e_Q\\e_\Phi\end{pmatrix}+\underbrace{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}}_{=:F_{R,p}}f_R+$$
$$+\underbrace{\begin{pmatrix}0\\0\\-1\\0\end{pmatrix}}_{=:E_{R,p}}e_R+\begin{pmatrix}0\\0\\0\\-1\end{pmatrix}f+\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e=0$$





$$H_L(x_Q,x_\Phi)=\frac{1}{2}\left(\frac{x_Q^2}{C_L}+\frac{x_\Phi^2}{L_L}\right)$$

$$R_L f_R+e_R=0$$

Energy-based control

- Let us consider a port-Hamiltonian system in *input-state-output* form:

$$\begin{cases} \dot{x}(t) = \big[J(x(t)) - R(x(t))\big]\dfrac{\partial H}{\partial x}(x(t)) + G(x(t))u(t) \\ y(t) = G^{\mathrm{T}}(x(t))\dfrac{\partial H}{\partial x}(x(t)) \end{cases}$$

$$\Downarrow$$

energy balancing

$$\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) = -\left(\frac{\partial H}{\partial x}(x(t))\right)^{\mathrm{T}} R(x(t))\frac{\partial H}{\partial x}(x(t)) + y^{\mathrm{T}}(t)u(t)$$

- Standard approach is to rely on *“energy considerations”* to obtain and prove asymptotic stability of equilibria
  - *Damping injection*
  - *Energy-shaping*
- Standard assumption is *H bounded from below*

Damping injection

- Suppose that *H has an isolated minimum at a desired equilibrium*

$$\frac{\partial H}{\partial x}(x^\star) = 0 \qquad \frac{\partial^2 H}{\partial x^2}(x^\star) > 0$$

- The idea is to *dissipate energy* until the minimum is reached
  - Asymptotic stability if there is “enough dissipation”
  - Zero-state detectability
  - La Salle’s Invariance principle
- The control action is

$$u(t) = -K_D y(t), \qquad K_D = K_D^{\mathrm{T}} \geq 0$$

$$\Downarrow$$

$$\frac{\mathrm{d}H}{\mathrm{d}t}(x(t)) = -\left(\frac{\partial H}{\partial x}(x(t))\right)^{\mathrm{T}} \big(R(x(t)) + K_D\big) \frac{\partial H}{\partial x}(x(t)) \leq 0$$

Energy-balancing control

- In general, it is necessary to *shape the open-loop Hamiltonian* to introduce a minimum at the desired equilibrium
- From the energy-balance relation we have

$$H(x(t)) - H(x(0)) = \int_0^t y^{\mathrm{T}}(\tau)u(\tau)\,\mathrm{d}\tau - d(t)$$

- The standard formulation of passivity-based control requires to *determine a control action*

$$u(t) = \beta(x(t)) + u'(t)$$

such that the *closed-loop dynamics* satisfies:

$$H_d(x(t)) - H_d(x(0)) = \int_0^t {y'}^{\mathrm{T}}(\tau)u'(\tau)\,\mathrm{d}\tau - d_d(t)$$

new energy balancing

- H_d is a desired energy function, while d_d replaces the natural dissipation
  - Energy-shaping *plus* damping injection

Energy-balancing control

- A large class of dynamical systems can be stabilised by requiring that the *supplied energy is a function of the state of the plant*

$$-\int_0^t y^{\mathrm{T}}(\tau)\beta(x(\tau))\,\mathrm{d}\tau = H_a(x(t)) + \kappa$$

- We require that along *all* system trajectories

$$-y^{\mathrm{T}}(t)\beta(x(t)) = \frac{\partial^{\mathrm{T}} H_a}{\partial x}(x(t))\dot{x}(t)$$

- The *"desired" closed-loop Hamiltonian* is then

$$H_d(x(t)) = H(x(t)) + H_a(x(t))$$

- The previous PDE provides the *class of H_a* and the *control actions,* while stability analysis follows from the energy-balance relation
  - *u' can be used to add damping*

Energy-balancing control

- The methodology can be applied to generic nonlinear systems

$$\begin{cases} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{cases}$$

- From *KYP lemma,* passivity is equivalent to the *existence of a function H(x)* such that

$$\left(\frac{\partial H}{\partial x}(x)\right)^{\mathrm{T}} f(x) \leq 0 \qquad\qquad h(x) = g^{\mathrm{T}}(x)\frac{\partial H}{\partial x}(x)$$

- *Matching equation:*

supplied power

$$\left(\frac{\partial H_a}{\partial x}(x)\right)^{\mathrm{T}}\Big[f(x) + g(x)\beta(x)\Big] = -h^{\mathrm{T}}(x)\beta(x)$$

- At the equilibrium:

$$f(x^\star) + g(x^\star)\beta(x^\star) = 0 \quad \Rightarrow \quad h^{\mathrm{T}}(x^\star)\beta(x^\star) = 0$$

dissipation obstacle

Control with state-modulated source

- The idea is to compute a state feed-back action

$$u(t) = \beta(x(t)) + u'(t)$$

so that *the open-loop system is mapped into a new one,* but with a desired Hamiltonian

$$H_d(x(t)) = H(x(t)) + H_a(x(t))$$

$$\Downarrow$$

$$\dot{x}(t) = \left[J(x(t)) - R(x(t))\right] \frac{\partial H_d}{\partial x}(x(t)) + G(x(t))u'(t)$$

- A direct computation leads to

$$G(x)\beta(x) = \left[J(x) - R(x)\right] \frac{\partial H_a}{\partial x}(x)$$

matching condition

- A further generalisation leads to the *IDA-PBC control technique,* where we shape
  - Hamiltonian
  - Interconnection and resistive structure

Dirac structures & control synthesis

- *Energy-balancing control* in the general case:

$$-\beta^{\mathrm{T}}(x(t))f_C(t) = \left(\frac{\partial H_a}{\partial x}(x(t))\right)^{\mathrm{T}}\dot{x}(t)$$

$(y,u) = (f_C, e_C)$

$$\Downarrow$$

$$\left(-\frac{\partial^{\mathrm{T}} H_a}{\partial x}E_S^{\mathrm{T}} + \beta^{\mathrm{T}}E_C^{\mathrm{T}}\right)\lambda = 0$$

$$\Downarrow$$

$$-E_S\frac{\partial H_a}{\partial x} + E_C\beta = 0$$

- A (necessary and) sufficient condition is that

$$\begin{pmatrix} 0 \\ 0 \\ 0 \\ -\frac{\partial H_a}{\partial x} \\ 0 \\ \beta \end{pmatrix} \in \mathrm{Im}\begin{pmatrix} E_S^{\mathrm{T}} \\ E_R^{\mathrm{T}} \\ E_C^{\mathrm{T}} \\ F_S^{\mathrm{T}} \\ F_R^{\mathrm{T}} \\ F_C^{\mathrm{T}} \end{pmatrix} \equiv \mathcal{D}$$

dissipation obstacle???

Dirac structures & control synthesis

- Let us consider at first the *finite element model* of a transmission line, which is characterised by a Dirac structure with matrices

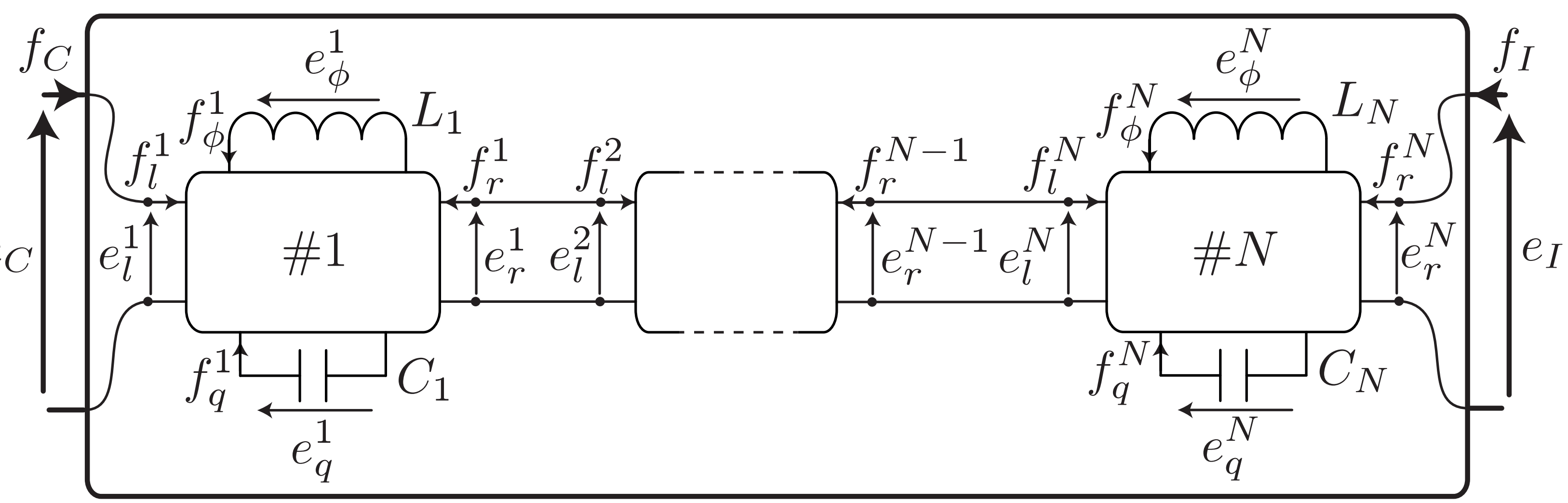
$$F_\infty = \begin{pmatrix} F_{S,\infty} & F_{C,\infty} & F_{I,\infty} \end{pmatrix} \qquad E_\infty = \begin{pmatrix} E_{S,\infty} & E_{C,\infty} & E_{I,\infty} \end{pmatrix}$$

and an Hamiltonian

$$H_\infty(x_\infty) = \frac{1}{2} \sum_{i=1}^{N} \left(\frac{{x_q^i}^2}{C_i} + \frac{{x_\phi^i}^2}{L_i} \right)$$

$$x_\infty = \begin{pmatrix} x_q^1 & x_\phi^1 & \dots & x_q^N & x_\phi^N \end{pmatrix}^{\mathrm{T}}$$

controller



load

The plant is a finite dimensional port-Hamiltonian system with control port *(f_C, e_C)*

Dirac structures & control synthesis

- For the complete system we have

$$x = \begin{pmatrix} x_Q & x_\Phi & x_\infty \end{pmatrix}^{\mathrm{T}} \qquad H(x) = H_\infty(x_\infty) + H_L(x_Q, x_\Phi)$$

- Simple physical considerations lead to the *desired equilibrium:*

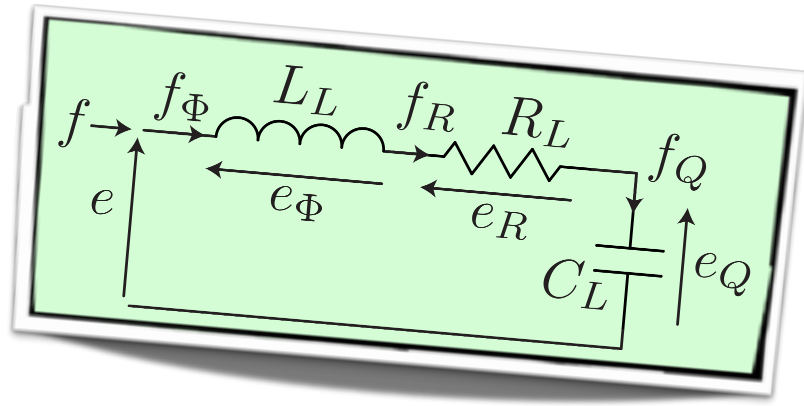
$$\frac{x_Q^\star}{C_L} = \frac{x_q^{i,\star}}{C_i} = e^\star \qquad \frac{x_\Phi^\star}{L_L} = \frac{x_\phi^{i,\star}}{L_i} = 0$$

- The *energy-balance controller* follows if it exists λ such that

$$-\frac{\partial H_a}{\partial x} = {F_S}^{\mathrm{T}}\lambda$$

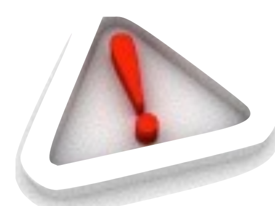
$$\beta = {F_C}^{\mathrm{T}}\lambda$$

$$0 = E_S^{\mathrm{T}}\lambda = E_R^{\mathrm{T}}\lambda = F_R^{\mathrm{T}}\lambda = E_C^{\mathrm{T}}\lambda$$



$$H_a(x) = H_a(\xi)\Big|_{\xi = x_Q + \sum_{i=1}^N x_q^i}$$

$$\beta(x) = -\frac{\partial H_a}{\partial \xi}\Big|_{\xi = x_Q + \sum_{i=1}^N x_q^i}$$



$$H_a(\xi) = \frac{1}{2}\frac{\xi^2}{C_C} - e^\star\left(1 + \frac{C_L}{C_C} + \sum_{i=1}^N \frac{C_i}{C_C}\right)\xi + \kappa$$

Dirac structures & control synthesis

- Finding the *EB regulator* means finding a state dependent control action able to shape the open-loop Hamiltonian, in such a way that closed loop and target dynamics have the *same behaviour at the storage, resistive and control ports*
  - ***Very strong requirement!***
  - Let us ask less: *just a matching* between open-loop *plus* controller, and target dynamics (with desired stability properties)

$$
\begin{aligned}
-\dot{x} &= E_S^{\mathrm{T}} \lambda \\
\frac{\partial H}{\partial x} &= F_S^{\mathrm{T}} \lambda \\
0 &= \left(R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}} \right) \lambda \\
e_C &= F_C^{\mathrm{T}} \lambda
\end{aligned}
$$

controlled system

$$e_C = \beta(x) + e'_C$$

$$
\begin{aligned}
-\dot{x} &= E_S^{\mathrm{T}} \lambda' \\
\frac{\partial H_d}{\partial x} &= F_S^{\mathrm{T}} \lambda' \\
0 &= \left(R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}} \right) \lambda' \\
e'_C &= F_C^{\mathrm{T}} \lambda'
\end{aligned}
$$

"desired" system

$$H_d(x) = H(x) + H_a(x)$$

Dirac structures & control synthesis

- Since trajectories are required to be the same

$$
\begin{aligned}
0 &= E_S^{\mathrm{T}}(\lambda - \lambda') \\
-\frac{\partial H_a}{\partial x} &= F_S^{\mathrm{T}}(\lambda - \lambda') \\
0 &= \left(R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}}\right)(\lambda - \lambda') \\
\beta &= F_C^{\mathrm{T}}(\lambda - \lambda')
\end{aligned}
\quad \Rightarrow \quad
\begin{pmatrix} 0 \\ -\frac{\partial H_a}{\partial x} \\ 0 \\ \beta \end{pmatrix} \in \mathrm{Im} \begin{pmatrix} E_S^{\mathrm{T}} \\ F_S^{\mathrm{T}} \\ R_f E_R^{\mathrm{T}} + R_e F_R^{\mathrm{T}} \\ F_C^{\mathrm{T}} \end{pmatrix}
$$

- It is possible to prove that *the open-loop system is mapped into the desired closed-loop one,* for which the Hamiltonian function H_d is selected so that "nice" stability properties are satisfied
  - *Asymptotic stability follows as in case of energy-balancing regulators*

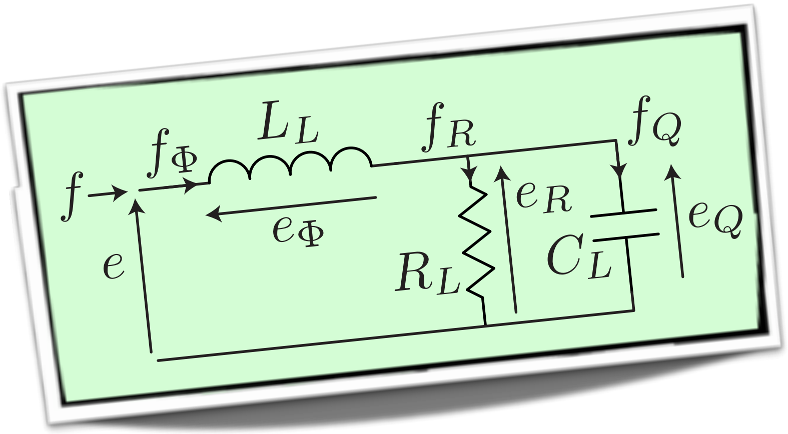
Dirac structures & control synthesis

- In case of pRLC with the fem of the line, the *desired equilibrium* is

$$\frac{x_Q^\star}{C_L} = \frac{x_q^{i,\star}}{C_i} = e^\star \qquad \frac{x_\Phi^\star}{L_L} = \frac{x_\phi^{i,\star}}{L_i} = \frac{e^\star}{R_L}$$

- The control synthesis requires to find λ, such that

$$-\frac{\partial H_a}{\partial x} = {F_S}^{\mathrm{T}}\lambda$$
$$\beta = {F_C}^{\mathrm{T}}\lambda$$
$$0 = E_S^{\mathrm{T}}\lambda = (R_L E_R^{\mathrm{T}} + F_R^{\mathrm{T}})\lambda$$



$$H_a(x) = H_a(\xi)\Big|_{\xi = x_\Phi + R_L x_Q + \sum_{i=1}^N \left(x_\phi^i + R_L x_q^i\right)}$$

$$\beta(x) = -R_L \frac{\partial H_a}{\partial \xi}\Big|_{\xi = x_\Phi + R_L x_Q + \sum_{i=1}^N \left(x_\phi^i + R_L x_q^i\right)}$$

- A possible *choice for H_a* can be the following:

$$H_a(\xi) = \frac{1}{2}\frac{(\xi - \xi^\star)^2}{L_C} - \frac{e^\star}{R_L}\xi + \kappa$$

Asymptotic stability is a consequence of the *energy dissipation inequality*

Invitation to CPDE 2016

2nd IFAC Workshop on
Control of Systems Governed by Partial Differential Equations

IFAC

CPDE'16, Bertinoro, Italy – June 13-15, 2016

Hosting institution

University of Bologna

Location

Centro Residenziale Universitario, on the hilltop of Bertinoro (FC), Italy

Sponsored by

IFAC International Federation of Automatic Control,
IFAC TC Distributed Parameter Systems

Co-sponsored by

IFAC TC Non Linear Control Systems

INVITATION

The second IFAC Workshop on Control of Systems Governed by Partial Differential Equations will be held in Bertinoro (Italy) from Monday to Wednesday 13-15 June 2016 at the Centro Residenziale Universitario of the University of Bologna.

The workshop will address new and state-of-the-art developments in modelling and control of distributed parameter systems and their applications. Since the control design for these systems resides at the intersection of mathematics, systems and control theory, control systems technology, computer and information science, it is essential to provide a joint forum to foster and evolve this important and emerging field of research.

CPDE'16 aims at providing this forum under the IFAC flagship.

SCOPE

Distributed parameter systems, which are mathematically described by partial differential equations, impose a formidable challenge in many applications coming from classical industrial fields as well as emerging sectors related to energy, transport, communication or medical science. Herein, the distri[…] essential ingredient of the modelling and ana[…] tribution of the system variables cannot be r[…] distributed parameter systems essentially reli[…] and estimation strategies to influence the system dynamics, and to enlarge the dynamic operating range.

Starting from these observations, new approaches to the control of distributed para-

http://www.cpde2016.org

ALMA MATER STUDIORUM
UNIVERSITÀ DI BOLOGNA

Alessandro Macchelli

Department of Electrical, Electronic, and Information Engineering (DEI)

alessandro.macchelli@unibo.it

http://www.unibo.it