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NEARLY EUCLIDEAN THURSTON MAPS

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ABSTRACT. We take an in-depth look at Thurston's combinatorial characterization of rational functions for a particular class of maps we call nearly Euclidean Thurston maps. These are orientation-preserving branched maps $f\colon S^2 \to S^2$ whose local degree at every critical point is 2 and which have exactly four postcritical points. These maps are simple enough to be tractable, but are complicated enough to have interesting dynamics.

In this work, we take an in-depth look at Thurston's combinatorial characterization of rational functions for a particular class of maps we call *nearly Euclidean Thurston* (NET) maps. Suppose $f\colon\ S^2 \to S^2$ is an orientation-preserving branched map. Following Thurston, we define $\nu_f\colon\ S^2 \to \mathbb{Z}_+ \cup \{\infty\}$ by

$$\nu_f(x) = \begin{cases} \operatorname{lcm}(D_f(x)) & \text{if } D_f(x) \text{ is finite,} \\ \infty & \text{if } D_f(x) \text{ is infinite,} \end{cases}$$

where $D_f(x) = \{n \in \mathbb{Z}_+ :\ \text{there exists } m \in \mathbb{Z}_+ \text{ and } y \in S^2 \text{ such that } f^{\circ m}(y) = x \text{ and } f^{\circ m} \text{ has degree } n \text{ at } y\}$. The points $x \in S^2$ with $\nu_f(x) > 1$ are called *postcritical points*, and the set of postcritical points is denoted by P_f. The map f is *postcritically finite* if P_f is finite. A Thurston map is an orientation-preserving branched map $f\colon\ S^2 \to S^2$ which is postcritically finite. In this case, we denote by $\mathcal{T}$ the Teichmüller space of the orbifold (S^2, ν_f). The map f induces a map $\Sigma_f\colon\ \mathcal{T} \to \mathcal{T}$ by pulling back complex structures.

In a CBMS Conference in 1983, Thurston [11] addressed the problem of determining when a Thurston map $f\colon\ S^2 \to S^2$ is equivalent to a rational map, where $f \sim g$ if there is an orientation-preserving homeomorphism $h\colon\ S^2 \to S^2$ such that $h(P_f) = P_g$, $(h \circ f)\big|_{P_f} = (g \circ h)\big|_{P_f}$, and $h \circ f$ is isotopic, rel P_f, to $g \circ h$. His main theorems were (1) that f is equivalent to a rational map exactly if Σ_f has a fixed point, and (2) if (S^2, ν_f) is hyperbolic, then Σ_f has a fixed point exactly if there are no Thurston obstructions (these will be defined next). Thurston didn't publish his proofs of the theorems, but proofs were given later by Douady and Hubbard in [3].

Now we define Thurston obstructions. A *multicurve* Γ is a finite collection of pairwise disjoint simple closed curves in $S^2 \setminus P_f$ such that each element of Γ is nontrivial, each element of Γ is nonperipheral, and distinct elements of Γ are not isotopic. A multicurve Γ is *invariant* or *f-stable* if each element of $f^{-1}(\Gamma)$ is either trivial, peripheral, or isotopic to an element of Γ. If Γ is an invariant multicurve,

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then the Thurston matrix $A^\Gamma\colon \mathbb{R}^\Gamma \to \mathbb{R}^\Gamma$ is defined in coordinates by

$$A^\Gamma_{\gamma\delta} = \sum_\alpha \frac{1}{\deg(f\colon\ \alpha \to \delta)},$$

where the sum is taken over connected components α of $f^{-1}(\delta)$ which are isotopic to γ in $S^2 \setminus P_f$. If Γ is an invariant multicurve, the spectral radius (eigenvalue of largest norm) of A^Γ is called the Thurston multiplier of Γ. An invariant multicurve is a *Thurston obstruction* if its Thurston multiplier is at least one.

Unfortunately, checking whether or not Thurston obstructions exist is very difficult primarily because there are infinitely many multicurves to consider. Our motivation in this work is to better understand Thurston obstructions and the issue of conformality of finite subdivision rules (for which, see [2]). We were led to a class of Thurston maps which are as simple as possible but yet nontrivial in this regard. We call these maps nearly Euclidean Thurston maps. These are simple generalizations of Lattès maps.

In [6] Milnor characterizes a Lattès map as a rational map from the Riemann sphere to itself such that each of its critical points is simple (local degree 2) and it has exactly four postcritical points, none of which is also critical. We say that a Thurston map is Euclidean if it is a straightforward generalization of this: a Thurston map is Euclidean if its degree is at least 2, its local degree at every critical point is 2 and it has at most four postcritical points, none of which is also critical. (Lemma 1.3 shows that if there are at most four postcritical points, then there are exactly four.) A nearly Euclidean Thurston (NET) map allows postcritical points to be critical: a Thurston map is nearly Euclidean if its local degree at every critical point is 2 and it has exactly four postcritical points. (Now at most four does not imply four, as is the case for the map $z \mapsto z^2$.) If f is a Euclidean Thurston map, then the orbifold (S^2, ν_f) is Euclidean. On the other hand, if a NET map f is not Euclidean, that is, some postcritical point is a critical point, then the orbifold (S^2, ν_f) is hyperbolic. These are the simplest Thurston maps with hyperbolic orbifolds and nontrivial Teichmüller spaces.

Our ultimate goal is to thoroughly understand Thurston obstructions for NET maps. This paper is devoted to developing the first properties of these maps.

Section 1 presents definitions and basic facts concerning NET maps. These basic facts involve lifting properties of NET maps. Every NET map lifts to a map from one torus to another. From such a lift we obtain a lift from $\mathbb{R}^2$ to itself.

Section 2 deals with twists of NET maps. We twist a NET map by postcomposing it with a suitable homeomorphism. We find that every NET map is a twist of a Euclidean Thurston map.

Section 3 presents two examples. The first of these is our main example. The finite subdivision rule associated to this example is the germ of this paper. Everything in this paper arose from studying this example. The second example in Section 3 shows that the rational function, which appears in the proof of statement 2 of Theorem 1.1 of [1], is a NET map.

Computing Thurston matrices involves degrees and numbers of components of pullbacks of invariant multicurves. These degrees and numbers of components are described rather completely for NET maps in Section 4.

Every homotopy class of simple closed curves in a 4-punctured sphere is assigned a slope in $\widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$ which characterizes the homotopy class. Taking pullbacks, a NET map f induces a self-map $\sigma_f \colon \widehat{\mathbb{Q}} \cup \{o\} \to \widehat{\mathbb{Q}} \cup \{o\}$, where o denotes the union of the classes of inessential and peripheral curves. In Section 4, we give an algorithm (Theorem 5.3 and the following discussion) for computing σ_f. The slope of a Thurston obstruction is a fixed point for σ_f.

Section 6 begins our study of the induced map on Teichmüller space. Theorem 6.7, the half-space theorem, shows how knowledge of the pullback of a given curve under f translates into an interval of slopes in which the slope of a Thurston obstruction cannot lie. At the end of Section 6, we use this result to show that there are no Thurston obstructions for the main example.

Section 7 discusses Dehn twists in the present context. Section 8 discusses reflections. Section 9 presents a common framework for the results of the previous two sections. Taken together, these three sections allow us to compute explicit "functional equations" satisfied by the map on Teichmüller space induced by a NET map.

Section 10 shows how the results of the previous sections can be applied to the study of maps on Teichmüller space induced by NET maps. Section 10 begins the characterization of those NET maps whose induced maps on Teichmüller space are constant. Theorem 10.2 reduces this characterization to a purely algebraic problem concerning finite Abelian groups generated by two elements. We then obtain partial results for this algebraic problem. Saenz Maldonado extends these results concerning this algebraic problem in his thesis [9], although a complete solution is not yet in hand.

1. Definitions and lifts

A Thurston map is an orientation-preserving branched covering map from the 2-sphere to itself which is postcritically finite.

Definition 1.1. A Thurston map is *Euclidean* if its degree is at least 2, its local degree at each of its critical points is 2, it has at most four postcritical points and none of its postcritical points is critical.

Definition 1.2. A Thurston map is *nearly Euclidean* (NET) if its local degree at each of its critical points is 2 and it has exactly four postcritical points.

Note that the definition of Euclidean used here is stronger than the condition of the orbifold being Euclidean. Although the definition only requires that a Euclidean Thurston map have at most four postcritical points, the first statement of the following lemma shows that it actually has exactly four. It follows that a Euclidean Thurston map is characterized by the property that its orbifold is the $(2,2,2,2)$-orbifold.

Lemma 1.3. (1) *Every Euclidean Thurston map has exactly four postcritical points, and so every Euclidean Thurston map is nearly Euclidean.*
(2) *Let $f \colon S^2 \to S^2$ be a NET map with postcritical set P_f. Then $f^{-1}(P_f)$ contains exactly four points which are not critical points. The map f is Euclidean if and only if these four points are the points of P_f.*

Proof. To prove statement (1), let $f \colon S^2 \to S^2$ be a Euclidean Thurston map with postcritical set P_f and degree d. Every point of S^2 has d preimages under f counting multiplicity. Hence $f^{-1}(P_f)$ has $d\,|P_f|$ points counting multiplicity. The Riemann-Hurwitz formula shows that f has $2d-2$ critical points. These points map to P_f with multiplicity 2 and they are distinct from the points of P_f. Combining these facts yields the inequality $4d-4+|P_f| \leq d\,|P_f|$. Hence $(4-|P_f|)(d-1) \leq 0$. Since $d>1$, we have that $|P_f| \geq 4$. Since $|P_f| \leq 4$ by assumption, it follows that $|P_f| = 4$. This proves statement (1) of Lemma 1.3.

To prove statement (2), we let f now be a NET map and argue as in the previous paragraph. If n is the number of points in $f^{-1}(P_f)$ which are not critical, then $4d-4+n = 4d$. Hence $n=4$.

This proves Lemma 1.3. □

A Lattès map as in Milnor's paper [6] (the definition in [7] is more general) is a rational function which is a Euclidean Thurston map, and so NET maps are closely related to Lattès maps. An important property of Lattès maps is that they lift to maps of tori in a special way. The next theorem shows that NET maps lift to maps of tori in a more general way and, in fact, this property characterizes NET maps. The proof uses the fact that given four points in S^2, there exists a double cover (unique up to isomorphism) of S^2 ramified over exactly these four points.

Theorem 1.4. *Let $f\colon S^2 \to S^2$ be a Thurston map. Then f is nearly Euclidean if and only if there exist branched covering maps $p_1\colon T_1 \to S^2$ and $p_2\colon T_2 \to S^2$ with degree 2 from tori T_1 and T_2 to S^2 such that the set of branch points of p_2 is the postcritical set of f and there exists a continuous map $\widetilde{f}\colon T_1 \to T_2$ such that $p_2 \circ \widetilde{f} = f \circ p_1$. If f is nearly Euclidean, then f is Euclidean if and only if the set of branch points of p_1 is the postcritical set of f.*

Proof. We begin by proving the backward implication of the first assertion. Let p_1, p_2 and $\widetilde{f}$ be maps as stated. It follows that $\widetilde{f}$ is a branched covering map. Two applications of the Riemann-Hurwitz formula show that $\widetilde{f}$ is unramified and that p_1 and p_2 are both ramified at exactly four points. Hence the postcritical set of f has exactly four points. Now we combine the equation $p_2 \circ \widetilde{f} = f \circ p_1$ with the facts that local degrees multiply under composition of functions, that the local degree of $\widetilde{f}$ at every point is 1 and that the local degree of p_2 at every point is either 1 or 2. We conclude that the local degree of f at every point is either 1 or 2. In other words, the local degree of f at each of its critical points is 2. This proves the backward implication of the first assertion.

To prove the forward implication of the first assertion, suppose that $f\colon S^2 \to S^2$ is a NET map with postcritical set P_2. Statement (2) of Lemma 1.3 implies that four points of $f^{-1}(P_2)$ are not critical points of f. Let P_1 be this set of four points. Now let T_1 and T_2 be tori, and let $p_1\colon T_1 \to S^2$ and $p_2\colon T_2 \to S^2$ be branched covering maps with degree 2 such that the set of branch points of p_1 is P_1 and the set of branch points of p_2 is P_2. Then the restriction of p_2 to $T_2 \setminus p_2^{-1}(P_2)$ is a covering map to $S^2 \setminus P_2$, and the restriction of $f \circ p_1$ to $T_1 \setminus p_1^{-1}(f^{-1}(P_2))$ is a continuous map to $S^2 \setminus P_2$. The fundamental group $\pi_1(S^2 \setminus P_2)$ is generated by the homotopy classes of four loops about the elements of P_2. There exists a group homomorphism from $\pi_1(S^2 \setminus P_2)$ to $\mathbb{Z}/2\mathbb{Z}$ which sends these homotopy classes to the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$. The kernel of this group homomorphism is the image of $\pi_1(T_2 \setminus p_2^{-1}(P_2))$ in $\pi_1(S^2 \setminus P_2)$. Now we see that because the elements of P_1 are branch points of p_1 and the remaining elements of $f^{-1}(P_2)$ are critical points of f, the image of $\pi_1(T_1 \setminus p_1^{-1}(f^{-1}(P_2)))$ in $\pi_1(S^2 \setminus P_2)$ is contained in the image of $\pi_1(T_2 \setminus p_2^{-1}(P_2))$. The standard lifting theorem from covering space theory now implies that there exists a lift from $T_1 \setminus p_1^{-1}(f^{-1}(P_2))$ to $T_2 \setminus p_2^{-1}(P_2)$, and this

$$\begin{tikzcd}
\Lambda_1 \arrow[r, "incl"] \arrow[d, "incl"'] & \Lambda_2 \arrow[d, "incl"'] \\
\mathbb{R}^2 \arrow[r, "id"] \arrow[d, "q_1"'] & \mathbb{R}^2 \arrow[d, "q_2"'] \\
T_1 \arrow[r, "\widetilde{f}"] \arrow[d, "p_1"'] & T_2 \arrow[d, "p_2"'] \\
S^2 \arrow[r, "f"] & S^2
\end{tikzcd}$$

FIGURE 1. The standard commutative diagram.

lift extends to a lift $\widetilde{f}\colon T_1 \to T_2$ such that $p_2 \circ \widetilde{f} = f \circ p_1$. This proves the forward implication of the first assertion.

The second assertion concerning Euclidean Thurston maps is now clear.

This proves Theorem 1.4. $\square$

We continue this section with a discussion of NET maps. Let $f\colon S^2 \to S^2$ be a NET map. Let $p_1\colon T_1 \to S^2$ and $p_2\colon T_2 \to S^2$ be covering maps with degree 2 from tori T_1 and T_2 to S^2 and let $\widetilde{f}\colon T_1 \to T_2$ be a continuous map as in Theorem 1.4 such that $p_2 \circ \widetilde{f} = f \circ p_1$. Let P_j be the set of branch points of p_j in S^2 for $j \in \{1,2\}$. We sometimes use the notation $P_j(f)$ instead of P_j to avoid possible confusion when dealing with more than one NET map. The set P_2 is the postcritical set of f.

Let $j \in \{1,2\}$. Let $q_j\colon \mathbb{R}^2 \to T_j$ be a universal covering map. The map $p_j \circ q_j\colon \mathbb{R}^2 \to S^2$ is a branched covering map whose local degree at every ramified point is 2. Let $\Lambda_j \subseteq \mathbb{R}^2$ be the set of these ramification points. It is furthermore true that $p_j \circ q_j$ is regular. Let Γ_j be its group of deck transformations. By choosing q_j appropriately, we may assume that Γ_j is generated by the set of all Euclidean rotations of order 2 about the points of Λ_j. Given rotations $x \mapsto 2\lambda - x$ and $x \mapsto 2\mu - x$ of order 2 about the points $\lambda, \mu \in \Lambda_j$, their composition, the second followed by the first, is the translation $x \mapsto x + 2(\lambda - \mu)$. We may, and do, normalize so that $0 \in \Lambda_j$. It follows that Λ_j is a lattice in $\mathbb{R}^2$ and that the elements of Γ_j are the maps of the form $x \mapsto 2\lambda \pm x$ for some $\lambda \in \Lambda_j$.

The map $\widetilde{f}$ lifts to a continuous map $\widetilde{\widetilde{f}}\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $q_2 \circ \widetilde{\widetilde{f}} = \widetilde{f} \circ q_1$. Since $\widetilde{f}$ is a covering map, so is $\widetilde{\widetilde{f}}$. Hence $\widetilde{\widetilde{f}}$ is a homeomorphism. We replace q_1 by $q_1 \circ \widetilde{\widetilde{f}}^{-1}$. As a result, $\widetilde{f}$ lifts to the identity map. Because $\widetilde{f}$ lifts to the identity map, $\Lambda_1 \subseteq \Lambda_2$ and $\Gamma_1 \subseteq \Gamma_2$. We obtain the standard commutative diagram in Figure 1, where the map from $\mathbb{R}^2$ to itself is the identity map and the maps from Λ_1 and Λ_2 are inclusion maps.

The group Γ_j contains the group of deck transformations of q_j. It is the subgroup with index 2 consisting of translations of the form $x \mapsto 2\lambda + x$ with $\lambda \in \Lambda_j$. Thus we identify T_j with $\mathbb{R}^2/2\Lambda_j$. The standard commutative diagram implies that $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$ are both identified with S^2. Thus there is an *identification map* $\phi\colon \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$. To be precise, to evaluate f at some point x, we view x as an element of $\mathbb{R}^2/\Gamma_1$. We lift it to $\mathbb{R}^2$, then project it to $\mathbb{R}^2/\Gamma_2$ and then apply the identification map ϕ to obtain $f(x)$. For Euclidean NET maps, we usually construct this identification map using an affine automorphism of $\mathbb{R}^2$ which restricts to an affine isomorphism from Λ_2 to Λ_1.

In this paragraph we discuss how the identification map ϕ might arise from an affine isomorphism Φ. Let $\Phi\colon \mathbb{R}^2 \to \mathbb{R}^2$ be an affine isomorphism such that $\Phi(\Lambda_2) = \Lambda_1$. Every rotation in Γ_2 has the form $x \mapsto 2\lambda - x$ for some $\lambda \in \Lambda_2$. Suppose that $\Phi(x) = Ax + B$, where $A \in \mathrm{GL}(2,\mathbb{R})$ and $B \in \mathbb{R}^2$. Let $\lambda \in \Lambda_2$. Then

$$\Phi(2\lambda - x) = A(2\lambda - x) + B = 2(A\lambda + B) - (Ax + B) = 2\Phi(\lambda) - \Phi(x).$$

Because $\Phi(\Lambda_2) = \Lambda_1$, this implies that Φ induces a map $\phi\colon \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$, and one checks that it is a homeomorphism. In this way we obtain an identification map ϕ.

Because $\widetilde{f}$ lifts to the identity map, when we interpret in terms of group theory, we see that $\widetilde{f}$ is the canonical group homomorphism from $\mathbb{R}^2/2\Lambda_1$ to $\mathbb{R}^2/2\Lambda_2$. Its kernel is $2\Lambda_2/2\Lambda_1 \cong \Lambda_2/\Lambda_1$. Thus $\deg(f) = \deg(\widetilde{f}) = |\Lambda_2/\Lambda_1|$.

Let $i \in \{1, 2\}$. We have that P_i is the set of branch points of $p_i \circ q_i$, that Λ_i is the set of ramification points of $p_i \circ q_i$ and that $q_i^{-1}(p_i^{-1}(P_i)) = \Lambda_i$. In this paragraph we show that $q_1^{-1}(p_1^{-1}(P_2)) \subseteq \Lambda_2$. For this, let $x \in P_2$. Let $y \in p_1^{-1}(x)$. Then $p_2(\widetilde{f}(y)) = f(p_1(y)) = f(x) \in P_2$. So $\widetilde{f}(y)$ is one of the four points of T_2 at which p_2 is ramified. We conclude that $p_1^{-1}(P_2)$ is contained in the set of $4\deg(f)$ points of T_1 which $\widetilde{f}$ maps to a ramification point of p_2. This implies that $q_1^{-1}(p_1^{-1}(P_2)) \subseteq \Lambda_2$. More precisely, if P_2 contains m elements in P_1 and n elements not in P_1, so that $m + n = 4$, then $q_1^{-1}(p_1^{-1}(P_2))$ consists of $m + 2n$ cosets of $2\Lambda_1$ in Λ_2.

2. Twists

In this section we consider "twists" of NET maps. That is, we consider how to obtain new NET maps from known ones by postcomposing with homeomorphisms. Suppose that $g\colon S^2 \to S^2$ is a NET map. Also suppose that $h\colon S^2 \to S^2$ is an orientation-preserving homeomorphism such that $h(P_g) \subseteq g^{-1}(P_g)$. Then the map $f = h \circ g$ is a NET map if it has at least four postcritical points. Indeed, it is an orientation-preserving branched map, the local degree at each of its critical points is 2, and its set of postcritical points is contained in $h(P_g)$, a set with four elements. If f is a NET map, then in the usual notation, $P_1(f) = P_1(g) = P_2(g)$ and $P_2(f) = h(P_2(g))$.

We continue the discussion of the previous paragraph by considering conditions under which the map $f = h \circ g$ has at least four postcritical points. We begin with the observation that if every element of P_g is the image under g of a critical point of g, then every element of $h(P_g)$ is the image under f of a critical point of f, and so f has at least four postcritical points. Statement (2) of Lemma 1.3 shows that $g^{-1}(P_g)$ contains exactly four points which are not critical points of g. So if some point x of P_g is not the image under g of a critical point of g, then $g^{-1}(x)$ contains at most four points and so $\deg(g) \leq 4$. We conclude that if $\deg(g) \geq 5$, then f has at least four postcritical points. If $\deg(g) = 3$, then the preimage under g of every element of P_g contains three points counting multiplicity. There cannot be two critical points in such a preimage because then the preimage would have at least four points counting multiplicity. It easily follows that every such preimage contains one point which is critical and one point which is not. So if $\deg(g) = 3$, then f has at least four postcritical points. Thus if either $\deg(g) = 3$ or $\deg(d) \geq 5$, then $f = h \circ g$ has at least four postcritical points. This may fail if either $\deg(g) = 2$ or $\deg(g) = 4$.

In this paragraph we consider the converse to the discussion of the previous two paragraphs. Let f be a NET map with $P_1 = P_1(f)$ and $P_2 = P_2(f)$ as usual. Let $h\colon S^2 \to S^2$ be any orientation-preserving homeomorphism which maps P_1 to P_2. Let $g = h^{-1} \circ f$. Then $\deg(g) = \deg(f) \geq 2$, the local degree of g at each of its critical points is 2, and the postcritical points of g are contained in $h^{-1}(P_2) = P_1$, a set with four elements containing no critical points of g. This means that g is a Euclidean Thurston map.

We have proved the following theorem.

Theorem 2.1. (1) *If $g\colon S^2 \to S^2$ is a NET map and $h\colon S^2 \to S^2$ is an orientation-preserving homeomorphism such that $h(P_g) \subseteq g^{-1}(P_g)$, then $f = h \circ g$ is a NET map if it has at least four postcritical points.*

(2) *Let f be a NET map with $P_1 = P_1(f)$ and $P_2 = P_2(f)$ as usual. Let $h\colon S^2 \to S^2$ be any orientation-preserving homeomorphism with $h(P_1) = P_2$. Then $f = h \circ g$, where $g\colon S^2 \to S^2$ is a Euclidean Thurston map with $P_g = P_1$ and $P_2 \subseteq g^{-1}(P_g)$, so that $h(P_g) \subseteq g^{-1}(P_g)$.*

3. Construction of examples

Let g be a NET map in the setting of Section 1. Let $h\colon S^2 \to S^2$ be an orientation-preserving homeomorphism such that $h(P_g) \subseteq g^{-1}(P_g)$. Statement (1) of Theorem 2.1 implies that the map $f = h \circ g$ is a NET map if it has at least four postcritical points. Also suppose that g is the subdivision map of a finite subdivision rule $\mathcal{Q}$ and that h maps the 1-skeleton of S^2 into the 1-skeleton of its first subdivision $\mathcal{Q}(S^2)$, taking vertices of S^2 to vertices of $\mathcal{Q}(S^2)$. Then f is the subdivision map of a finite subdivision rule $\mathcal{R}$. The subdivision complex of $\mathcal{R}$ is S^2 with cell structure the image under h of the original cell structure. Even though the subdivision complexes of $\mathcal{Q}$ and $\mathcal{R}$ are probably different, their first subdivisions are identical. As noted in the introduction, if g is Euclidean and if h does not stabilize the postcritical set of g, then the orbifold structure of S^2 for f is hyperbolic. These observations allow us to easily construct finite subdivision rules whose subdivision maps are NET maps whose orbifolds are hyperbolic.

Example 3.1. This takes us to our main example. It will be a NET map of the form $f = h \circ g$, where g is Euclidean. In the process of defining g and f, we will show that each inherits the additional structure of being a subdivision map for a finite subdivision rule. To define f we first define g and then we define h.

We begin the definition of g by setting $\Lambda_2 = \mathbb{Z}^2$. Let $\mathcal{S}_2$ be the tiling of the plane by 2×1 rectangles so that four rectangles meet at every lattice point (x, y) for which x is even as in Figure 2. Since every such rectangle contains six elements of Λ_2, every such rectangle should be viewed as a hexagon rather than a quadrilateral. Recall that Γ_i is the group generated by 180 degree rotations about the lattice points of Λ_i for $i \in \{1, 2\}$. Every such rectangle is a fundamental domain for the action of Γ_2 on $\mathbb{R}^2$. Let F_2 be the rectangle which has as corners $(0, 0)$, $(2, 0)$ and $(0, 1)$.

Let $\Lambda_1 = \langle (2, -1), (0, 5) \rangle$, the sublattice of Λ_2 generated by $(2, -1)$ and $(0, 5)$. A fundamental domain F_1 for the action of Γ_1 on $\mathbb{R}^2$ is hatched in Figure 2. We give

F_1 a cell structure so that the boundary of F_1 is its 1-skeleton and its vertices are at $(0,0)$, $(2,-1)$, $(4,-2)$, $(4,3)$, $(2,4)$ and $(0,5)$. We regard the hatched region in Figure 2 as a subdivision of F_1. Let $\mathcal{S}_1$ be the tiling of the plane by the images of F_1 under the elements of Γ_1.

We next construct an identification map $\phi\colon \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$ for g. Because we wish to preserve cell structure, instead of directly using an affine automorphism of $\mathbb{R}^2$ as in Section 1, we proceed as follows.

Let $j \in \{1,2\}$. Let $T_j = \mathbb{R}^2/2\Lambda_j$. Let $q_j\colon \mathbb{R}^2 \to T_j$ be the canonical quotient map from $\mathbb{R}^2$ to T_j, and let $p_j\colon T_j \to \mathbb{R}^2/\Gamma_j$ be the canonical quotient map from T_j to $\mathbb{R}^2/\Gamma_j$. The tiling $\mathcal{S}_j$ induces a tiling of $\mathbb{R}^2/\Gamma_j$ with one tile. Because F_2 is cellularly homeomorphic to F_1 in a way which respects the edge pairings induced by Γ_2 and Γ_1, there exists an orientation-preserving cellular homeomorphism $\phi\colon \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$ which maps $p_2(q_2(0,0))$ to $p_1(q_1(0,0))$. Such a homeomorphism ϕ can be constructed as follows. Define ϕ to map the four points $p_2(q_2(1,0))$, $p_2(q_2(0,0))$, $p_2(q_2(0,1))$ and $p_2(q_2(1,1))$ to the four points $p_1(q_1(2,-1))$, $p_1(q_1(0,0))$, $p_1(q_1(0,5))$ and $p_1(q_1(2,4))$ in order. The image of ∂F_2 in $\mathbb{R}^2/\Gamma_2$ is an arc joining the first four points in order, and the image of ∂F_1 in $\mathbb{R}^2/\Gamma_1$ is an arc joining the second four points in order. We extend ϕ to a homeomorphism from the first arc to the second arc. Finally, we extend this map to an orientation-preserving homeomorphism $\phi\colon \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$. We use this homeomorphism to identify these two spaces and we identify the result with S^2. With this identification, the set of branch points of p_1 equals the set of branch points of p_2. This identification map is isotopic to the one induced by the linear automorphism of $\mathbb{R}^2$ whose matrix with respect to the standard basis is $\left[\begin{smallmatrix} 2 & 0 \\ -1 & 5 \end{smallmatrix}\right]$.

Let $\widetilde{g}\colon T_1 \to T_2$ be the canonical map, and let $g\colon S^2 \to S^2$ be the map which it induces. Then g is a Euclidean Thurston map. Its postcritical set P_g is the set of branch points of p_1 and p_2. It is also the subdivision map of a finite subdivision rule $\mathcal{Q}$. The subdivision complex of $\mathcal{Q}$ is S^2 with cell structure the push forward of $\mathcal{S}_1$ under $p_1 \circ q_1$. This is the same as the push forward of $\mathcal{S}_2$ under $p_2 \circ q_2$. Its first subdivision is the push forward of $\mathcal{S}_2$ under $p_1 \circ q_1$. Figure 3 indicates the action of g. The right portion of Figure 3 shows the push forward of $\mathcal{S}_1$ under $p_1 \circ q_1$ in S^2 and the left portion of Figure 3 shows the push forward of $\mathcal{S}_2$ under $p_1 \circ q_1$ in S^2. Most of the vertices in the left portion are labeled with preimages in $\mathbb{R}^2$.

Thus far we have the map g. For $h\colon S^2 \to S^2$ we choose an orientation-preserving homeomorphism which takes the 1-skeleton of the push forward of $\mathcal{S}_1$ into the 1-skeleton of the push forward of $\mathcal{S}_2$ such that h fixes the images of $(1,0),(0,0),\dots,(0,5),(1,5)$ and h maps the image of $(2,4)$ to the image of $(2,5)$ and the image of $(2,-1)$ to the image of $(2,0)$. Let $f = h \circ g$. The action of f is indicated in Figure 4. The map f preserves the edge labels which are given. We see that $h(P_g) \subseteq g^{-1}(P_g)$. As discussed in the beginning of this section, it follows that f is a NET map and it is the subdivision map of a finite subdivision rule $\mathcal{R}$. The single tile type of $\mathcal{R}$ is a hexagon. The subdivision of the hexagon is shown in Figure 5. It is easy to check that $\mathcal{R}$ has bounded valence and that the mesh of $\mathcal{R}$ approaches 0 combinatorially. The mapping scheme of f is shown in Figure 6, where points are labeled by their preimages in F_1 under $p_1 \circ q_1$. This example was designed to make it difficult to determine the invariant multicurves for possible Thurston obstructions.

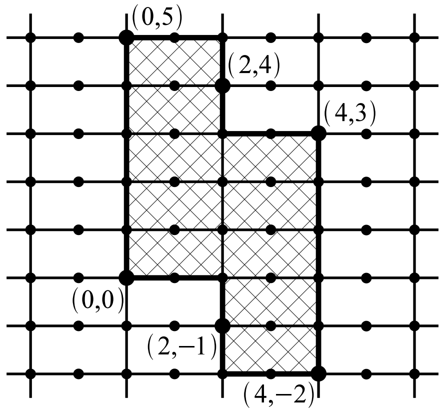

FIGURE 2. A fundamental domain for Γ_1 of the main example.

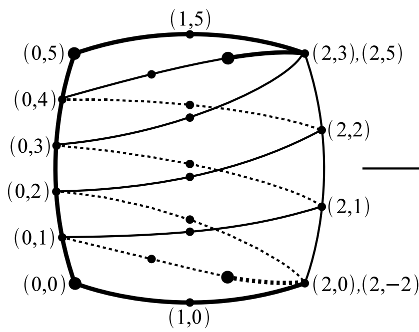

FIGURE 3. The map g of the main example.

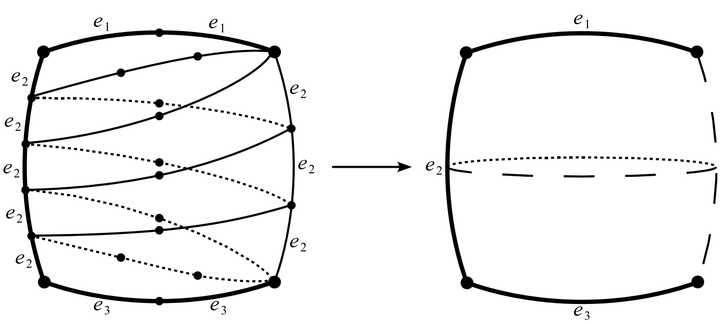

FIGURE 4. The map f of the main example.

Example 3.2. We begin this example by finding a Lattès map g which as a Euclidean Thurston map has lattices $\Lambda_2 = \left\langle 1, \frac{1+\sqrt{-3}}{2} \right\rangle$, $\Lambda_1 = \sqrt{-3}\Lambda_2$ and identification map induced by the linear automorphism given by $\Phi(z) = \sqrt{-3}z$. Because Φ is a conformal affine map, our Riemann spheres $\mathbb{C}/\Gamma_1$ and $\mathbb{C}/\Gamma_2$ have the same

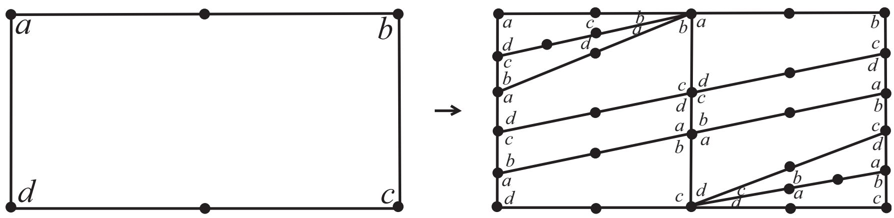

FIGURE 5. The subdivision of the tile type for the finite subdivision rule $\mathcal{R}$ of the main example.

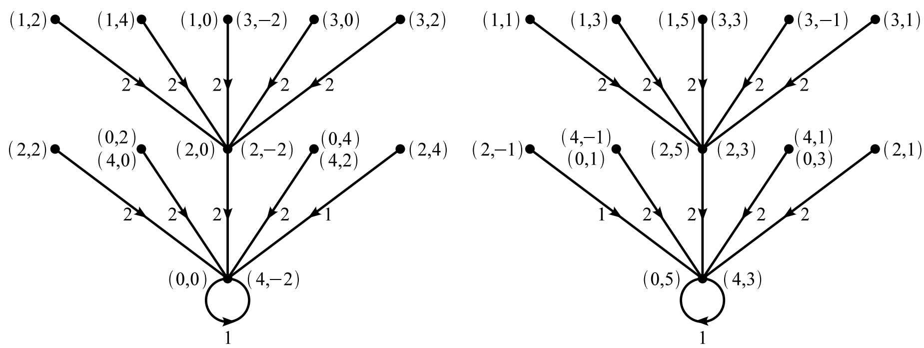

FIGURE 6. The mapping scheme for the main example.

conformal structure. Let $\tau = \frac{1+\sqrt{-3}}{2}$; see Figure 7. The parallelogram F_2 with vertices 0, 2, τ and $2+\tau$ is a fundamental domain for the action of Γ_2. The image F_1 of F_2 under Φ is a fundamental domain for the action of Γ_1. Both F_1 and F_2 are shown in Figure 7.

The matrix of Φ with respect to the ordered basis $(1,\tau)$ of $\mathbb{R}^2$ is $\left[\begin{smallmatrix}-1 & -2\\ 2 & 1\end{smallmatrix}\right]$. So $\Phi(\alpha) \equiv \alpha \mod 2\Lambda_2$ for every $\alpha \in \Lambda_2$. The four elements of Λ_2 in the interior of F_1 are $\alpha_0 = -2+2\tau$, $\alpha_1 = -3+4\tau$, $\alpha_\tau = -2+3\tau$ and $\alpha_{1+\tau} = -1+\tau$. The images of these four lattice points under $p_1 \circ q_1$ are the critical points of g. One verifies that $\alpha_\lambda \equiv \lambda \mod 2\Lambda_2$ for every $\lambda \in \{0,1,\tau,1+\tau\}$. Using the fact that Φ is the identity map modulo $2\Lambda_2$, it follows that $g(p_1(q_1(\alpha_\lambda))) = p_2(q_2(\lambda))$ for every $\lambda \in \{0,1,\tau,1+\tau\}$.

Let $\omega = \frac{-1+\sqrt{-3}}{2}$. Because the map $z \mapsto \omega z$ stabilizes Λ_1, it determines an analytic homeomorphism from $\widehat{\mathbb{C}} = \mathbb{C}/\Gamma_1$ to itself. In the same way, the map $z \mapsto \omega z$ determines an analytic homeomorphism from $\widehat{\mathbb{C}} = \mathbb{C}/\Gamma_2$ to itself. Because the map $z \mapsto \omega z$ commutes with the map Φ which induces the identification map, our two maps from $\widehat{\mathbb{C}}$ to itself are equal. Let $\psi\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be this map, which is a Möbius transformation. The map $z \mapsto \omega z$ permutes the cosets of $2\Lambda_1$ in Λ_1, and it permutes the cosets of $2\Lambda_1$ in Λ_2. Hence it permutes the cosets $\pm\alpha_\lambda + 2\Lambda_1$ for $\lambda \in \{0,1,\tau,1+\tau\}$. The congruence $\alpha_\lambda \equiv \lambda \mod 2\Lambda_2$ implies that $\pm\alpha_\lambda + 2\Lambda_1 \subseteq \lambda + 2\Lambda_2$ for $\lambda \in \{0,1,\tau,1+\tau\}$. So the action of $z \mapsto \omega z$ on these cosets of $2\Lambda_1$ is the same as its action on these cosets of $2\Lambda_2$. This gives the following congruences

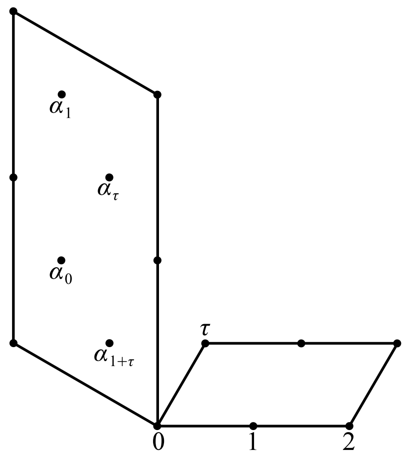

FIGURE 7. Computing the Lattès map g.

modulo $2\Lambda_1$,

$$\omega\alpha_0 \equiv \pm\alpha_0, \quad \omega\alpha_1 \equiv \pm\alpha_{1+\tau}, \quad \omega\alpha_{1+\tau} \equiv \pm\alpha_\tau, \quad \omega\alpha_\tau \equiv \pm\alpha_1.$$

So ψ is a Möbius transformation with order 3 which fixes $p_1(q_1(0))$ and $p_1(q_1(\alpha_0))$ and cyclically permutes $p_1(q_1(\alpha_1))$, $p_1(q_1(\alpha_{1+\tau}))$ and $p_1(q_1(\alpha_\tau))$.

We identify $\mathbb{C}/\Gamma_1$ with $\widehat{\mathbb{C}}$ so that $0 \in \mathbb{C}$ maps to ∞, the point α_0 maps to 0 and $\sqrt{-3}$ maps to $-\frac{1}{2}$. Since ψ is a Möbius transformation with order 3 which fixes 0 and ∞ and our identifications preserve orientation, $\psi(z) = \omega z$. Since the maps $z \mapsto \sqrt{-3}z$ and $z \mapsto \omega z$ commute, so do g and ψ. Hence $g(\omega z) = \omega g(z)$ for every $z \in \widehat{\mathbb{C}}$. Because $\sqrt{-3}\cdot 0 = 0$ and 0 maps to ∞ in $\widehat{\mathbb{C}}$, we see that $g(\infty) = \infty$. Because α_0 maps to $0 \in \widehat{\mathbb{C}}$, the point 0 is a critical point of g with $g(0) = \infty$. Because $\sqrt{-3}$ maps to $-\frac{1}{2} \in \widehat{\mathbb{C}}$, it follows that $-\frac{1}{2}$ is fixed by g and it is the image of a critical point of g.

Now we finally determine g. Since the square of the modulus of $\sqrt{-3}$ is 3, the map g is a cubic rational function. Since it has poles at 0 and ∞ with 0 being a critical point, we may assume that its denominator is z^2. Because $g(\omega z) = \omega g(z)$, we may assume that its numerator is $az^3 + b$ for some $a, b \in \mathbb{C}$: $g(z) = \frac{az^3+b}{z^2}$. Since $-\frac{1}{2}$ is fixed by g, the polynomial $2az^3 + z^2 + 2b$ has a root at $-\frac{1}{2}$: $-\frac{a}{4} + \frac{1}{4} + 2b = 0$. Hence $b = \frac{1}{8}(a-1)$ and $2az^3 + z^2 + 2b = (2z+1)(az^2 + \frac{1}{2}(1-a)z + \frac{1}{4}(a-1))$. Since $-\frac{1}{2}$ is the image under g of a critical point, the second factor is a square, and so its discriminant is 0: $0 = \frac{1}{4}(a-1)^2 - a(a-1) = -\frac{1}{4}(3a+1)(a-1)$. If $a = 1$, then $b = 0$, which is impossible. Thus $a = -\frac{1}{3}$, $b = -\frac{1}{6}$ and $g(z) = -\frac{1}{6}\frac{2z^3+1}{z^2}$.

The critical points of g are 0, 1, ω and $\overline{\omega}$. These are mapped to ∞, $-\frac{1}{2}$, $-\frac{1}{2}\omega$ and $-\frac{1}{2}\overline{\omega}$ by g in order. The map $h\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ defined by $h(z) = -\frac{1}{2z}$ maps the latter four points to the former four points. So $f(z) = h(g(z)) = \frac{3z^2}{2z^3+1}$ is a NET map which maps its set of four critical points bijectively to itself. Figure 8 shows the mapping scheme of f. This is the rational function which appears in the proof of statement 2 of Theorem 1.1 of [1].

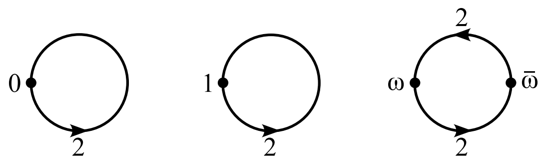

FIGURE 8. The mapping scheme of $f(z) = \frac{3z^2}{2z^3+1}$.

In [5] Russell Lodge computes the slope function σ_f of f introduced in Section 5. His methods are different from those of Section 5; see Remark 7.2 for a bit more on this.

4. Pullbacks of simple closed curves

In this section we investigate pullbacks of simple closed curves under NET maps. We begin by reviewing well-known facts about simple closed curves in tori and 4-punctured spheres. A good reference for this material is the book [4] by Farb and Margalit; see Propositions 1.5 and 2.6 in [4].

Let $f\colon S^2 \to S^2$ be a NET map. We maintain the setting of Section 1.

Let $j \in \{1, 2\}$. Let (λ_j, μ_j) be an ordered basis of Λ_j. Let p and q be relatively prime integers. The universal covering map q_j maps every line in $\mathbb{R}^2$ with parametrization of the form $(x, y) = v + t(q\lambda_j + p\mu_j)$ to a simple closed curve in T_j, which is said to have slope $\frac{p}{q} \in \widehat{\mathbb{Q}} = \mathbb{Q} \cup \{\infty\}$. The resulting map from slopes to curves establishes a bijective correspondence between $\widehat{\mathbb{Q}}$ and the set of nontrivial homotopy classes of simple closed curves in T_j.

A simple closed curve in $S^2 \setminus P_j$ is peripheral if it is homotopic to a very small closed curve around an element of P_j. A simple closed curve in $S^2 \setminus P_j$ is essential if it is not null homotopic. If γ is an essential, nonperipheral simple closed curve in $S^2 \setminus P_j$, then $p_j^{-1}(\gamma)$ consists of two disjoint simple closed curves in T_j. They are not null homotopic. Being disjoint, they are homotopic to each other and hence have the same slope. This establishes a bijection between $\widehat{\mathbb{Q}}$ and the set of homotopy classes of essential, nonperipheral simple closed curves in $S^2 \setminus P_j$. We must take care that this bijection is very uncanonical.

Here is a slightly different point of view. We use the fact that $p_j \circ q_j\colon \mathbb{R}^2 \setminus \Lambda_j \to S^2 \setminus P_j$ is a regular covering map with a group of deck transformations Γ_j. Let α be an essential, nonperipheral simple closed curve in $S^2 \setminus P_j$. Suppose that α has a lift to $\mathbb{R}^2$ which joins points x and y. Since α is not null homotopic, $x \neq y$. Because the deck transformations of $p_j \circ q_j$ are Euclidean isometries and because this lift of α is also a lift of a closed curve in T_j, we have that $y = \gamma(x)$ for some translation γ in Γ_j. It follows that the slope of the line through x and y is independent of the choice of lift of α to $\mathbb{R}^2$. The slope of such a line relative to the ordered basis (λ_j, μ_j) of Λ_j is the slope of α.

If γ is an essential, nonperipheral simple closed curve in $S^2 \setminus P_j$, then γ separates two points, x and y, of P_j from the other two points of P_j. We call an arc in S^2 joining x and y which is disjoint from γ a *core arc* for γ. Giving a homotopy class of essential, nonperipheral simple closed curves in $S^2 \setminus P_j$ is equivalent to giving such a core arc.

In this paragraph we make a definition to prepare for the next theorem. Let A be a finite Abelian group. Let H be a subset of A which is the disjoint union of four inverse pairs $\{\pm h_1\}$, $\{\pm h_2\}$, $\{\pm h_3\}$ and $\{\pm h_4\}$. (It is possible that $h_i = -h_i$.) Let B be a subgroup of A such that A/B is cyclic, and let a be an element of A whose image in A/B generates A/B. Let n be the order of A/B. For every $k \in \{1, 2, 3, 4\}$ exactly one coset $ca + B$ of B in A contains either h_k or $-h_k$, where c is an integer with $0 \leq c \leq n/2$. Let c_1, c_2, c_3, c_4 be these four integers ordered so that $c_1 \leq c_2 \leq c_3 \leq c_4$. (The integer c_k need not correspond to $\pm h_k$.) We call c_1, c_2, c_3, c_4 the *coset numbers* for H relative to B and a or relative to B and the generator $a+B$ of A/B. We are interested in coset numbers when $A = \Lambda_2/2\Lambda_1$, where Λ_1 and Λ_2 are the lattices in Section 1. If λ and μ form a basis of Λ_2, then the image of λ in A generates a cyclic subgroup B and the image of μ in A is an element a whose image in A/B generates A/B. Recall from the end of Section 1 that $q_1^{-1}(p_1^{-1}(P_2)) \subseteq \Lambda_2$. Thus we may speak of coset numbers for $H = p_1^{-1}(P_2)$ relative to B and a. We also call these coset numbers the coset numbers for $q_1^{-1}(p_1^{-1}(P_2))$ relative to λ and μ. The coset number of $\eta \in q_1^{-1}(p_1^{-1}(P_2))$ is the smallest nonnegative integer c for which there exists an integer b such that $\pm\eta \in b\lambda + c\mu + 2\Lambda_1$.

This takes us to the main result of this section.

Theorem 4.1. *Let f be a NET map in the setting of Section* 1. *Let δ be an essential, nonperipheral simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$, where p and q are relatively prime integers. Let $\lambda = q\lambda_2 + p\mu_2 \in \Lambda_2$. Let d be the order of the image of λ in Λ_2/Λ_1. Let d' be the positive integer such that $dd' = |\Lambda_2/\Lambda_1| = \deg(f)$. Since p and q are relatively prime, there exists $\mu \in \Lambda_2$ such that λ and μ form another basis of Λ_2. Let c_1, c_2, c_3, c_4 be the coset numbers for the elements of $q_1^{-1}(p_1^{-1}(P_2))$ relative to λ and μ. Then the following statements hold.*

(1) *Every connected component of $f^{-1}(\delta)$ maps to δ with degree d.*
(2) *The number of essential, nonperipheral components in $f^{-1}(\delta)$ is $c_3 - c_2$.*
(3) *The number of peripheral components in $f^{-1}(\delta)$ is $c_2 - c_1 + c_4 - c_3$.*
(4) *The number of null homotopic components in $f^{-1}(\delta)$ is $c_1 - c_4 + d'$.*
(5) *The lines in $\mathbb{R}^2$ with slope $\frac{p}{q}$ relative to the basis (λ_2, μ_2) of Λ_2 which map under $p_1 \circ q_1$ to essential, nonperipheral simple closed curves in $S^2 \setminus P_2$ are exactly the Γ_1-translates of the lines with parametric forms $(x, y) = t\lambda + u\mu$ with parameter t and $c_2 < u < c_3$.*

Proof. Because δ has slope $\frac{p}{q}$, it is homotopic to the image in $S^2 \setminus P_2$ under $p_2 \circ q_2$ of a line in $\mathbb{R}^2$ with a parametrization of the form $(x, y) = v + t\lambda$. Without loss of generality, we assume that δ is the image in $S^2 \setminus P_2$ under $p_2 \cdot q_2$ of such a line. Because p and q are relatively prime, the line segment joining v and $v + 2\lambda$ maps injectively to T_2 and hence to $S^2 \setminus P_2$ except for its endpoints.

Such a line segment is shown in Figure 9 together with some of its Γ_2-translates, drawn as dashed line segments. The dots in Figure 9 are elements of Λ_2 with the larger ones being in Λ_1. Each of the smallest parallelograms bounded by solid line segments in Figure 9 is a fundamental domain for the action of Γ_2 on $\mathbb{R}^2$. The entire parallelogram subdivided by these small parallelograms is a fundamental domain F for the action of Γ_1 on $\mathbb{R}^2$. As such, it contains exactly one lift to $\mathbb{R}^2$ under $p_1 \circ q_1$ of every connected component of $f^{-1}(\delta)$.

Because the lift of f to $\mathbb{R}^2$ is the identity map, the line segment joining v and $v + 2\lambda$ is a lift of δ in F under the map $f \circ p_1 \circ q_1$. The Γ_2-translates of this lift

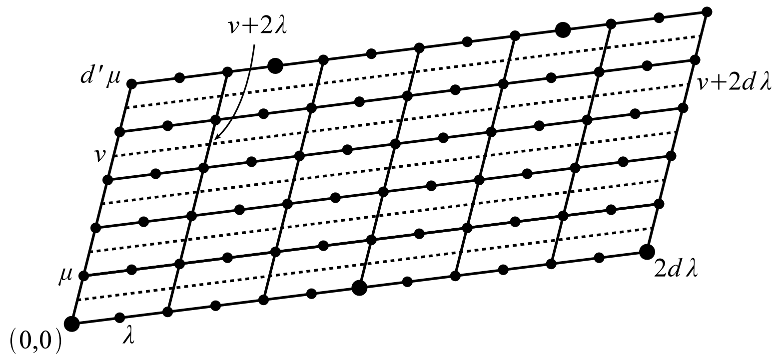

FIGURE 9. Proving Theorem 4.1.

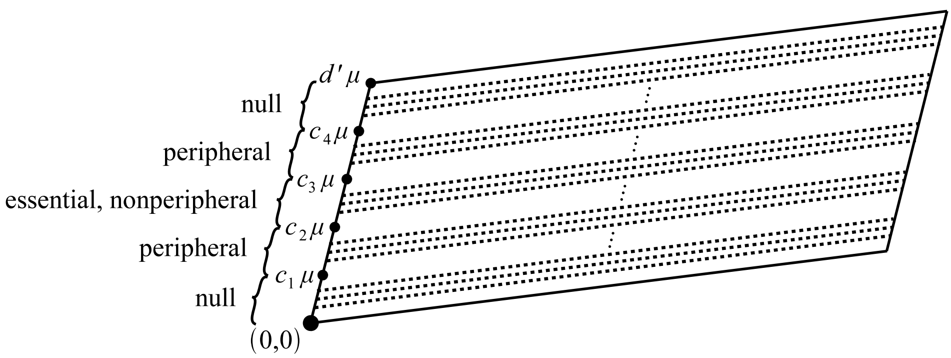

FIGURE 10. Illustrating Theorem 4.1.

are other lifts of δ. A concatenation of such line segments is the lift of a closed curve in $S^2 \setminus P_1$ if and only if the difference between its endpoints is an element of $2\Lambda_1$ which is not a nontrivial multiple of an element of $2\Lambda_1$. In other words, this difference is the smallest multiple of 2λ which lies in $2\Lambda_1$. This is the order of the image of 2λ in $2\Lambda_2/2\Lambda_1$, which equals the order d of the image of λ in Λ_2/Λ_1. This proves the first statement of Theorem 4.1.

Figure 10 illustrates statements (2) through (4). The coset numbers c_1, c_2, c_3, c_4 determine a partition of the line segment joining $(0,0)$ and $d'\mu$. The components of $f^{-1}(\delta)$ corresponding to the first and last of these subsegments are null homotopic. The components corresponding to the subintervals adjacent to these are peripheral. The remaining components are essential and nonperipheral.

Now we prove these statements. Since the restriction of f to every connected component of $f^{-1}(\delta)$ has degree d and $dd' = \deg(f)$, the number of these components is d'. Let α be one of the d' line segments in F which is a lift of a connected component of $f^{-1}(\delta)$. The action of Γ_1 on the boundary of F identifies two halves of the bottom of F by a rotation of order 2. In general, the identification of the top of F is only slightly more complicated because the top two corners of F are not necessarily elements of Λ_1. The two sides of F are identified by a translation.

Let U and V be the connected components of the complement of α in F. Both U and V map to open disks in S^2 under $p_1 \circ q_1$ and the image of α separates these two disks. So for the image of α to be essential and nonperipheral, two elements of U must map to distinct elements of P_2 and two elements of V must map to distinct elements of P_2. Thus the number of essential, nonperipheral components of $f^{-1}(\delta)$ is $c_3 - c_2$. This proves statement (2). Statements (3) and (4) can be proven similarly. Statement (5) is now clear.

This proves Theorem 4.1. □

The following lemma provides a way to compute the coset numbers c_1, c_2, c_3, c_4 in Theorem 4.1.

Lemma 4.2. *Maintain the setting of Theorem* 4.1*. Let r and s be integers, and let $\eta = r\lambda_2 + s\mu_2 \in \Lambda_2$. Let b and c be integers such that $c \geq 0$ and c is as small as possible such that $\pm\eta \in b\lambda + c\mu + 2\Lambda_1$. Then c, the coset number of η with respect to λ and μ, is the smallest nonnegative integer congruent to $\pm(pr - qs)$ modulo $2d'$.*

Proof. We begin by finding a useful basis of Λ_1. Not every element of Λ_1 is a multiple of λ. So there exist integers l and m with $m > 0$ such that $l\lambda + m\mu \in \Lambda_1$. Let μ' be an element of Λ_1 such that $\mu' = l\lambda + m\mu$ with $m > 0$ and m as small as possible. We claim that $d\lambda$ and μ' form a basis of Λ_1.

To prove this, let $\nu \in \Lambda_1$. It suffices to prove that ν is an integral linear combination of $d\lambda$ and μ'. There exist integers x and y such that $\nu = x\lambda + y\mu$. Subtracting an appropriate multiple of μ' from ν obtains an element $\nu' \in \Lambda_1$ with $\nu' = x'\lambda + y'\mu$ and $0 \leq y' < m$. The choice of m implies that $y' = 0$. Hence $\nu' = x'\lambda$. The choice of d implies that $d|x'$. Thus every element of Λ_1 is an integral linear combination of $d\lambda$ and μ', and so $d\lambda$ and μ' form a basis of Λ_1. Since the determinant of the matrix $\left[\begin{smallmatrix} d & l \\ 0 & m \end{smallmatrix}\right]$ is $|\Lambda_2/\Lambda_1| = \deg(f)$, it follows that $m = d'$.

Now let x and y be the integers such that $\eta = x\lambda + y\mu$. We seek the nonnegative integer c which is as small as possible such that there exists an integer b for which $\pm\eta \in b\lambda + c\mu + 2\Lambda_1$. Equivalently, $b\lambda + c\mu = 2\lambda' \pm (x\lambda + y\mu)$ for some $\lambda' \in \Lambda_1$. The previous paragraph shows that λ' is an integral linear combination of $d\lambda$ and $\mu' = l\lambda + d'\mu$. It follows that c is the smallest nonnegative integer congruent to $\pm y$ modulo $2d'$.

So now we determine y. Let t and u be the integers such that $\mu = t\lambda_2 + u\mu_2$. Using the fact that $\eta = r\lambda_2 + s\mu_2 = x\lambda + y\mu$ and multilinearity of determinants, we see that

$$\begin{vmatrix} q & r \\ p & s \end{vmatrix} = x \begin{vmatrix} q & q \\ p & p \end{vmatrix} + y \begin{vmatrix} q & t \\ p & u \end{vmatrix} = \pm y,$$

the last determinant being ± 1 because λ and μ form a basis of Λ_2. So $y = \pm(pr - qs)$. Therefore c is the smallest nonnegative integer congruent to $\pm(pr - qs)$ modulo $2d'$.

This proves Lemma 4.2. □

Remark 4.3. The element η in Lemma 4.2 can be any element of Λ_2. However, if r and s are relatively prime, then we have the following interpretation. If r and s are relatively prime, then η determines a simple closed curve γ in T_2 with slope $\frac{s}{r}$. Let $\widetilde{\delta}$ be a lift of δ to T_2, a simple closed curve with slope $\frac{p}{q}$. As in Section 1.2.3 of [4] by Farb and Margalit, the intersection number $\iota(\widetilde{\delta}, \gamma)$ is $|pr - qs|$. So in this case c is the smallest nonnegative integer which is congruent to $\pm$ this intersection number modulo $2d'$.

q mod 20	$2p+q$ mod 5	deg	c_1,c_2,c_3,c_4	essl	perl	null
±1, ±3, ±5, ±7, ±9	±1, ±2	10	0, 0, 1, 1	1	0	0
±1, ±9	0	2	0, 1, 4, 5	3	2	0
±3, ±7	0	2	0, 2, 3, 5	1	4	0
0, ±2, ±4, ±6, ±8, 10	±1, ±2	5	0, 0, 2, 2	2	0	0
±2	0	1	0, 2, 8, 10	6	4	0
±4	0	1	0, 0, 6, 6	6	0	4
±6	0	1	0, 4, 6, 10	2	8	0
±8	0	1	0, 0, 2, 2	2	0	8

TABLE 1. Degrees, coset numbers and numbers of components for the main example.

We continue with one more general observation about computations. Lemma 4.2 implies that the coset numbers c_1, c_2, c_3, c_4 depend in a simple way on p and q and that to determine them it suffices to determine them modulo $2d'$. So suppose that the coset numbers c_1, c_2, c_3, c_4 arise from slope $\frac{p}{q}$. Let $\frac{p'}{q'}\in\widehat{\mathbb{Q}}$ and suppose that, just as for $\frac{p}{q}$, the order of the image of $q'\lambda_2+p'\mu_2$ in Λ_2/Λ_1 is d. Also suppose that there exists an integer u which is a unit modulo $2d'$ such that $p'\equiv up \mod 2d'$ and $q'\equiv uq \mod 2d'$. The point of this discussion is that if c_1', c_2', c_3', c_4' are the coset numbers as in Theorem 4.1 for $\frac{p'}{q'}$, then c_1', c_2', c_3', c_4' are congruent to $\pm uc_1$, $\pm uc_2$, $\pm uc_3$, $\pm uc_4$ modulo $2d'$, not necessarily in order.

Now we apply Theorem 4.1 to the main example. Table 1 displays the results. The first column gives q modulo 20. The second column gives $2p+q$ modulo 5. The third column gives the degree d of the restriction of f to every connected component of the inverse image of an essential simple closed curve in $S^2\setminus P_f$ with slope $\frac{p}{q}$. The next column gives the coset numbers c_1, c_2, c_3, c_4 which appear in Theorem 4.1. The last three columns give the numbers of essential nonperipheral components, peripheral components and null homotopic components in this inverse image.

For the main example $\Lambda_2=\mathbb{Z}^2$ and $\Lambda_1=\langle(2,-1),(0,5)\rangle$. Let p and q be relatively prime integers. We begin with an essential simple closed curve δ in $S^2\setminus P_f$ with slope $\frac{p}{q}$. Let d be the degree of the restriction of f to any connected component of $f^{-1}(\delta)$.

Statement (1) of Theorem 4.1 implies that d is the smallest positive integer such that there exist integers x and y for which $x(2,-1)+y(0,5)=d(q,p)$. Solving for x and y, we find that

$$x=\frac{dq}{2}\quad\text{and}\quad y=\frac{d(2p+q)}{10}.$$

Suppose that $q\equiv 0 \mod 2$. Then x is an integer for every d. For y to be an integer, we see that the only condition on d is that $d\equiv 0 \mod 5$ if $2p+q\not\equiv 0 \mod 5$. Now suppose that $q\not\equiv 0 \mod 2$. Considering x shows that $d\equiv 0 \mod 2$. Considering y shows that, as before, $d\equiv 0 \mod 5$ if $2p+q\not\equiv 0 \mod 5$. This leads to the values of d given in Table 1.

Now we determine the remaining entries of the table. The elements $(0,0)$, $(2,0)$, $(0,5)$ and $(2,5)$ of Λ_2 map to the four elements of P_2 under $p_1\circ q_1$. To apply Lemma 4.2, we calculate $pr-qs$ for these four elements and obtain 0, $2p$, $-5q$ and $2p-5q$. In what follows, we find the reduced residues of ±1 times these values modulo $2d'$. It is then easy to determine the remaining entries in Table 1 using Theorem 4.1.

First suppose that $d = 10$ and $d' = 1$. According to Table 1, the integer q is odd. So reducing 0, $2p$, $-5q$ and $2p - 5q$ modulo 2 yields 0, 0, 1, 1. This completes the computation for $d = 10$.

Next suppose that $d = 5$ and $d' = 2$. Then q is even and p is odd. Regardless of whether $q \equiv 0 \mod 4$ or $q \equiv 2 \mod 4$, our four values reduce to 0, 0, 2, 2 modulo 4. This completes the computation for $d = 5$.

Next suppose that $d = 2$ and $d' = 5$. Table 1 shows that $q \not\equiv 0 \mod 2$. It also shows that $2p + q \equiv 0 \mod 5$, and so $q \not\equiv 0 \mod 5$. So q is a unit modulo 10. Suppose that $q \equiv 1 \mod 5$. Then $p \equiv 2 \mod 5$. So up to a sign, our values reduce to 0, 1, 4, 5 modulo 10. According to the observation after Lemma 4.2, multiplying q by a unit modulo 10 amounts to multiplying these four values by the same unit. The units modulo 10 are represented by ± 1 and ± 3. Since multiplication by -1 does nothing, we need only consider multiplication by 3. We obtain 0, 2, 3, 5. This completes the computation for $d = 2$.

Finally, suppose that $d = 1$ and $d' = 10$. Table 1 shows that $q \equiv 0 \mod 2$ and $2p + q \equiv 0 \mod 5$. Hence $p \not\equiv 0 \mod 2$ and $q \not\equiv 0 \mod 5$. Up to multiplication by a unit, either $q \equiv 2 \mod 20$ or $q \equiv 4 \mod 20$. Suppose that $q \equiv 2 \mod 20$. Then $p \equiv -1, 9 \mod 20$. So up to a sign, our four values are 0, 2, 8, 10. If $q \equiv 4 \mod 20$, then $p \equiv 3, 13 \mod 20$. Now we obtain 0, 0, 6, 6. The units modulo 20 are ± 1, ± 3, ± 7, ± 9. Up to a sign, ± 1 and ± 9 stabilize both $\{0, 2, 8, 10\}$ and $\{0, 0, 6, 6\}$. Multiplying by 3 yields $\{0, 4, 6, 10\}$ and $\{0, 0, 2, 2\}$. This completes the computation for $d = 1$.

5. The slope function

Let f be a NET map in the setting of Section 1. We let o denote the union of the classes of inessential and peripheral curves in $S^2 \setminus P_2$, and we define a *slope function* $\sigma_f \colon \widehat{\mathbb{Q}} \to \widehat{\mathbb{Q}} \cup \{o\}$ as follows. As at the beginning of Section 4, we fix an ordered basis (λ_2, μ_2) of Λ_2 by which we define slopes of essential, nonperipheral simple closed curves in $S^2 \setminus P_2$. Let p and q be relatively prime integers, so that $\frac{p}{q} \in \widehat{\mathbb{Q}}$. Let δ be an essential, nonperipheral simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$. Every connected component of $f^{-1}(\delta)$ is in $S^2 \setminus P_2$. If no connected component of $f^{-1}(\delta)$ is essential and nonperipheral in $S^2 \setminus P_2$, then set $\sigma_f(\frac{p}{q}) = o$. Suppose that some connected component α of $f^{-1}(\delta)$ is essential and nonperipheral in $S^2 \setminus P_2$. In this case we let $\sigma_f(\frac{p}{q})$ be the slope of α in $S^2 \setminus P_2$. This defines σ_f, independent of the choices of δ and α. The main goal of this section is to describe a method to compute σ_f.

According to statement (2) of Theorem 2.1, it is possible to factor f as a composition $f = h \circ g$ of functions, where $g \colon S^2 \to S^2$ is a Euclidean Thurston map and $h \colon S^2 \to S^2$ is any orientation-preserving homeomorphism such that $h(P_1) = P_2$. We choose h so that h fixes $P_1 \cap P_2$.

In this paragraph we construct four arcs in S^2 and their inverse images in $\mathbb{R}^2$. Suppose that $P_1 = \{x_1, x_2, x_3, x_4\}$. For every $k \in \{1, 2, 3, 4\}$ let β_k be an arc in S^2 which joins x_k and $h(x_k)$. Because h fixes $P_1 \cap P_2$, we may choose these arcs so that they are disjoint. Every connected component of $q_j^{-1}(p_j^{-1}(\beta_k))$ contains exactly one element of Λ_j for $j \in \{1, 2\}$. If β_k is nontrivial, then the restriction of

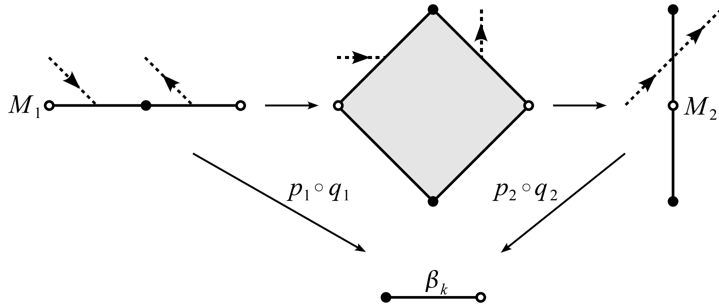

FIGURE 11. The map ω near the spin mirror M_1.

$p_j \circ q_j$ to such a component is a branched covering map onto β_k with degree 2. We call every such connected component a *spin mirror* for $p_j \circ q_j$. This terminology will be explained soon. We emphasize that these spin mirrors depend on the choice of four arcs in S^2 and these arcs depend in turn on the homeomorphism h. We furthermore assume that every spin mirror for $p_1 \circ q_1$ is a piecewise linear arc in $\mathbb{R}^2$.

Let $B = \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4$, and let $B_j = q_j^{-1}(p_j^{-1}(B))$ for $j \in \{1, 2\}$. Recall that $p_j \circ q_j \colon \mathbb{R}^2 \to S^2$ is a branched covering map ramified at exactly the points of Λ_j with local degree 2 at every such point. Hence the restriction of $p_1 \circ q_1$ to $\mathbb{R}^2 \setminus B_1$ is a covering map onto $S^2 \setminus B$ which is equivalent to the covering map obtained by restricting $p_2 \circ q_2$ to $\mathbb{R}^2 \setminus B_2$. This means that there exists a homeomorphism $\omega \colon \mathbb{R}^2 \setminus B_1 \to \mathbb{R}^2 \setminus B_2$ such that $p_2 \circ q_2 \circ \omega = p_1 \circ q_1$ as functions from $\mathbb{R}^2 \setminus B_1$ to $S^2 \setminus B$.

In general the map ω does not extend to a continuous map from $\mathbb{R}^2$ to $\mathbb{R}^2$, but nonetheless it does determine a bijection from the set of spin mirrors in B_1 to the set of spin mirrors in B_2. Suppose that $k \in \{1, 2, 3, 4\}$ such that β_k is nontrivial. Let M_1 be a spin mirror in B_1 which maps to β_k, and let M_2 be the corresponding spin mirror in B_2. Both M_1 and M_2 are branched double covering spaces of β_k with M_1 branched over the point of P_1 in β_k and M_2 branched over the point of P_2 in β_k. Figure 11 illustrates the behavior of ω near M_1. Figure 11 is an idealized drawing. The spin mirrors need not be line segments, and ω need not be piecewise linear. We might think in terms of cutting $\mathbb{R}^2$ open along M_1. We obtain a hole bounded by four arcs as in the middle of Figure 11. The inverse operation is to identify two pairs of adjacent arcs in the boundary of this hole. To obtain M_2, we identify the other two pairs of adjacent arcs. We might imagine a photon traveling through $\mathbb{R}^2$ and crossing M_2 as indicated by the dashed line segment in the rightmost part of Figure 11. The photon's inverse image under ω is indicated by the two dashed line segments in the leftmost part of Figure 11. When the photon's inverse image strikes M_1, it spins about the center of M_1 and thereby reverses direction. This property gives spin mirrors their name.

Now we consider computing the slope function σ_f. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$. Let L be a line in $\mathbb{R}^2 \setminus \Lambda_2$ with slope $\frac{p}{q}$ relative to the ordered basis (λ_2, μ_2) of Λ_2. Then $\delta = p_2(q_2(L))$ is an essential, nonperipheral simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$. Theorem 4.1 provides the means to determine whether or not $f^{-1}(\delta)$ contains an essential, nonperipheral component; that is, it allows us to determine whether or not $\sigma_f(\frac{p}{q}) = o$. So suppose that $f^{-1}(\delta)$ contains an essential, nonperipheral component α. Because the lift of f to $\mathbb{R}^2$ is the identity map, we may, and do, assume that L is one connected component of $q_1^{-1}(p_1^{-1}(\alpha))$. Let $\lambda = q\lambda_2 + p\mu_2$. Let d be the degree with which f maps α to δ. Theorem 4.1 shows that d is the order of the image of λ in Λ_2/Λ_1. Choose $v \in L$ so that v is not contained in a spin mirror for $p_1 \circ q_1$. Then the segment of L joining v and $v + 2\lambda$ is a lift to $\mathbb{R}^2$ of δ under $f \circ p_1 \circ q_1$ and the segment S of L joining v and $w = v + 2d\lambda$ is a lift to $\mathbb{R}^2$ of α under $p_1 \circ q_1$. Recall that every spin mirror for $p_1 \circ q_1$ is a piecewise linear arc. We choose L so that its intersection with every such spin mirror is transverse. So S meets B_1, the union of the spin mirrors for $p_1 \circ q_1$, transversely in finitely many points.

Let S' be the lift to $\mathbb{R}^2$ of α under $p_2 \circ q_2$ based at $\omega(v)$. Suppose that S meets B_1 in n points. Let $S_1, \dots, S_{n+1}$ be the line segments in order from v to $w = v + 2d\lambda$ so that $S_1 \cup \cdots \cup S_{n+1} = S \setminus B_1$. Let $S_1', \dots, S_{n+1}'$ be the corresponding arcs in S'. For every $j \in \{1, \dots, n\}$ the closures of S_j and S_{j+1} meet at a spin mirror. Let $\lambda_j \in \Lambda_1$ be the midpoint of this spin mirror.

Standard covering space theory implies that $S_1' = \omega(S_1)$. The discussion which explains the naming of spin mirrors shows that $S_2' = \omega(2\lambda_1 - S_2)$. Next, $S_3' = \omega(2\lambda_1 - (2\lambda_2 - S_3)) = \omega(2\lambda_1 - 2\lambda_2 + S_3)$. Inductively, it follows that

$$S_j' = \omega\left((-1)^{j+1}S_j + 2\sum_{i=1}^{j-1}(-1)^{i+1}\lambda_i\right)$$

for $j \in \{1, \dots, n+1\}$. Set $w' = (-1)^n w + 2\sum_{i=1}^{n}(-1)^{i+1}\lambda_i$. It follows that S' joins $\omega(v)$ and $\omega(w')$. So $\sigma_f(\frac{p}{q})$ is the slope of the line segment joining $\omega(v)$ and $\omega(w')$ relative to the ordered basis (λ_2, μ_2) of Λ_2.

It remains to interpret this in terms of the line segment joining v and w'. We use the fact that the restriction of $p_j \circ q_j$ to $\mathbb{R}^2 \setminus B_j$ is a regular covering map with a group of deck transformations Γ_j for $j \in \{1, 2\}$. The map ω induces a group isomorphism from Γ_1 to Γ_2, hence a group isomorphism from $2\Lambda_1$ to $2\Lambda_2$ and hence a group isomorphism from Λ_1 to Λ_2. The map ω is not uniquely determined by the choice of spin mirrors, but it is unique up to postcomposing with an element of Γ_2. So the isomorphism from Γ_1 to Γ_2 is unique up to conjugation by an element of Γ_2. One checks that the isomorphism from Λ_1 to Λ_2 is therefore unique up to multiplication by ± 1. This does not affect slopes. So the choice of ordered basis (λ_2, μ_2) of Λ_2 together with the choice of spin mirrors determines two ordered bases of Λ_1 of the form (λ_1, μ_1) and $(-\lambda_1, -\mu_1)$. Then $\sigma_f(\frac{p}{q})$ is the slope of the line segment joining v and w' relative to either of these ordered bases of Λ_1. We emphasize that the correspondence between this basis of Λ_2 and these two bases of Λ_1 involves both the identification map $\phi\colon \mathbb{R}^2/\Gamma_2 \to \mathbb{R}^2/\Gamma_1$ and the choice of spin mirrors.

We have proved the following theorem.

Theorem 5.1. *Let f be a NET map in the setting of Section 1. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$. Let δ be an essential simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$ relative to the basis (λ_2, μ_2) of Λ_2. Suppose that α is an essential, nonperipheral component of $f^{-1}(\delta)$ in $S^2 \setminus P_2$. Let d be the degree with which f maps α to δ. Let $\lambda = q\lambda_2 + p\mu_2$. Let v be any point in $\mathbb{R}^2$ such that $p_1(q_1(v)) \in \alpha$ and v is not contained in a spin mirror for $p_1 \circ q_1$. It*

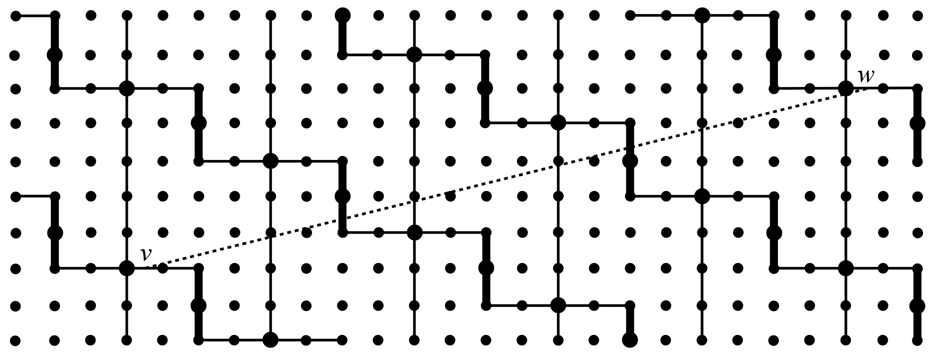

FIGURE 12. Spin mirrors for $p_1 \circ q_1$ of the main example.

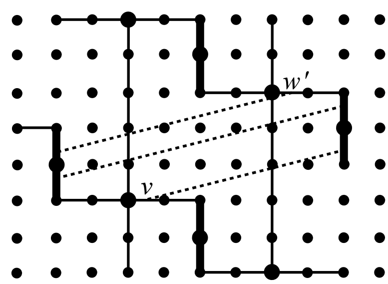

FIGURE 13. Illustrating Theorem 5.1.

is possible to choose δ so that the line segment joining v and $w = v + 2d\lambda$ is a lift of α to $\mathbb{R}^2$ under $p_1 \circ q_1$ and it intersects the spin mirrors for $p_1 \circ q_1$ transversely in finitely many points. Let S be the line segment joining v and w. Let $\lambda_1, \ldots, \lambda_n$ be the midpoints of the spin mirrors which meet S in order. Then $\sigma_f(\frac{p}{q})$ is the slope of the line segment joining v and $w' = (-1)^n w + 2\sum_{i=1}^{n}(-1)^{i+1}\lambda_i$ relative to either of the two ordered bases of Λ_1 determined by (λ_2, μ_2) and the choice of spin mirrors.

Remark 5.2. If the spin mirrors for $p_1 \circ q_1$ are invariant under translation by the elements of Λ_1, as in the main example, then the element w in Theorem 5.1 may be taken to be $v + d\lambda$ instead of $v + 2d\lambda$.

Figures 12 and 13 illustrate Theorem 5.1 for the main example. Figure 12 shows the tiling of $\mathbb{R}^2$ by the Γ_1-translates of the fundamental domain for Γ_1 shown in Figure 2. The spin mirrors for $p_1 \circ q_1$ are drawn with thick line segments. We take the standard basis $\lambda_2 = (1,0)$ and $\mu_2 = (0,1)$ for $\Lambda_2 = \mathbb{Z}^2$. We choose $\frac{p}{q} = \frac{1}{4}$. Hence $\lambda = (4,1)$, and we may take $\mu = (1,0)$. Table 1 shows for this slope that $d = 5$ and $c_1 = c_2 = 0$. Thus we may take $v = \frac{1}{2}\mu = (\frac{1}{2}, 0)$. By Remark 5.2 we may take $w = v + d\lambda = (\frac{41}{2}, 5)$. The dashed line segment in Figure 12 joins v and w and has slope $\frac{1}{4}$. It meets two spin mirrors. The resulting spin reflections are shown in Figure 13. Since the ordered basis of Λ_1 consisting of $(2,-1)$ and $(0,5)$ corresponds to the ordered basis of Λ_2 consisting of $(1,0)$ and $(0,1)$, it follows that $\sigma_f(\frac{1}{4})$ is the slope of the line through $v = (\frac{1}{2}, 0)$ and $w' = (\frac{9}{2}, 3)$ relative to the basis $(2, -1)$ and $(0, 5)$ of Λ_1. Thus $\sigma_f(\frac{1}{4}) = \frac{1}{2}$.

Theorem 5.1 provides a way to compute the slope function as in the previous paragraph, but the method leaves something to be desired. Although it might not be obvious, the next theorem provides an improvement.

Theorem 5.3. *Let f be a NET map in the setting of Section 1. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$. As in Theorem 5.1, let $\lambda = q\lambda_2 + p\mu_2$ and let μ be an element of Λ_2 such that λ and μ form a basis of Λ_2. Also let c_1, c_2, c_3, c_4 be the coset numbers for $q_1^{-1}(p_1^{-1}(P_2))$ relative to λ and μ. We assume that $\sigma_f(\frac{p}{q}) \neq o$, equivalently, $c_2 \neq c_3$ by Theorem 4.1. Let L be a line in $\mathbb{R}^2$ which has a Γ_1-translate given in parametric form by either $(x, y) = t\lambda + c_2\mu$ or $(x, y) = t\lambda + c_3\mu$. Let v and w be distinct elements of $L \cap q_1^{-1}(p_1^{-1}(P_2))$ such that no element of $q_1^{-1}(p_1^{-1}(P_1 \cup P_2))$ is strictly between v and w. Let S be the closed line segment which joins v and w. We assume that the interior of S intersects the spin mirrors for $p_1 \circ q_1$ transversely in finitely many points. Let $\lambda_1, \ldots, \lambda_n$ be the midpoints of these spin mirrors which meet the interior of S in order. Since $v, w \in q_1^{-1}(p_1^{-1}(P_2))$, both v and w are contained in spin mirrors for $p_1 \circ q_1$. Let λ_0 and λ_{n+1} be the midpoints of these two spin mirrors. Then $\sigma_f(\frac{p}{q})$ is the slope of the line segment joining 0 and $\sum_{i=0}^{n}(-1)^i(\lambda_{i+1} - \lambda_i)$ relative to either of the two ordered bases of Λ_1 determined by (λ_2, μ_2) and the choice of spin mirrors.*

Proof. It suffices to prove the theorem for the case in which L is given in parametric form by either $(x, y) = t\lambda + c_2\mu$ or $(x, y) = t\lambda + c_3\mu$, and so we assume that L has this form. Let ϵ be a positive real number. If L is given in parametric form by $(x, y) = t\lambda + c_2\mu$, then let L_ϵ be the line with parametric form $(x, y) = t\lambda + (c_2 + \epsilon)\mu$. If L is given in parametric form by $(x, y) = t\lambda + c_3\mu$, then let L_ϵ be the line with parametric form $(x, y) = t\lambda + (c_3 - \epsilon)\mu$. Statement (5) of Theorem 4.1 shows that if ϵ is small enough, then $p_1(q_1(L_\epsilon))$ is an essential, nonperipheral simple closed curve in $S^2 \setminus P_2$. Of course, f maps $p_1(q_1(L_\epsilon))$ to an essential simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$. So $\sigma_f(\frac{p}{q})$ is the slope of $p_1(q_1(L_\epsilon))$ in $S^2 \setminus P_2$.

The assumptions imply that the interior of $p_1(q_1(S))$ avoids the branch points of $p_2 \circ q_2$ and maps injectively to $S^2 \setminus P_2$. Let S' be a lift of $p_1(q_1(S))$ to $\mathbb{R}^2$ under $p_2 \circ q_2$. It follows that there exists $v_\epsilon \in L_\epsilon$ near v and $w_\epsilon \in L_\epsilon$ near w such that $p_1(q_1(v_\epsilon))$ and $p_1(q_1(w_\epsilon))$ lift under $p_2 \circ q_2$ to points near the endpoints of S' and these lifts differ by the same nontrivial element of Λ_2. Expressing this element of Λ_2 in terms of the basis (λ_2, μ_2) determines $\sigma_f(\frac{p}{q})$. If ϵ is small enough, then computing this element of Λ_2 from S, which joins v and w, is the same as computing this element of Λ_2 from the segment of L_ϵ which joins v_ϵ and w_ϵ.

The discussion before Theorem 5.1 now essentially completes the proof of Theorem 5.3. The only difference now is that the endpoints v and w of S lie in spin mirrors for $p_1 \circ q_1$. Computing the relevant element of Λ_1 by means of the isomorphism between Λ_1 and Λ_2 requires replacing v and w by the midpoints of the spin mirrors which contain them. Hence $\sigma_f(\frac{p}{q})$ is the slope of the line segment joining λ_0 and $(-1)^n\lambda_{n+1} + 2\sum_{i=1}^{n}(-1)^{n+1}\lambda_i$ relative to either of the two ordered bases of Λ_1 determined by (λ_2, μ_2) and the choice of spin mirrors. This is equivalent to the desired conclusion.

This proves Theorem 5.3. $\square$

q mod 4	$2p+q$ mod 5	v	w
0	0	$(0,0)$	(q,p)
0	±1, ±2	$(0,0)$	$(5q,5p)$
2	0	$(2,0)$	$(2+2q,2p)$
2	±1	$(0,0)$	$(3q,3p)$
2	±2	$(0,0)$	(q,p)
±1	0	$(2,0)$	$(2+4q,4p)$
±1	±1	$(0,0)$	$(2q,2p)$
±1	±2	$(0,0)$	$(6q,6p)$

TABLE 2. Choosing v and w for the main example.

One advantage of Theorem 5.3 over Theorem 5.1 is that in Theorem 5.3 both v and w are in Λ_2, whereas in Theorem 5.1 neither is. Another advantage is that in Theorem 5.3 the line segment joining v and w is shorter than the one in Theorem 5.1, resulting in a shorter computation.

We next show for the main example that Theorem 5.3 provides an algorithm for computing the slope function which is easy to implement by computer and which can even be used by hand in simple cases. This will occupy the rest of this section.

Let p and q be relatively prime integers. We wish to compute $\sigma_f(\frac{p}{q})$ for the main example using Theorem 5.3. Table 1 shows that $c_2 \neq c_3$, so $\sigma_f(\frac{p}{q}) \neq o$. We begin by choosing appropriate lattice points v and w as in Theorem 5.3. Since $\Lambda_2 = \mathbb{Z}^2$ for the main example, v and w are simply standard lattice points in $\mathbb{R}^2$. Table 2 gives our choices. The first column gives q modulo 4. The second column gives $2p+q$ modulo 5. The last two columns give v and w.

We justify our choices of v and w beginning with this paragraph. Our basis for Λ_2 is the standard basis $\lambda_2 = (1,0)$ and $\mu_2 = (0,1)$. Hence $\lambda = (q,p)$. Moreover, $\Lambda_1 = \langle (2,-1),(0,5)\rangle$. The set $q_1^{-1}(p_1^{-1}(P_1 \cup P_2))$ is a union of cosets of $2\Lambda_1$ in Λ_2, and the following elements are distinct representatives for these cosets,

$$(0,0),(0,5),(2,-1),(2,4),(2,0),(2,-2),(2,3),(2,5).$$

The first two coset representatives are in $q_1^{-1}(p_1^{-1}(P_1 \cap P_2))$, the next two are in $q_1^{-1}(p_1^{-1}(P_1 \setminus P_2))$ and the last four are in $q_1^{-1}(p_1^{-1}(P_2 \setminus P_1))$.

We first determine all cases in which it is possible to choose $v = (0,0)$. We see that $(0,0) \in q_1^{-1}(p_1^{-1}(P_2))$, as required by Theorem 5.3. We also need $(0,0)$ to be in the line L of Theorem 5.3. This is equivalent to the condition that $c_2 = 0$. Table 1 shows that this is in turn equivalent to the condition that $q \equiv 0 \mod 4$ if $2p+q \equiv 0 \mod 5$.

With $v = (0,0)$ the element w is an integer multiple of $\lambda = (q,p)$, and without loss of generality we take this integer x to be positive. We want w to be in $q_1^{-1}(p_1^{-1}(P_2))$ with no element of $q_1^{-1}(p_1^{-1}(P_1 \cup P_2))$ between v and w. Let (r,s) be one of our eight coset representatives. We are interested in the congruence $(r,s) \equiv x(q,p) \mod 2\Lambda_1$. Hence we are interested in integers y and z such that

$$(r,s) = x(q,p) + y(4,-2) + z(0,10).$$

The following equations give y and z as rational numbers:

$$y = \frac{1}{4}(r - xq) \quad \text{and} \quad z = \frac{1}{20}(2s + r - x(2p+q))$$

Thus y and z are integers if and only if the following three congruences are satisfied,

$$r \equiv xq \mod 4 \quad s \equiv xp \mod 2 \quad 2s + r \equiv x(2p+q) \mod 5. \tag{5.4}$$

So, assuming that $q \equiv 0 \mod 4$ if $2p + q \equiv 0 \mod 5$, then we may take $v = (0,0)$ and $w = x(q,p)$, where x is the smallest positive integer which satisfies (5.4) for some choice of (r,s).

First suppose that $q \equiv 0 \mod 4$. (5.4) implies that $r \equiv 0 \mod 4$. Thus either $(r,s) = (0,0) \in \Lambda_1$ or $(r,s) = (0,5) \in \Lambda_1$. It follows that x is the order of the image of $\lambda = (q,p)$ in Λ_2/Λ_1. Theorem 4.1 and Table 1 now imply that $x = 1$ if $2p + q \equiv 0 \mod 5$ and $x = 5$ if $2p + q \equiv \pm 1, \pm 2 \mod 5$. This gives the first two lines of Table 2.

Next suppose that $q \equiv 2 \mod 4$. Then $p \equiv 1 \mod 2$. We consider solutions to (5.4) with $x = 1$. There is such a solution if and only if $r \equiv 2 \mod 4$, $s \equiv 1 \mod 2$ and $2s + r \equiv 2p + q \mod 5$. These congruences have a solution with $(r,s) \in q_1^{-1}(p_1^{-1}(P_2))$ if and only if $(r,s) \in \{(2,3),(2,5)\}$ and $2p + q \equiv \pm 2 \mod 5$. This obtains line 5 of Table 2. If $x = 2$, then (5.4) shows that $r \equiv 0 \mod 4$ and $s \equiv 0 \mod 2$. Hence $(r,s) = (0,0)$. Hence $2(q,p) \in 2\Lambda_1$, hence $(q,p) \in \Lambda_1$ and so (5.4) has a solution with $x = 1$. Thus there is no acceptable value of w with $x = 2$. Finally, we verify that (5.4) always has a solution for $x = 3$, $q \equiv 2 \mod 4$ and $2p+q \equiv \pm 1 \mod 5$ by taking $(r,s) \in \{(2,3),(2,5)\}$. This obtains line 4 of Table 2.

Next suppose that $q \equiv \pm 1 \mod 4$. If x is odd, then the first congruence in line 5.4 shows that r is also odd. This is impossible. So x is even. We now proceed as in the last paragraph to obtain lines 7 and 8 of Table 2.

We have thus far handled every case in which it is possible to choose $v = (0,0)$. We are left with the values of p and q for which $q \not\equiv 0 \mod 4$ and $2p + q \equiv 0 \mod 5$. Table 1 shows that these are precisely the cases in which $c_1 < c_2 < c_3 < c_4$. In this situation the elements of $q_1^{-1}(p_1^{-1}(P_1 \cap P_2))$ correspond to c_1 and c_4 while the elements of $q_1^{-1}(p_1^{-1}(P_2 \setminus P_1))$ correspond to c_2 and c_3. Thus we may choose the line L of Theorem 5.3 so that it contains $(2,0)$. As for (5.4), replacing $(0,0)$ by $(2,0)$ has the effect of replacing r by $r-2$ to obtain the following:

$$r - 2 \equiv xq \mod 4 \quad s \equiv xp \mod 2 \quad 2s + r - 2 \equiv x(2p+q) \mod 5.$$

First suppose that $q \not\equiv 0 \mod 4$, $2p + q \equiv 0 \mod 5$ and $x \equiv 1 \mod 2$. Then $r \not\equiv 2 \mod 4$. Hence $(r,s) \in \{(0,0),(0,5)\} \subseteq \Lambda_1$. As discussed in the previous paragraph, the elements of Λ_1 correspond to c_1 and c_4 not c_2 or c_3. Thus $x \equiv 0 \mod 2$. Now we verify that the congruences in the last display are solved by choosing $q \equiv 2 \mod 4$, $2p + q \equiv 0 \mod 5$, $x = 2$ and $(r,s) = (2,0)$. On the other hand, if $q \equiv \pm 1 \mod 4$, $2p+q \equiv 0 \mod 5$ and $x = 2$, then $r \equiv 0 \mod 4$ and $s \equiv 0 \mod 2$, hence $(r,s) = (0,0)$ and so $2s + r - 2 \equiv -2 \not\equiv 0 \equiv x(2p+q) \mod 5$. Thus there is no solution in this case. Finally, taking $q \equiv \pm 1 \mod 4$, $2p + q \equiv 0 \mod 5$, $x = 4$ and $(r,s) = (2,0)$ gives a solution. This completes the verification of Table 2.

Now that we have v and w, we find the lattice points $\lambda_1, \ldots, \lambda_n$ which appear in Theorem 5.3. A point $(x,y) \in \mathbb{R}^2$ is the center of a spin mirror for $p_1 \circ q_1$ if and only if x is an integer congruent to 2 modulo 4 and there exists an integer Q_x such that $x + 2y = 10Q_x$. A point $(x,y) \in \mathbb{R}^2$ is in a spin mirror for $p_1 \circ q_1$ if and only if x is an integer congruent to 2 modulo 4 and there exists an integer Q_x and a real number R_x with $|R_x| \leq 2$ such that $x + 2y = 10Q_x + R_x$.

So suppose that $v=(0,0)$. Let $w=(w_1,w_2)$. The line segment joining v and w is in the line given by $y=\frac{p}{q}x$. For every integer x, define Q_x and R_x so that

$$\left(1+\frac{2p}{q}\right)x=10Q_x+R_x \quad \text{with } Q_x\in\mathbb{Z}, R_x\in\mathbb{Q} \text{ and } -5<R_x\leq 5.$$

Let $0<x_1<x_2<x_3<\cdots<x_n<w_1$ be those integers congruent to 2 modulo 4 such that $|R_{x_i}|<2$. Set $x_0=0$ and $x_{n+1}=w_1$. If λ_i is the center of the ith spin mirror as in Theorem 5.3, then

$$\lambda_i=(x_i,\tfrac{10Q_{x_i}-x_i}{2})=\tfrac{1}{2}x_i(2,-1)+Q_{x_i}(0,5) \quad \text{for } i\in\{0,\dots,n+1\}.$$

Set

$$N=\sum_{i=0}^{n}(-1)^i(Q_{x_{i+1}}-Q_{x_i})$$

and

$$D=\frac{1}{2}\sum_{i=0}^{n}(-1)^i(x_{i+1}-x_i).$$

Assembling what we have, Theorem 5.3 implies that

$$\sigma_f\left(\frac{p}{q}\right)=\frac{N}{D}.$$

The situation is similar if $v=(2,0)$. Again let $w=(w_1,w_2)$. The line segment joining v and w is in the line given by $y=\frac{p}{q}x-\frac{2p}{q}$. For every integer x define Q_x and R_x so that

$$\left(1+\frac{2p}{q}\right)x-\frac{4p}{q}=10Q_x+R_x \quad \text{with } Q_x\in\mathbb{Z}, R_x\in\mathbb{Q} \text{ and } -5<R_x\leq 5.$$

Let $2<x_1<x_2<x_3<\cdots<x_n<w_1$ be those integers congruent to 2 modulo 4 such that $|R_{x_i}|<2$. Set $x_0=2$ and $x_{n+1}=w_1$. Set

$$N=\sum_{i=0}^{n}(-1)^i(Q_{x_{i+1}}-Q_{x_i})$$

and

$$D=\frac{1}{2}\sum_{i=0}^{n}(-1)^i(x_{i+1}-x_i).$$

Then

$$\sigma_f\left(\frac{p}{q}\right)=\frac{N}{D}.$$

We illustrate this formula for $\sigma_f(\frac{p}{q})$ by calculating $\sigma_f(\frac{3}{2})$. Table 2 shows that $v=(0,0)$ and $w=(2,3)$. Since $(1+\frac{2\cdot 3}{2})\cdot 2=10\cdot 1-2$, we simply have that $n=0$, $x_0=0$, $x_1=2$, $Q_{x_0}=0$ and $Q_{x_1}=1$. So $N=Q_{x_1}-Q_{x_0}=1$ and $D=\frac{1}{2}(x_1-x_0)=1$. Therefore $\sigma_f(\frac{3}{2})=\frac{N}{D}=1$.

The formulas just derived for the slope function σ_f of the main example were used to create the graph of σ_f shown in Figure 14. The prominent horizontal lines in the graph indicate that σ_f is often infinite-to-one, which can be proved using the functional equations in Sections 7, 8 and 9. The less prominent vertical lines indicate that σ_f does not extend continuously to the Thurston boundary, which can also be proved using functional equations. The prominent fuzzy line with positive slope less than 1 indicates that iteration of σ_f is probably contracting. It seems

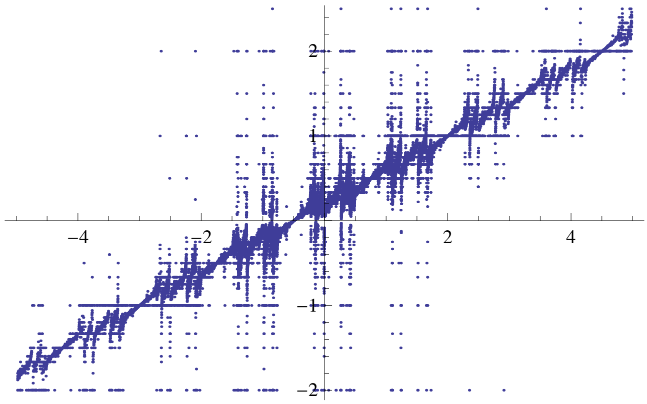

FIGURE 14. The graph of σ_f for the main example.

very possible that under iteration of σ_f, every element of $\widehat{\mathbb{Q}}$ eventually lies in a finite set of cycles, although we do not know this. See Remark 9.4 for an equation of the prominent fuzzy line.

6. THE HALF-SPACE THEOREM

We maintain the setting of Section 1. Let $\mathbb{H}$ denote the upper half complex plane, so that $\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. We identify $\mathbb{H}$ with the Teichmüller space of $S^2 \setminus P_2$ as follows. Let $\tau \in \mathbb{H}$. Recall that we have chosen an ordered basis (λ_2, μ_2) of the lattice Λ_2. There exists a unique $\mathbb{R}$-linear isomorphism $\varphi\colon \mathbb{C} \to \mathbb{R}^2$ so that $\varphi(1) = 2\lambda_2$ and $\varphi(\tau) = 2\mu_2$. The map $p_2 \circ q_2 \circ \varphi\colon \mathbb{C} \to S^2 \setminus P_2$ induces a complex structure on $S^2 \setminus P_2$. As τ varies over $\mathbb{H}$, the resulting isotopy classes of complex structures on $S^2 \setminus P_2$ are distinct, and every complex structure on $S^2 \setminus P_2$ is isotopic to one of them. In this way we regard $\mathbb{H}$ as the Teichmüller space of $S^2 \setminus P_2$.

In this section we relate horoballs in $\mathbb{H}$ to moduli of curve families. The main goal of this section is to prove Theorem 6.7, the half-space theorem. We begin by determining convenient equations for the horocycles in $\mathbb{H}$.

The horocycles in $\mathbb{H}$ at ∞ are simply horizontal lines, which are given by equations of the form $\mathrm{Im}(z) = m$ for positive real numbers m. Now let p and q be relatively prime integers with $q \neq 0$. We consider horocycles in $\mathbb{H}$ at $\frac{p}{q}$. Figure 15 shows a horocycle in $\mathbb{H}$ at $\frac{p}{q}$ with Euclidean diameter D. Two similar right triangles also appear in Figure 15, from which we conclude the following:

$$\frac{B}{A} = \frac{A}{D} \iff \frac{B}{A^2} = \frac{1}{D} \iff \frac{\mathrm{Im}(z)}{|z - p/q|^2} = \frac{1}{D} \iff \frac{\mathrm{Im}(z)}{|qz - p|^2} = m.$$

Here $m = \frac{1}{q^2 D}$ and $D = \frac{1}{q^2 m}$. Thus if p and q are relatively prime integers, then the horoballs in $\mathbb{H}$ at $\frac{p}{q} \in \widehat{\mathbb{Q}}$ are the subsets of the form $\{z \in \mathbb{H} : \frac{\mathrm{Im}(z)}{|qz-p|^2} > m\}$ for positive real numbers m.

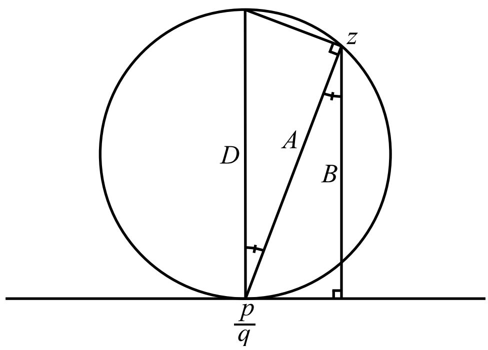

FIGURE 15. Finding an equation for a horocycle.

We view $\mathrm{PSL}(2,\mathbb{Z})$ as a subgroup of the group of orientation-preserving isometries of $\mathbb{H}$; that is,

$$\mathrm{PSL}(2,\mathbb{Z}) = \{z \mapsto \frac{az+b}{cz+d} : \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathrm{SL}(2,\mathbb{Z})\}.$$

If $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is an element of $\mathrm{GL}(2,\mathbb{Z})$ with determinant -1, then $z \mapsto \frac{a\overline{z}+b}{c\overline{z}+d}$ is an orientation-reversing isometry of $\mathbb{H}$. This allows us to view $\mathrm{PGL}(2,\mathbb{Z})$ as a subgroup of the group of isometries of $\mathbb{H}$. We next consider the action of $\mathrm{PGL}(2,\mathbb{Z})$ on the horoballs of $\mathbb{H}$.

Lemma 6.1. *Let $\varphi \in PGL(2,\mathbb{Z})$, let $\frac{p}{q} \in \widehat{\mathbb{Q}}$ and suppose that $\varphi(\frac{p}{q}) = \frac{p'}{q'} \in \widehat{\mathbb{Q}}$. Then*

$$\frac{\mathrm{Im}(\varphi(z))}{|q'\varphi(z)-p'|^2} = \frac{\mathrm{Im}(z)}{|qz-p|^2}.$$

Proof. If the lemma is true for $\varphi_1, \varphi_2 \in \mathrm{PGL}(2,\mathbb{Z})$, then it is true for the composition $\varphi_1 \circ \varphi_2$. It follows that since $z \mapsto -\overline{z}$, $z \mapsto z+1$ and $z \mapsto -\frac{1}{z}$ generate $\mathrm{PGL}(2,\mathbb{Z})$, to prove the lemma, it suffices to prove it for these three transformations.

First suppose that $\varphi(z) = -\overline{z}$. Then $\frac{p'}{q'} = -\frac{p}{q}$, and so we may assume that $p' = p$ and $q' = -q$. We easily see that the lemma is true in this case. Next suppose that $\varphi(z) = z+1$. Then $\frac{p'}{q'} = \frac{p}{q}+1$, and so we may assume that $p' = p+q$ and $q' = q$. Since $\mathrm{Im}(z+1) = \mathrm{Im}(z)$, the lemma is true in this case. If $\varphi(z) = -\frac{1}{z}$, then $\frac{p'}{q'} = -\frac{q}{p}$, and so we may assume that $p' = -q$ and $q' = p$. Since $\mathrm{Im}(-\frac{1}{z}) = \frac{\mathrm{Im}(z)}{|z|^2}$, the lemma is true in this case too.

This proves Lemma 6.1. □

Corollary 6.2. *Let $\varphi \in PGL(2,\mathbb{Z})$, let $\frac{p}{q} \in \widehat{\mathbb{Q}}$ and suppose that $\varphi(\frac{p}{q}) = \frac{p'}{q'} \in \widehat{\mathbb{Q}}$. Then φ maps the horoball $\{z \in \mathbb{H} : \frac{\mathrm{Im}(z)}{|qz-p|^2} > m\}$ bijectively to the horoball $\{z \in \mathbb{H} : \frac{\mathrm{Im}(z)}{|q'z-p'|^2} > m\}$ for every positive real number m.*

Now we turn our attention to moduli of curve families. Let $\tau \in \mathbb{H}$, let $\Lambda_\tau = \langle 1, \tau \rangle$, and let $T_\tau = \mathbb{C}/\Lambda_\tau$. We view T_τ as a torus with complex structure. We calculate slopes of simple closed curves in T_τ using the ordered basis $(1,\tau)$ of Λ_τ. We denote by $\Gamma_{\frac{p}{q},\tau}$ the set of simple closed curves in T_τ with slope $\frac{p}{q}$, although in this paragraph we abbreviate $\Gamma_{\frac{p}{q},\tau}$ to $\Gamma_{\frac{p}{q}}$. By abuse of notation we write dz for the 1-form on T_τ induced by the standard 1-form on $\mathbb{C}$. For a nonnegative Borel measurable function ρ on T_τ, define

$$L_\rho(\Gamma_{\frac{p}{q}}) = \inf_{\gamma\in\Gamma_{\frac{p}{q}}} \int_\gamma \rho\,|dz| \quad \text{and} \quad A_\rho = \iint\limits_{T_\tau} \rho^2\,|dz|^2.$$

The extremal length of the curve family $\Gamma_{\frac{p}{q}}$ on T_τ is

$$\sup_\rho \frac{L_\rho^2(\Gamma_{\frac{p}{q}})}{A_\rho}.$$

The modulus of the curve family $\Gamma_{\frac{p}{q}}$ on T_τ is the reciprocal of this:

$$\mathrm{mod}_\tau(\tfrac{p}{q}) = \inf_\rho \frac{A_\rho}{L_\rho^2(\Gamma_{\frac{p}{q}})}.$$

The modulus is a conformal invariant: if $\phi : T_\tau \to T_{\tau'}$ is a conformal isomorphism sending $\Gamma_{\frac{p}{q},\tau}$ to $\Gamma_{\frac{p'}{q'},\tau'}$, then

$$\mathrm{mod}_\tau(\tfrac{p}{q}) = \mathrm{mod}_{\tau'}(\tfrac{p'}{q'}).$$

Lemma 6.3. *For every $\frac{p}{q} \in \widehat{\mathbb{Q}}$ and $\tau \in \mathbb{H}$ we have that*

$$\mathrm{mod}_\tau(\tfrac{p}{q}) = \frac{\mathrm{Im}(\tau)}{|p\tau+q|^2}.$$

Proof. We first prove the lemma for $\frac{p}{q} = 0$. Let $\tau \in \mathbb{H}$. Let $h = \mathrm{Im}(\tau)$. The parallelogram P with vertices 0, 1, τ and $1+\tau$ is a fundamental domain for the action of Λ_τ on $\mathbb{C}$. Every element of $\Gamma_{0,\tau}$ is homotopic to the image in T_τ of a horizontal line segment joining the left and right sides of this fundamental domain. Let ρ be the Borel function on T_τ which equals 1 everywhere. Then A_ρ is the Euclidean area of P, and so $A_\rho = \mathrm{Im}(\tau) = h$. Now we consider $L_\rho(\Gamma_{0,\tau})$. Let $\gamma \in \Gamma_{0,\tau}$. Every lift of γ to $\mathbb{C}$ has the property that its endpoints have equal imaginary parts and even that the Euclidean distance between these endpoints is 1. Since $\int_\gamma \rho\,|dz|$ is the Euclidean length of every such lift, it follows that $L_\rho(\Gamma_{0,\tau}) = 1$. So $\mathrm{mod}_\tau(0) \le \frac{A_\rho}{L_\rho^2(\Gamma_{0,\tau})} = h$.

Now let ρ be any nonnegative Borel measurable function on T_τ. Let $\widetilde{\rho}$ be the lift of ρ to $\mathbb{C}$. Let $l = L_\rho(\Gamma_{0,\tau})$. We use the fact that the rectangle R whose vertices are 0, 1, hi and $1+hi$ is a fundamental domain for the action of Λ_τ on $\mathbb{C}$. Horizontal line segments in R give rise to elements of $\Gamma_{0,\tau}$; that is, for every $y \in [0,h]$, the curve γ_y parametrized by $\gamma(x) = x + yi \mod \Lambda_\tau$ from the closed interval $[0,1]$ to T_τ is an element of $\Gamma_{0,\tau}$. Hence $l \le \int_0^1 \widetilde{\rho}(x,y)\,dx$ for every $y \in [0,h]$, and so

$$hl \le \int_0^h \int_0^1 \widetilde{\rho}\,dxdy \le \left(\iint\limits_R dxdy \iint\limits_R \widetilde{\rho}^2 dxdy \right)^{1/2} = (hA_\rho)^{1/2}.$$

Thus

$$\frac{A_\rho}{L_\rho^2(\Gamma_{0,\tau})} = \frac{A_\rho}{l^2} \ge h.$$

Since $\mathrm{mod}_\tau(0)$ is the infimum of such terms, $\mathrm{mod}_\tau(0) \geq h$. This inequality and the conclusion of the previous paragraph imply that $\mathrm{mod}_\tau(0) = h = \mathrm{Im}(\tau)$. This proves Lemma 6.3 if $\frac{p}{q} = 0$.

Now suppose that $\frac{p}{q} \neq 0$. Since p and q are relatively prime, there exist integers r and s such that $rq - sp = 1$. Hence $(p\tau + q, r\tau + s)$ is an ordered basis of Λ_τ. Setting $\tau' = \frac{r\tau+s}{p\tau+q} \in \mathbb{H}$, we find that the conformal map $\widetilde{\phi}\colon \mathbb{C} \to \mathbb{C}$ given by

$$\widetilde{\phi}(z) = \frac{z}{p\tau + q}$$

sends the lattice Λ_τ to $\Lambda_{\tau'}$ and sends the ordered basis $(p\tau + q, r\tau + s)$ to $(1, \tau')$. Hence it descends to a conformal isomorphism $\phi : T_\tau \to T_{\tau'}$. Moreover,

$$\phi(\Gamma_{\frac{p}{q},\tau}) = \Gamma_{0,\tau'},$$

and so since the modulus is a conformal invariant, we have that

$$\mathrm{mod}_\tau(\tfrac{p}{q}) = \mathrm{mod}_{\tau'}(0).$$

This and the previous paragraph imply that

$$\mathrm{mod}_\tau(\tfrac{p}{q}) = \mathrm{Im}(\tau').$$

Finally, Lemma 6.1 with $\frac{p}{q}$ there replaced by $-\frac{q}{p}$ shows that $\mathrm{mod}_\tau(\frac{p}{q}) = \frac{\mathrm{Im}(\tau)}{|p\tau+q|^2}$.

This proves Lemma 6.3. □

For every $\frac{p}{q} \in \widehat{\mathbb{Q}}$ and every positive real number m, we set

$$B_m(\tfrac{p}{q}) = \{\tau \in \mathbb{H} : \mathrm{mod}_\tau(\tfrac{p}{q}) > m\}.$$

Corollary 6.4. *If $\frac{p}{q} \in \widehat{\mathbb{Q}}$ and if m is a positive real number, then*

$$B_m(\tfrac{p}{q}) = \{\tau \in \mathbb{H} : \tfrac{\mathrm{Im}(\tau)}{|p\tau+q|^2} > m\},$$

a horoball in $\mathbb{H}$ at $-\frac{q}{p}$.

Let S_τ be the quotient space of T_τ determined by the map $z \mapsto -z$. Let $p_\tau\colon T_\tau \to S_\tau$ be the corresponding degree 2 branched covering map. Let P_τ be the set of four branch points of p_τ in S_τ. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$. By definition, the set of essential simple closed curves in $S_\tau \setminus P_\tau$ with slope $\frac{p}{q}$ lifts under p_τ to $\Gamma_{\frac{p}{q},\tau}$. We define the modulus of this family of curves just as we defined $\mathrm{mod}_\tau(\frac{p}{q})$. Because lengths of curves do not change when pulling back from S_τ to T_τ but area doubles, this new modulus is $\frac{1}{2}\mathrm{mod}_\tau(\frac{p}{q})$.

Let f be a NET map. We define a function $\delta_f\colon \widehat{\mathbb{Q}} \to \mathbb{Q}$ as follows. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$. Let γ be an essential simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$. Let d be the degree with which f maps every connected component of $f^{-1}(\gamma)$ to γ, and let c be the number of these connected components which are essential and nonperipheral. Then $\delta_f(\frac{p}{q}) = \frac{c}{d}$. The multicurve Γ whose only element is γ is f-stable if and only if either $\sigma_f(\frac{p}{q}) = \frac{p}{q}$ or $\sigma_f(\frac{p}{q}) = o$. If Γ is f-stable, then the Thurston matrix A^Γ is 1×1 with entry $\delta_f(\frac{p}{q})$. Thus Γ is a Thurston obstruction if and only if $\frac{p}{q} \in \mathrm{Fix}(\sigma_f)$ and $\delta_f(\frac{p}{q}) \geq 1$.

We maintain the setting of the previous paragraph. Recall from the introduction that $\Sigma_f\colon \mathbb{H} \to \mathbb{H}$ is the map on Teichmüller space induced by f. Let $\tau \in \mathbb{H}$, let

$\tau' = \Sigma_f(\tau)$, let $\frac{p'}{q'} = \sigma_f(\frac{p}{q})$ and let $\delta = \delta_f(\frac{p}{q})$. Then an argument based on the subadditivity of moduli proves that

$$\mathrm{mod}_{\tau'}(\tfrac{p'}{q'}) \geq \delta \mathrm{mod}_\tau(\tfrac{p}{q}).$$

Hence

$$\Sigma_f(B_m(\tfrac{p}{q})) \subseteq B_{\delta m}(\tfrac{p'}{q'}) \tag{6.5}$$

for every positive real number m.

Remark 6.6. For a sphere with four marked points, the Teichmüller and hyperbolic metrics coincide. There is another natural metric on Teichmüller space, the so-called Weil-Petersson (WP) metric. The WP metric is incomplete. For a sphere with four marked points, the WP boundary, as a topological space, is discrete and as a set is $\widehat{\mathbb{Q}}$. A neighborhood basis element for a boundary point $-\frac{q}{p}$ is a horoball tangent to $-\frac{q}{p}$ union the singleton $\{-\frac{q}{p}\}$. Selinger [10] shows that Σ_f is $\sqrt{\deg(f)}$-Lipschitz with respect to the WP metric, and so extends to the WP completion. It easily follows that for NET maps, computation of σ_f is the computation of the boundary values of Σ_f on the WP completion of Teichmüller space.

We next prove the following theorem in the above setting. Let $d(\cdot,\cdot)$ denote the hyperbolic metric for $\mathbb{H}$. If H is a half-space in $\mathbb{H}$, then we let $\partial_\infty H$ denote the set of points in the boundary of $\mathbb{H}$ which are limits of sequences in H.

Theorem 6.7 (Half-space theorem).

(1) *If $\frac{p}{q} \neq \frac{p'}{q'}$, then for every sufficiently large m, the closed horoballs $B = \overline{B_m(\frac{p}{q})}$ and $B' = \overline{B_{\delta m}(\frac{p'}{q'})}$ are disjoint. When they are disjoint, the set $H = \{\tau \in \mathbb{H} : d(\tau, B) < d(\tau, B')\}$ is an open hyperbolic half-space which is independent of m.*

(2) *If $\frac{r}{s} \in Fix(\sigma_f)$ and $-\frac{s}{r} \in \partial_\infty H$, then $\delta_f(\frac{r}{s}) < 1$; that is, there is no Thurston obstruction with slope $\frac{r}{s}$.*

(3) *If $\tau_0 \in Fix(\Sigma_f)$, then $\tau_0 \notin H$.*

Proof. We first prove statement (1). It is clear from the description of horoballs above that if m is sufficiently large, then B and B' are disjoint. We assume that m is this large. For every $m > 0$, every $\frac{a}{b} \in \widehat{\mathbb{Q}}$ and every $t > 0$, the hyperbolic distance between the horocycles $\partial B_m(\frac{a}{b})$ and $\partial B_{tm}(\frac{a}{b})$ is equal to $|\ln(t)|$; in particular, this is independent of m. Let l be the hyperbolic geodesic joining $-\frac{q}{p}$ and $-\frac{q'}{p'}$, let $l_m \subseteq l$ be the closure of the geodesic segment lying outside $B \cup B'$ and let $l_m^\perp$ be its perpendicular bisector. Then $l_m^\perp$ is independent of m, and so H is an open hyperbolic half-space independent of m. This proves statement (1).

To prove statement (2), suppose that $\frac{r}{s} \in \mathrm{Fix}(\sigma_f)$, that $-\frac{s}{r} \in \partial_\infty H$ and to the contrary that $\delta_f(\frac{r}{s}) \geq 1$. We now choose m so that B and B' intersect in a single point. Figure 16 shows B and B' with $\frac{p}{q} = 0$. It also shows $l_m^\perp$. Now we choose $m^* > 0$ so that B and $\overline{B_{m^*}(\frac{r}{s})}$ also intersect in a single point η. The assumption that $-\frac{s}{r} \in \partial_\infty H$ implies that $\overline{B_{m^*}(\frac{r}{s})} \cap B' = \emptyset$. (6.5) implies that $\Sigma_f(B) \subseteq B'$ and $\Sigma_f(\overline{B_{m^*}(\frac{r}{s})}) \subseteq \overline{B_{m^*}(\frac{r}{s})}$ because $\delta_f(\frac{r}{s}) \geq 1$. But then $\Sigma_f(\eta) \in B' \cap \overline{B_{m^*}(\frac{r}{s})}$, which is impossible. This proves statement (2).

To prove statement (3), suppose that $\tau_0 \in \mathrm{Fix}(\Sigma_f)$. Let $\tau_1 \in B$ realize the distance between τ_0 and B. (6.5) implies that $\Sigma_f(B) \subseteq B'$ and Σ_f is distance

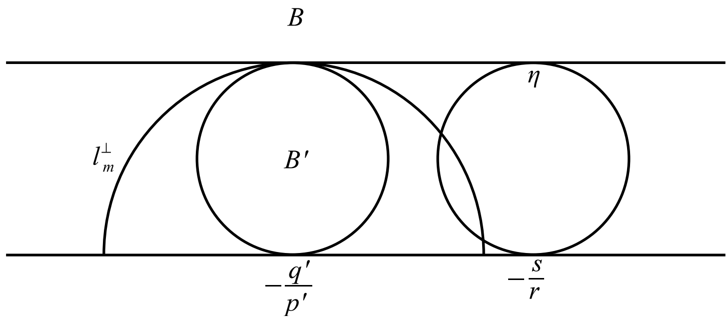

FIGURE 16. Proving Theorem 6.7.

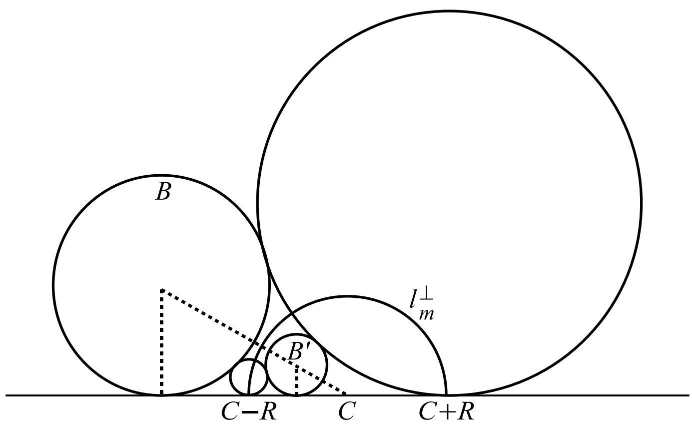

FIGURE 17. The generic situation in Theorem 6.7.

nonincreasing, so

$$d(\tau_0, B) = d(\tau_0, \tau_1) \geq d(\tau_0, \Sigma_f(\tau_1)) \geq d(\tau_0, B').$$

Hence $\tau_0 \notin H$, proving statement (3).

This proves Theorem 6.7. □

Figure 17 shows the generic situation in Theorem 6.7. The horocycle boundaries of B and B' are shown, tangent to the real line at $-\frac{q}{p}$ and $-\frac{q'}{p'}$, respectively. The geodesic $l_m^\perp$ in the proof of Theorem 6.7 is shown, with Euclidean center C and Euclidean radius R. Two other horocycles are shown. The larger one is tangent to B and the real line at $C+R$. The smaller one is tangent to B and the real line at $C-R$. Because l is the geodesic joining $-\frac{q}{p}$ and $-\frac{q'}{p'}$ and $l_m^\perp$ is the perpendicular bisector of the segment of l joining B and B', inversion about $l_m^\perp$ stabilizes these two horocycles and interchanges B and B'. So these two horocycles are also tangent to B'.

The proof of the main part of Theorem 6.7 can be restated as follows. Suppose that f has a Thurston obstruction with slope r. (6.5) implies that Σ_f maps every horoball at r into itself. Let B_r be the closed horoball at r tangent to B. Then

$\Sigma_f(B) \cap B_r \neq \emptyset$, because Σ_f maps B_r into itself. But then $B' \cap B_r \neq \emptyset$. So given B and B', it follows that $C - R \leq r \leq C + R$.

To apply Theorem 6.7, it is useful to have concrete formulas for C and R. We find such formulas now. The dotted line segments in Figure 17 identify two similar right triangles. The height of the larger one times 2 is the Euclidean diameter of B, which by the argument involving Figure 15 is $D = \frac{1}{mp^2}$. The width of this right triangle is $C + \frac{q}{p}$. Corresponding statements hold for the smaller right triangle, and so

$$\frac{C + \frac{q}{p}}{\frac{1}{mp^2}} = \frac{C + \frac{q'}{p'}}{\frac{1}{\delta mp'^2}} \iff p^2 C + pq = \delta p'^2 C + \delta p' q' \iff C = \frac{-pq + \delta p' q'}{p^2 - \delta p'^2}.$$

Because $-\frac{q}{p}$ and $-\frac{q'}{p'}$ are interchanged by inversion about $l_m^\perp$, it follows that $R^2 = (C + \frac{q}{p})(C + \frac{q'}{p'})$. The last display shows that $C + \frac{q'}{p'} = \frac{p^2}{\delta p'^2}(C + \frac{q}{p})$. So $R^2 = \frac{p^2}{\delta p'^2}(C + \frac{q}{p})^2$. We calculate:

$$C + \frac{q}{p} = \frac{-pq + \delta p' q'}{p^2 - \delta p'^2} + \frac{q}{p} = \frac{-p^2 q + \delta pp' q' + p^2 q - \delta p'^2 q}{p(p^2 - \delta p'^2)} = \frac{\delta p'(pq' - p'q)}{p(p^2 - \delta p'^2)}.$$

So

$$R = \left| \frac{(pq' - p'q)\sqrt{\delta}}{p^2 - \delta p'^2} \right|.$$

These formulas for C and R hold if neither $\frac{p}{q}$ nor $\frac{p'}{q'}$ is 0 and the Euclidean radii of B and B' are unequal. If either $\frac{p}{q} = 0$ or $\frac{p'}{q'} = 0$, then one may either directly verify that these formulas still hold or one may apply a continuity argument. We note that if $\delta > \frac{p^2}{p'^2}$, then B has larger Euclidean radius than B' as in Figure 17 and H is the region in $\mathbb{H}$ outside the Euclidean circle with center C and radius R. If $\delta < \frac{p^2}{p'^2}$, then H is the region in $\mathbb{H}$ within this circle.

Finally, we consider the case in which $\delta = \frac{p^2}{p'^2}$. In this case B and B' have the same Euclidean radius and $l_m^\perp$ is a Euclidean half-line. One endpoint of $l_m^\perp$ is ∞ and the other is $-\frac{1}{2}(\frac{q}{p} + \frac{q'}{p'})$. So if $\delta = \frac{p^2}{p'^2}$, then

$$H = \{\tau \in \mathbb{H} : \mathrm{Re}(\tau) < -\frac{1}{2}(\frac{q}{p} + \frac{q'}{p'})\} \text{ if } \frac{p}{q} < \frac{p'}{q'}$$

and

$$H = \{\tau \in \mathbb{H} : \mathrm{Re}(\tau) > -\frac{1}{2}(\frac{q}{p} + \frac{q'}{p'})\} \text{ if } \frac{p}{q} > \frac{p'}{q'}$$

with the convention that every rational number is less than ∞.

Example 6.8. In this example we apply these ideas to the Thurston map f of the main example. Table 3 contains values of $\frac{p}{q}$, $\frac{p'}{q'}$, δ, C, and R and the last column states whether or not H is bounded in the Euclidean metric. The values of p' and q' can be computed using the results at the end of Section 5. The values of δ come from the values of c and d in Table 1.

Figure 18 shows the corresponding half-spaces H in $\mathbb{H}$. The intersection of their complements is shaded. Statement (2) of Theorem 6.7 and the fact that this intersection is a bounded subset of $\mathbb{H}$ imply that f has no Thurston obstructions. Thus the Teichmüller map of f has a fixed point in $\mathbb{H}$, the map f is equivalent

$\frac{p}{q}$	$\frac{p'}{q'}$	δ	C	R	H bounded?
$-\frac{1}{2}$	0	6	2	$\sqrt{6}$	yes
$-\frac{1}{4}$	$\frac{1}{6}$	$\frac{2}{5}$	$\frac{32}{3}$	$\frac{10\sqrt{10}}{3}$	yes
$\frac{1}{8}$	$\frac{1}{4}$	2	0	$4\sqrt{2}$	no
$\frac{1}{4}$	$\frac{1}{2}$	$\frac{2}{5}$	$-\frac{16}{3}$	$\frac{2\sqrt{10}}{3}$	yes
$\frac{1}{3}$	0	$\frac{1}{2}$	-3	$\frac{1}{\sqrt{2}}$	yes
$\frac{7}{16}$	$\frac{1}{4}$	6	$-\frac{88}{43}$	$\frac{12\sqrt{6}}{43}$	yes
$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{5}$	$-\frac{4}{3}$	$\frac{\sqrt{10}}{3}$	yes
$\frac{3}{4}$	$\frac{1}{2}$	6	0	$\frac{2\sqrt{6}}{3}$	yes

Table 3. Half-space data for the main example.

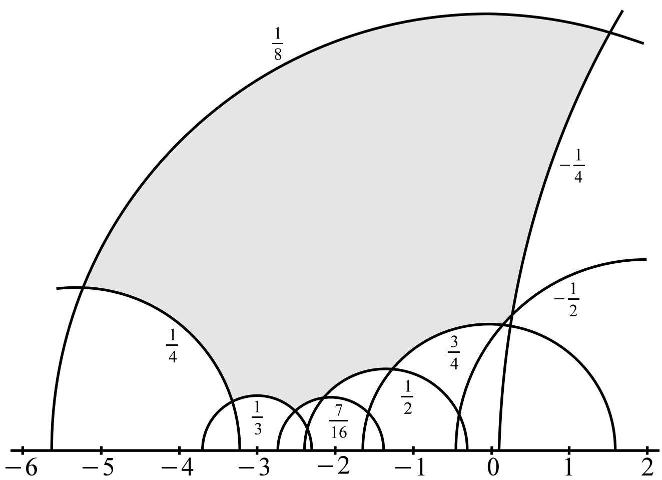

Figure 18. Some half-spaces for the main example.

to a rational map and the finite subdivision rule of the main example is combinatorially conformal. Statement (3) of Theorem 6.7 implies that the fixed point of the Teichmüller map of f is in the shaded region of Figure 18. More half-spaces are drawn in Figure 18 than is necessary to obtain these results; the half-spaces corresponding to $\frac{7}{16}$ and $-\frac{1}{2}$ are not necessary. These two half-spaces are included because computations suggest that the half-space corresponding to each value of $\frac{p}{q} \in \widehat{\mathbb{Q}}$ is contained in one of the eight shown.

7. Dehn twists

We fix our conventions concerning Dehn twists in this paragraph. Let γ be a simple closed curve in an oriented surface S. Let A be a closed regular neighborhood of γ in S, so that there exists an orientation-preserving homeomorphism

$g\colon \{z \in \mathbb{C} : 1 \le |z| \le 2\} \to A$. A (right-handed) Dehn twist about γ is a homeomorphism from S to S which is the identity map outside of A and on A it has the form $g \circ h \circ g^{-1} : A \to A$, where $h(re^{2\pi i\theta}) = re^{2\pi i(\theta - r)}$. Thus for a fixed θ, as r traverses the closed interval $[1, 2]$ in either direction, $h(re^{2\pi i\theta})$ bends to the right as it winds once around the annulus $\{z \in \mathbb{C} : 1 \le |z| \le 2\}$.

Now let γ be a simple closed curve in $S^2 \setminus P_2$, and let $t\colon S^2 \setminus P_2 \to S^2 \setminus P_2$ be a right-handed Dehn twist about γ. We next determine the induced map $\Sigma_t\colon \mathbb{H} \to \mathbb{H}$ on Teichmüller space. If γ is inessential or peripheral, then t is homotopic to the identity map, and so Σ_t is the identity map. So suppose that γ is essential with slope $\frac{p}{q}$. Then γ is homotopic to the image in $S^2 \setminus P_2$ under $p_2 \circ q_2$ of a line segment. Such a line segment is drawn with dashes in Figure 19. Here $\lambda = q\lambda_2 + p\mu_2$ and $\mu = s\lambda_2 + r\mu_2$, where r and s are integers with $ps - qr = 1$, so that λ and μ form a basis of Λ_2. Let $\sigma_t\colon \widehat{\mathbb{Q}} \to \widehat{\mathbb{Q}} \cup \{o\}$ be the induced map on slopes.

We first consider the case for which $\frac{p}{q} = 0$. Then $\sigma_t(0) = 0$. This and the results of Section 6 up to and including Lemma 6.3 imply that Σ_t maps every horocycle based at ∞ to itself. By considering Figure 19, we see that $\sigma_t(\infty) = -\frac{1}{2}$ and $\sigma_t(1) = -1$. This and the results of Section 6 imply that Σ_t maps horocycles based at 0, respectively -1, with Euclidean diameter D to horocycles based at 2, respectively 1, with Euclidean diameter D. Now let $z \in \mathbb{H}$. The element z determines horocycles based at ∞, 0 and -1. The map Σ_t takes the first horocycle to itself, it takes the second horocycle to a horocycle based at 2 maintaining Euclidean diameter and it takes the third horocycle to a horocycle based at 1 maintaining Euclidean diameter. Hence $\Sigma_t(z) = z + 2$ for every $z \in \mathbb{H}$.

Now consider a general value of $\frac{p}{q} \in \widehat{\mathbb{Q}}$. Let r and s be integers such that $ps - qr = 1$. The induced map on $\mathbb{H}$ of a Dehn twist about a simple closed curve with slope $\frac{p}{q}$ is conjugate to the induced map on $\mathbb{H}$ of a Dehn twist about a simple closed curve with slope 0. In terms of matrices, this conjugation has the form

$$\begin{aligned}\begin{bmatrix} -q & -s \\ p & r \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -q & -s \\ p & r \end{bmatrix}^{-1} &= \begin{bmatrix} -q & -s-2q \\ p & r+2p \end{bmatrix} \begin{bmatrix} r & s \\ -p & -q \end{bmatrix} \\ &= \begin{bmatrix} 1+2pq & 2q^2 \\ -2p^2 & 1-2pq \end{bmatrix}.\end{aligned}$$

The matrix $\left[\begin{smallmatrix} -q & -s \\ p & r \end{smallmatrix}\right]$ in $\mathrm{SL}(2,\mathbb{Z})$ represents a Möbius transformation which maps ∞ to $-\frac{q}{p}$. The matrix $\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$ represents the second power of a generator of the stabilizer of ∞ in $\mathrm{PSL}(2,\mathbb{Z})$, and it translates horocycles at ∞ in the counterclockwise direction. So Σ_t is the second power of a generator of the stabilizer of $-\frac{q}{p}$ in $\mathrm{PSL}(2,\mathbb{Z})$, and it translates horocycles at $-\frac{q}{p}$ in the counterclockwise direction.

Now we consider the effect of Dehn twists on a NET map f. Let γ be an essential, nonperipheral simple closed curve in $S^2 \setminus P_2$. Let t be a right-handed Dehn twist about γ. Let d be the degree with which f maps every connected component of $f^{-1}(\gamma)$ to γ. The number of connected components of $f^{-1}(\gamma)$ is $d' = \deg(f)/d$. Let $t_1, \dots, t_{d'}$ be right-handed Dehn twists about these simple closed curves in $S^2 \setminus P_2$. By considering Figure 9, we see that $t^d \circ f$ is homotopic to $f \circ t_1 \circ \cdots \circ t_{d'}$. Hence

$$\Sigma_f \circ \Sigma_t^d = \Sigma_{t_{d'}} \circ \cdots \circ \Sigma_{t_1} \circ \Sigma_f.$$

The Dehn twists among $t_1, \dots, t_{d'}$ which correspond to inessential or peripheral connected components act trivially on $\mathbb{H}$, and the rest induce the same map. We have proved the following theorem.

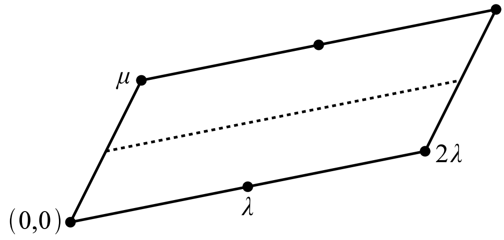

FIGURE 19. Determining Σ_t.

Theorem 7.1. *Let f be a NET map. Let $\frac{p}{q} \in \widehat{\mathbb{Q}}$, and let γ be an essential simple closed curve in $S^2 \setminus P_2$ with slope $\frac{p}{q}$. Let c be the number of connected components of $f^{-1}(\gamma)$ which are essential and nonperipheral in $S^2 \setminus P_2$, and suppose that f maps each of these components to γ with degree d. Suppose that each of these essential connected components has slope $\frac{p'}{q'} \in \widehat{\mathbb{Q}}$. Let*

$$\varphi(z) = \frac{(1+2pq)z + 2q^2}{-2p^2 z + 1 - 2pq} \quad \text{and} \quad \psi(z) = \frac{(1+2p'q')z + 2q'^2}{-2p'^2 z + 1 - 2p'q'}.$$

Then

$$\Sigma_f \circ \varphi^d = \psi^c \circ \Sigma_f.$$

Theorem 7.1 includes the possibility that $f^{-1}(\gamma)$ contains no essential components, in which case $c = 0$ and $\Sigma_f \circ \varphi^d = \Sigma_f$.

Remark 7.2. Every Thurston map $f\colon (S^2, P_f) \to (S^2, P_f)$ defines a virtual endomorphism

$$\phi_f\colon \operatorname{Mod}(S^2, P_f) \dashrightarrow \operatorname{Mod}(S^2, P_f)$$

as in [8]. For NET maps, computation of the slope function σ_f together with the mapping degree and number of preimages amounts to computation of ϕ_f on powers of Dehn twists. This is what Lodge exploits in [5] to calculate the slope function for the NET map of Example 3.2: he finds (in the terminology of Theorem 7.1) ψ^c in terms of φ^d, which allows him to find $\frac{p'}{q'}$ in terms of $\frac{p}{q}$.

Example 7.3. We apply Theorem 7.1 to the main example with slope ∞. It is easy to see that $\sigma_f(\infty) = \infty$; the methods of Section 5 are not needed. We have that $\varphi(z) = \psi(z) = \frac{z}{-2z+1}$. Table 1 shows that $c = 2$ and $d = 5$. The equation $\psi^{-c} \circ \Sigma_f = \Sigma_f \circ \varphi^{-d}$ yields that

$$\Sigma_f\left(\frac{z}{10z+1}\right) = \frac{\Sigma_f(z)}{4\Sigma_f(z)+1}.$$

8. REFLECTIONS

By a reflection ρ of a 2-sphere about a simple closed curve γ we mean a map which is topologically conjugate to the map of $\{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 1\}$ given by $(x,y,z) \mapsto (x,y,-z)$ and which has γ as the set of points fixed by ρ. We are interested in maps of Teichmüller space induced by reflections.

We consider these maps of Teichmüller space in this paragraph. We maintain the setting of Section 1. The boundary of the parallelogram with vertices 0, λ_2,

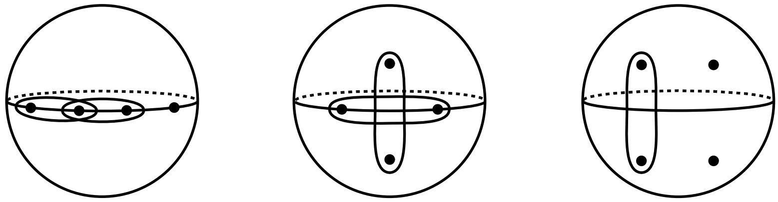

FIGURE 20. The fixed point set γ and P_2.

μ_2 and $\lambda_2+\mu_2$ maps under $p_2\circ q_2$ to a simple closed curve γ in S^2. Let ρ be a reflection of S^2 about γ. Let $\sigma_\rho\colon \widehat{\mathbb{Q}}\to\widehat{\mathbb{Q}}\cup\{o\}$ be the induced map on slopes, and let $\Sigma_\rho\colon \mathbb{H}\to\mathbb{H}$ be the induced map on Teichmüller space. Let $\tau\in\mathbb{H}$, and let $\tau'=\Sigma_\rho(\tau)$. We see that $\sigma_\rho(0)=0$. So $\mathrm{mod}_\tau(0)=\mathrm{mod}_{\tau'}(0)$. This and Lemma 6.3 and the discussion at the beginning of Section 6 imply that τ and τ' are on the same horocycle of $\mathbb{H}$ at ∞. In the same way τ and τ' are on the same horocycle of $\mathbb{H}$ at 0. But distinct horocycles meet in at most two points. So from this alone we conclude that either $\tau'=\tau$ or $\tau'=-\overline{\tau}$. Since in general $\sigma_\rho(x)=-x$, it follows that $\tau'=-\overline{\tau}$. So $\Sigma_\rho(\tau)=-\overline{\tau}$ for every $\tau\in\mathbb{H}$.

The discussion of the previous paragraph can be generalized as follows. Let γ be a simple closed curve in S^2, and let ρ be a reflection of S^2 about γ. Suppose that $\rho(P_2)=P_2$. The number of elements of P_2 fixed by ρ is either 4, 2 or 0. The three possibilities are illustrated in Figure 20 with γ drawn as an equator. In every case there exist two essential simple closed curves in $S^2\setminus P_2$ with distinct slopes which are fixed by ρ. Two such curves are drawn in the first two parts of Figure 20. In the third part, γ is one of these two curves. If these two curves have slopes $\frac{p}{q}$ and $\frac{r}{s}$, then σ_ρ is the reflection of $\widehat{\mathbb{Q}}$ which fixes $\frac{p}{q}$ and $\frac{r}{s}$. Similarly, Σ_ρ is the reflection of $\mathbb{H}$ which fixes the geodesic with endpoints $-\frac{q}{p}$ and $-\frac{s}{r}$.

Theorem 8.1. *Let f be a NET map in the setting of Section 1. Let $\frac{p}{q},\frac{r}{s}\in\widehat{\mathbb{Q}}$. Suppose that $\sigma_f(\frac{p}{q})\neq o$, $\sigma_f(\frac{r}{s})\neq o$ and $\sigma_f(\frac{p}{q})\neq\sigma_f(\frac{r}{s})$. Let $\lambda=q\lambda_2+p\mu_2$ and $\mu=s\lambda_2+r\mu_2$. Let d, respectively d', be the order of the image of λ, respectively μ, in Λ_2/Λ_1. Suppose that (λ,μ) is a basis of Λ_2 and that $(d\lambda,d'\mu)$ is a basis of Λ_1. Suppose that $q_1^{-1}(p_1^{-1}(P_2))$ is invariant under the reflection of $\mathbb{R}^2$ given by $x\lambda+y\mu\mapsto(2d-x)\lambda+y\mu$, where x and y are real numbers. Let ρ_1 be the reflection of $\mathbb{H}$ about the geodesic whose endpoints are $-\sigma_f(\frac{p}{q})^{-1}$ and $-\sigma_f(\frac{r}{s})^{-1}$. Let ρ_2 be the reflection of $\mathbb{H}$ about the geodesic whose endpoints are $-\frac{q}{p}$ and $-\frac{s}{r}$. Then*

$$\Sigma_f\circ\rho_2=\rho_1\circ\Sigma_f.$$

Proof. The situation is as in Figure 9, except that now the vertices of the large parallelogram are all elements of Λ_1. The discussion preceding Theorem 8.1 shows that ρ_2 is induced by a reflection r_2 of S^2 about the simple closed curve which is the image under $p_2\circ q_2$ of the boundary of the parallelogram whose vertices are 0, λ, μ and $\lambda+\mu$. Likewise let r_1 be a reflection of S^2 about the image under $p_1\circ q_1$ of the boundary of the parallelogram whose vertices are 0, $d\lambda$, $d'\mu$ and $d\lambda+d'\mu$. In other words, we may take r_1 to be the reflection of S^2 induced by the reflection of $\mathbb{R}^2$ given by $x\lambda+y\mu\mapsto(2d-x)\lambda+y\mu$, where x and y are real numbers. The assumptions imply that r_1 restricts to a homeomorphism of $S^2\setminus P_2$. It fixes essential simple closed curves with slopes $\sigma_f(\frac{p}{q})$ and $\sigma_f(\frac{r}{s})$, so the Teichmüller map which it induces is ρ_1. It is clear that $r_2 \circ f = f \circ r_1$. Thus $\Sigma_f \circ \rho_2 = \rho_1 \circ \Sigma_f$. This proves Theorem 8.1. $\square$

Corollary 8.2. *In the situation of Theorem* 8.1*, the Teichmüller map* Σ_f *maps the geodesic in* $\mathbb{H}$ *with endpoints* $-\frac{q}{p}$ *and* $-\frac{s}{r}$ *to the geodesic with endpoints* $-\sigma_f(\frac{p}{q})^{-1}$ *and* $-\sigma_f(\frac{r}{s})^{-1}$.

9. Aff(f) and functional equations

Let f be a NET map in the setting of Section 1. The results of Sections 7 and 8 show that Σ_f satisfies certain "functional equations". By this we mean that there are choices of Möbius transformations φ and ψ (which might reverse orientation) such that $\Sigma_f \circ \varphi = \psi \circ \Sigma_f$. The aim of this section is to unify and clarify these results.

Let $i \in \{1,2\}$. Every homeomorphism from $S^2 \setminus P_i$ to $S^2 \setminus P_i$ is isotopic to a homeomorphism which lifts via $p_i \circ q_i$ to an affine map from $\mathbb{R}^2$ to $\mathbb{R}^2$ taking Λ_i bijectively to Λ_i. This is essentially the content of Proposition 2.7 of [4] by Farb and Margalit. This leads us to define the *affine group* of f to be the set Aff(f) of all affine transformations of $\mathbb{R}^2$ which bijectively stabilize Λ_1, Λ_2 and $q_1^{-1}(p_1^{-1}(P_2))$. We define the *general linear group* GL(f) and the *special linear group* SL(f) of f analogously. If $\delta \in \text{Aff}(f)$ and if γ is a rotation of order 2 in Γ_1 which fixes $\lambda \in \Lambda_1$, then $\delta\gamma\delta^{-1}$ is a rotation of order 2 which fixes $\delta(\lambda)$. Since these rotations generate Γ_1, it follows that Aff(f) normalizes Γ_1. Likewise Aff(f) normalizes Γ_2. Because Aff(f) normalizes Γ_1 and Γ_2, its action on $\mathbb{R}^2$ induces actions on $\mathbb{R}^2/\Gamma_1$ and $\mathbb{R}^2/\Gamma_2$. Let $\delta \in \text{Aff}(f)$, let δ_1 be the map which it induces on $\mathbb{R}^2/\Gamma_1$ and let δ_2 be the map which it induces on $\mathbb{R}^2/\Gamma_2$. Since the lift of f to $\mathbb{R}^2$ is the identity map, this lift commutes with δ. Thus $f \circ \delta_1 = \delta_2 \circ f$. The assumptions imply that both δ_1 and δ_2 stabilize P_2, and so both δ_1 and δ_2 induce maps on the Teichmüller space of $S^2 \setminus P_2$. This leads to the functional equation $\Sigma_f \circ \Sigma_{\delta_2} = \Sigma_{\delta_1} \circ \Sigma_f$. Of course, both Σ_{δ_1} and Σ_{δ_2} are Möbius transformations. We have proved the first assertion of the following theorem. The second assertion concerning Σ_{δ_1} and Σ_{δ_2} follows from the discussion between here and Example 9.2.

Theorem 9.1. *Let f be a NET map in the setting of Section* 1*. Let $\delta \in$ Aff(f). Let δ_i be the map which δ induces on $\mathbb{R}^2/\Gamma_i$ for $i \in \{1,2\}$. Then $\Sigma_f \circ \Sigma_{\delta_2} = \Sigma_{\delta_1} \circ \Sigma_f$. If δ_i preserves, respectively reverses, orientation, then Σ_{δ_i} is a Möbius transformation from $PSL(2,\mathbb{Z})$, respectively $PGL(2,\mathbb{Z}) \setminus PSL(2,\mathbb{Z})$, for $i \in \{1,2\}$.*

In this and the next two paragraphs we discuss computations involving Theorem 9.1. Let f be a NET map in the setting of Section 1. Let $\delta \in \text{Aff}(f)$. We want explicit forms for the Möbius transformations Σ_{δ_1} and Σ_{δ_2}. For this it suffices to understand how Σ_{δ_1} and Σ_{δ_2} act on the boundary of $\mathbb{H}$.

We first consider δ_2. We express δ in terms of the chosen basis (λ_2,μ_2) of Λ_2, and we calculate slopes of lines in $\mathbb{R}^2$ with respect to this basis. Let p, q, p', q' be integers as usual so that δ^{-1} maps lines in $\mathbb{R}^2$ with slope $\frac{p}{q}$ to lines with slope $\frac{p'}{q'}$. Let $\tau \in \mathbb{H}$, and suppose that $\Sigma_{\delta_2}(\tau) = \tau'$. Then $\text{mod}_\tau(\frac{p}{q}) = \text{mod}_{\tau'}(\frac{p'}{q'})$. Using Lemma 6.3 and Corollary 6.2, we see that $\Sigma_{\delta_2}(-\frac{q}{p}) = -\frac{q'}{p'}$. If δ preserves orientation and if $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is the matrix of the linear part of δ, then the inverse of the

Möbius transformation $z \mapsto \frac{az+b}{cz+d}$ maps $\frac{q}{p}$ to $\frac{q'}{p'}$. Combining this with the previous statement yields that $\Sigma_{\delta_2}(z) = \frac{dz+b}{cz+a}$. If δ reverses orientation, then $\Sigma_{\delta_2}(z) = \frac{d\overline{z}+b}{c\overline{z}+a}$. This determines Σ_{δ_2} in terms of δ. In particular, Σ_{δ_2} comes from $\mathrm{PGL}(2,\mathbb{Z})$.

Whereas computing Σ_{δ_2} is easy, computing Σ_{δ_1} is usually not, although see Examples 9.2 and 9.3 for two easy special cases. Its computation has much in common with the computation of σ_f using spin mirrors. In general, we begin with an essential simple closed curve γ in $S^2 \setminus P_2$. We lift it to $\mathbb{R}^2$ using $p_1 \circ q_1$. Although we may assume that this lift is piecewise linear, we may not assume that it is a line segment. We wish to understand how δ_1 acts on slopes, so we apply δ to this lift. We then use spin mirrors as in Section 5 to compute the slope of $\delta_1(\gamma)$. Doing this for curves γ with slopes 0 and ∞ obtains two corresponding slopes $\frac{c}{a}$ and $\frac{d}{b}$, where (a,c) and (b,d) form a basis of $\mathbb{Z}^2$. We multiply one of these vectors by -1 if necessary so that $\left|\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right| = 1$. If δ preserves orientation, then its action on slopes is given by the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. If δ reverses orientation, then the matrix is $\left[\begin{smallmatrix} a & -b \\ c & -d \end{smallmatrix}\right]$. As for δ_2, it follows that $\Sigma_{\delta_1}(z) = \frac{dz+b}{cz+a}$ if orientation is preserved and $\Sigma_{\delta_1}(z) = -\frac{d\overline{z}+b}{c\overline{z}+a}$ if orientation is reversed. In particular, Σ_{δ_1} comes from $\mathrm{PGL}(2,\mathbb{Z})$.

In the following examples, we apply Theorem 9.1 to our main example of a NET map.

Example 9.2. For our main example of a NET map, we have that $\Lambda_2 = \mathbb{Z}^2$ and $\Lambda_1 = \langle(2,-1),(0,5)\rangle$. Furthermore, $q_1^{-1}(p_1^{-1}(P_2))$ consists of the six cosets of $2\Lambda_1$ represented by $(0,0)$, $(0,5)$, $(2,0)$, $(2,-2)$, $(2,3)$ and $(2,5)$. Let $\delta\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map with matrix $\left[\begin{smallmatrix} 1 & 0 \\ 5 & 1 \end{smallmatrix}\right]$ with respect to the standard basis. Then $\delta \in \mathrm{SL}(f)$. The discussion following Theorem 9.1 shows that $\Sigma_{\delta_2}(z) = \frac{z}{5z+1}$. Let B_1 be the set of usual spin mirrors relative to $p_1 \circ q_1$ for the main example. Because δ fixes $(0,1)$ and stabilizes every vertical line, it stabilizes B_1. Hence δ restricts to a homeomorphism of $\mathbb{R}^2 \setminus B_1$ to itself. It follows that the matrix which δ_1 determines for its action on slopes of essential curves in $S^2 \setminus P_2$ relative to the basis (λ_2, μ_2) is the same matrix which δ_1 determines for its action on Λ_1 relative to the basis (λ_1, μ_1). This matrix is $\left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$. We conclude that

$$\Sigma_f\left(\frac{z}{5z+1}\right) = \frac{\Sigma_f(z)}{2\Sigma_f(z)+1}.$$

The action of Σ_f and σ_f are related by conjugation by $z \mapsto -\frac{1}{z}$. Hence $\sigma_f(z-5) = \sigma_f(z) - 2$, and so $\sigma_f(z+5) = \sigma_f(z) + 2$. The square of δ_1 is a Dehn twist about an essential simple closed curve in $S^2 \setminus P_2$ with slope ∞. Compare this functional equation with that of Example 7.3.

Example 9.3. We continue with the main example. Let $\delta\colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map with matrix $\left[\begin{smallmatrix} -1 & 0 \\ 1 & 1 \end{smallmatrix}\right]$. Then $\delta(2,-1) = (-2,1)$ and $\delta(0,1) = (0,1)$. So δ reverses orientation and it fixes the lines generated by $(2,-1)$ and $(0,1)$. It stabilizes Λ_1 and Λ_2. Furthermore a glance at Figure 2 shows that it stabilizes $q_1^{-1}(p_1^{-1}(P_2))$. So $\delta \in \mathrm{GL}(f)$. We have that $\Sigma_{\delta_2}(z) = \frac{\overline{z}}{\overline{z}-1}$. The map on slopes of essential simple closed curves in $S^2 \setminus P_2$ induced by δ_1 fixes 0 and ∞ and takes 1 to -1. Hence $\Sigma_{\delta_1}(z) = -\overline{z}$. We conclude that

$$\Sigma_f\left(\frac{\overline{z}}{\overline{z}-1}\right) = -\overline{\Sigma}_f(z),$$

and

$$\sigma_f(-z-1) = -\sigma_f(z).$$

Remark 9.4. We return to the graph of the slope function σ_f for the main example which appears at the end of Section 5. A prominent fuzzy line appears with positive slope less than 1. Suppose that this line is given by $y = mx + b$. Example 9.2 implies that $\sigma_f(x+5) = \sigma_f(x)+2$. The line given by $y = mx+b$ should also satisfy this functional equation, which implies that $m = \frac{2}{5}$. Example 9.3 implies that $\sigma_f(-x-1) = -\sigma_f(x)$. The line should also satisfy this functional equation, and so $b = \frac{1}{5}$. So the special line with positive slope less than 1 is given by $y = \frac{2}{5}x + \frac{1}{5}$.

10. Constant Teichmüller maps

This section deals with NET maps whose associated Teichmüller maps are constant. We aim to find an algebraic formulation of what it means for the Teichmüller map of a NET map to be constant. The results of this section are extended in Saenz Maldonado's thesis [9]. We begin with the following lemma.

Lemma 10.1. *Let $\varphi\colon \mathbb{Z}^2 \to A$ be a surjective group homomorphism from $\mathbb{Z}^2$ to a finite Abelian group A. Let $a \in A$, and let B be a cyclic subgroup of A. Then the quotient group A/B is cyclic and the image of a in A/B generates A/B if and only if there exists a basis of $\mathbb{Z}^2$ consisting of elements α and α' with $\varphi(\alpha) = a$ and $\varphi(\alpha')$ a generator of B.*

Proof. It is clear that if α and α' form a basis of $\mathbb{Z}^2$ with $\varphi(\alpha) = a$ and $\varphi(\alpha')$ a generator of B, then A/B is cyclic and the image of a in A/B generates A/B.

To prove the converse, suppose that A/B is cyclic and that the image of a in A/B generates A/B. Since the kernel K of φ has finite index in $\mathbb{Z}^2$, there exists a basis of $\mathbb{Z}^2$ consisting of elements β_1 and β_2 and positive integers m and n with $m|n$ such that $m\beta_1$ and $n\beta_2$ form a basis of K. Without loss of generality we assume that $\beta_1 = (1,0)$ and $\beta_2 = (0,1)$, so that $(m,0)$ and $(0,n)$ form a basis of K. Hence $A \cong \mathbb{Z}/m\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Let $C = \mathbb{Z}/n\mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}$. Let $\varphi_1\colon \mathbb{Z}^2 \to C$ be the canonical group homomorphism. By the standard homomorphism theorems of group theory, there exists a group homomorphism $\varphi_2\colon C \to A$ such that $\varphi = \varphi_2 \circ \varphi_1$. Let c and c' be elements of C such that $\varphi_2(c) = a$ and the image of $\varphi_2(c')$ in A/B generates A/B. Suppose that c and c' are given in coordinates by $c = (c_1, c_2)$ and $c' = (c'_1, c'_2)$. The determinant $d = \begin{vmatrix} c_1 & c'_1 \\ c_2 & c'_2 \end{vmatrix}$ is then an element of $\mathbb{Z}/n\mathbb{Z}$. We aim to prove that it is possible to choose c and c' so that $d = 1$.

To begin this, let p be a prime such that $p|m$. The assumptions imply that $C/pC \cong A/pA \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$ and that the images of c and c' in A/pA generate A/pA. So the images of c and c' in C/pC are linearly independent elements of this vector space. Thus the image of d in $\mathbb{Z}/p\mathbb{Z}$ is not 0.

Now let p be a prime such that $p|n$ but $p \nmid m$. The assumptions imply that $C/pC \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/p\mathbb{Z}$, that $A/pA \cong \mathbb{Z}/p\mathbb{Z}$ and that the images of c and c' in A/pA generate A/pA. In this case the images of c and c' in C/pC generate a nonzero subspace of C/pC, the image of the kernel of φ_2 is a 1-dimensional subspace of C/pC and C/pC is the sum of these two subspaces. So we may modify c and c' by elements of the kernel of φ_2 if necessary to make the images of c and c' in C/pC linearly independent. Once this is done, the image of d in $\mathbb{Z}/p\mathbb{Z}$ is not 0.

We use the Chinese remainder theorem to modify c and c' as in the previous paragraph for every prime p such that $p|n$ but $p \nmid m$. Once this is done, d is a unit modulo n. Now we replace c' by $d^{-1}c'$. It is still true that $\varphi_2(b) = a$ and $\varphi_2(c')$ generates B. Furthermore the new determinant is 1. Now we use the fact that the canonical group homomorphism from $\mathrm{SL}(2,\mathbb{Z})$ to $\mathrm{SL}(2,\mathbb{Z}/n\mathbb{Z})$ is surjective. So there exists a basis of $\mathbb{Z}^2$ consisting of elements α and α' with $\varphi_1(\alpha) = c$ and $\varphi_1(\alpha') = c'$. Hence $\varphi(\alpha) = a$ and $\varphi(\alpha')$ generates B.

This proves Lemma 10.1. □

Now we find an algebraic formulation of what it means for the Teichmüller map of a NET map to be constant. Let f be a NET map in the setting of Section 1. Combining statements 1 and 4 of Theorem 5.1 of [1] implies that the Teichmüller map of f is constant if and only if for every essential, nonperipheral simple closed curve δ in $S^2 \setminus P_2$, every connected component of $f^{-1}(\delta)$ is either null or peripheral. This and Theorem 4.1 leads us to consider the following. Let λ and μ be elements of Λ_2 which form a basis of Λ_2. Let c_1, c_2, c_3, c_4 be the coset numbers for $q_1^{-1}(p_1^{-1}(P_2))$ relative to λ and μ. Statement (2) of Theorem 4.1 now shows that the Teichmüller map of f is constant if and only if $c_2 = c_3$ for every choice of λ and μ.

This leads us to make a definition. Let A be a finite Abelian group. We say that a subset H of A is *nonseparating* if and only if it satisfies the following conditions. First, H is a disjoint union of the form $H = \{\pm h_1\} \amalg \{\pm h_2\} \amalg \{\pm h_3\} \amalg \{\pm h_4\}$. (It is possible that $h_i = -h_i$.) Let B be a cyclic subgroup of A such that A/B is cyclic. Let c_1, c_2, c_3, c_4 be the coset numbers for H relative to B and some generator of A/B. The main condition is that $c_2 = c_3$ for every such choice of B and generator of A/B. We say that H is nonseparating because it never separates c_2 from c_3. Lemma 10.1 and the intervening discussion yield the following theorem.

Theorem 10.2. *Let f be a NET map in the setting of Section* 1. *Then the Teichmüller map of f is constant if and only if $p_1^{-1}(P_2)$ is a nonseparating subset of $\Lambda_2/2\Lambda_1$.*

Theorem 10.2 provides a strategy for constructing NET maps whose Teichmüller maps are constant. The first step, and only step which is not straightforward, is to construct a finite Abelian group A generated by two elements with $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ such that A has a nonseparating subset H. We then construct lattices $\Lambda_1 \subseteq \Lambda_2 \subseteq \mathbb{R}^2$ such that $\Lambda_2/2\Lambda_1 \cong A$ and use this isomorphism to identify $\Lambda_2/2\Lambda_2$ with A. We construct an isomorphism from Λ_2 to Λ_1, which in effect constructs a Euclidean Thurston map g corresponding to Λ_1 and Λ_2. We then construct an orientation-preserving homeomorphism $h\colon S^2 \to S^2$ such that $h(P_2) = p_1(H)$. As in Section 2, it follows that $f = h \circ g$ is a NET map if it has four postcritical points (which it usually does), and Theorem 10.2 shows that its Teichmüller map is constant. Since $|\Lambda_2/\Lambda_1| = \deg(f)$, we have that $|A| = 4\deg(f)$.

Examples 10.3 and 10.4 give examples of finite Abelian groups with nonseparating subsets.

Example 10.3. In this example we consider $A = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We show that $H = \{(0,0), \pm(1,0), (2,0), \pm(1,1)\}$ is a nonseparating subset of A. Let B be a cyclic subgroup of A such that A/B is cyclic. Then either $|B| = 4$ or $|B| = 2$. If $|B| = 4$, then either $B = \langle(1,0)\rangle$ or $B = \langle(1,1)\rangle$. One verifies in these cases that $c_1 = c_2 = c_3 = 0$ and $c_4 = 1$, and so $c_2 = c_3$, as desired. If $|B| = 2$, then either $B = \langle(0,1)\rangle$ or $B = \langle(2,1)\rangle$. One verifies in these cases that $c_1 = 0$, $c_2 = c_3 = 1$ and $c_4 = 2$. Thus H is a nonseparating subset of A.

Example 10.4. In this example we show that the set H of elements of order 3 is a nonseparating subset of $A = \mathbb{Z}/6\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$. The 3-torsion subgroup of A is isomorphic to $\mathbb{Z}/3\mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. It has eight elements of order 3, which are paired by inversion. So the set H contains four pairs of elements which are mutually inverse. To verify that H is a nonseparating subset for A, let B be a cyclic subgroup of A such that A/B is cyclic. Then $B \cong A/B \cong \mathbb{Z}/6\mathbb{Z}$. Elements of order 3 in A map to elements of order either 1 or 3 in A/B. Both B and A/B contain exactly one pair of mutually inverse elements of order 3. It follows that $c_1 = 0$ and $c_2 = c_3 = c_4 = 2$. Hence $c_2 = c_3$, and so H is a nonseparating subset of A.

The next lemma provides a simple but limited way to produce nonseparating subsets from known ones.

Lemma 10.5. *Let A be a finite Abelian group, and let $H = \{\pm h_1, \pm h_2, \pm h_3, \pm h_4\}$ be a nonseparating subset of A. Let h be an element of order 2 in A, and let $H' = H + h = \{\pm(h_1 + h), \pm(h_2 + h), \pm(h_2 + h), \pm(h_4 + h)\}$. Then H' is a nonseparating subset of A.*

Proof. Let B be a cyclic subgroup of A such that A/B is cyclic. Let c_1, c_2, c_3, c_4 be the coset numbers for H relative to B and a generator of A/B. Let c'_1, c'_2, c'_3, c'_4 be the corresponding coset numbers for H'. If $h \in B$, then $c'_k = c_k$ for $k \in \{1, 2, 3, 4\}$. If $h \notin B$, then the order of A/B is even, say, $2m$. In this case $c'_k = m - c_k$ for $k \in \{1, 2, 3, 4\}$. It follows that if $c_2 = c_3$, then $c'_2 = c'_3$. This proves Lemma 10.5. □

The next lemma shows that a nonseparating subset for a subgroup is a nonseparating subset for the group.

Lemma 10.6. *If A is a finite Abelian group and if A' is a subgroup of A, then every subset of A' which is nonseparating for A' is nonseparating for A.*

Proof. Let A be a finite Abelian group, let A' be a subgroup of A and suppose that H is a subset of A' which is nonseparating for A'. Let B be a cyclic subgroup of A such that A/B is cyclic. Then both $A' \cap B$ and $A'/A' \cap B$ are cyclic. Choosing a generator for A/B determines an ordering of the cosets of B in A. This determines an ordering of the cosets of $A' \cap B$ in A', and this ordering of the cosets of $A' \cap B$ is determined by a generator of $A'/A' \cap B$. It follows from this and the definition that H is nonseparating for A. This proves Lemma 10.6. □

Example 10.7. In this example we construct a rational function with degree 4 which is a NET map whose Teichmüller map is constant. We follow the strategy outlined immediately after Theorem 10.2.

We first construct an Abelian group A with order $4 \cdot 4 = 16$ which has a nonseparating subset. We take $A = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/4\mathbb{Z}$. The group A contains the subgroup $A' = \langle (1, 0), (0, 2) \rangle$, which is isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Example 10.3 implies that $H = \{(0, 0), \pm(1, 0), (2, 0), \pm(1, 2)\}$ is a nonseparating subset of A'. Hence Lemma 10.6 implies that H is a nonseparating subset of A.

Now we identify $\mathbb{R}^2$ with $\mathbb{C}$, and we let Λ_2 be any lattice in $\mathbb{C}$. We let $\Lambda_1 = 2\Lambda_2$, so that $\Lambda_2/2\Lambda_1 \cong A$. For an isomorphism from Λ_2 to Λ_1, we choose the map $z \mapsto 2z$. The lattices Λ_1, Λ_2 and the map $z \mapsto 2z$ determine a Lattès map $g\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ from the Riemann sphere to itself up to analytic conjugation.

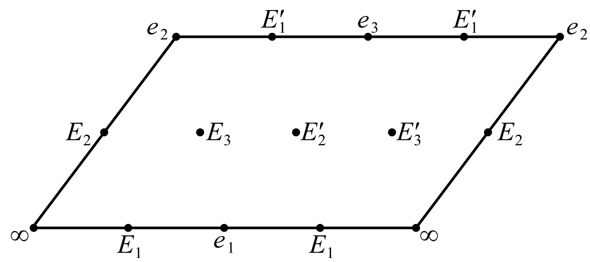

FIGURE 21. A fundamental domain for Γ_1.

In this paragraph we identify g. Let $\wp$ be the Weierstrass function with group of periods $2\Lambda_1$. Figure 21 shows a fundamental domain for the action of Γ_1 on $\mathbb{C}$. The dots are elements of Λ_2. The lower left corner is 0. Points are labeled by their images in $\widehat{\mathbb{C}}$ under $\wp$. For our usual branched covers $p_1 \circ q_1 \colon \mathbb{C} \to S^2$ and $p_2 \circ q_2 \colon \mathbb{C} \to S^2$, we take $p_1(q_1(z)) = \wp(z)$ and $p_2(q_2(z)) = \wp(2z)$. It has long been known that $\wp(2z)$ is a rational function of $\wp(z)$. The map g is this rational function. Thus $g(e_1) = g(e_2) = g(e_3) = g(\infty) = \infty$ and $g(E_k) = g(E_k') = e_k$ for $k \in \{1, 2, 3\}$. The critical points of g are E_1, E_1', E_2, E_2', E_3, E_3'. The postcritical set of g is $\{e_1, e_2, e_3, \infty\}$. According to line 3.41 of [12],

$$g(z) = \frac{(z^2 - s_2)^2 + 8s_3 z}{4(z^3 + s_2 z - s_3)},$$

where $s_2 = e_1e_2 + e_1e_3 + e_2e_3$ and $s_3 = e_1e_2e_3$. The only restriction on e_1, e_2 and e_3 is that they are distinct and that $e_1 + e_2 + e_3 = 0$.

We identify A with a subgroup of $\mathbb{C}/2\Lambda_1$ so that the images in $\widehat{\mathbb{C}}$ of the elements $(0,0)$, $(1,0)$, $(2,0)$, $(1,2)$ of S are ∞, E_1, e_1, E_1' in order. We let $h \colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be an orientation-preserving homeomorphism such that $h(e_1) = E_1'$, $h(e_2) = E_1$, $h(e_3) = e_1$ and $h(\infty) = \infty$, so that h takes the postcritical points of g to the points of $\widehat{\mathbb{C}}$ corresponding to H.

Let $f = h \circ g$. Then f is a Thurston map. Its critical points are E_1, E_1', E_2, E_2', E_3, E_3'. These points are taken by f to E_1', E_1, e_1. Moreover, $f(e_1) = \infty$. Now we see that f has exactly four postcritical points, and so it is a NET map whose Teichmüller map is constant.

To construct a specific rational map equivalent to g, we choose the simple case in which $e_1 = 1$, $e_2 = -1$ and $e_3 = 0$. Then $s_2 = -1$ and $s_3 = 0$. Hence

$$g(z) = \frac{(z^2+1)^2}{4(z^3 - z)}.$$

To find E_1 and E_1', we solve the equation $g(z) = e_1 = 1$. This leads to the equation $(z^2+1)^2 = 4(z^3 - z)$, then $z^4 - 4z^3 + 2z^2 + 4z + 1 = 0$ and finally $(z^2 - 2z - 1)^2 = 0$. We take $E_1 = 1 + \sqrt{2}$ and $E_1' = 1 - \sqrt{2}$. Because g is an odd function, we may take $E_2 = -1 - \sqrt{2}$ and $E_2' = -1 + \sqrt{2}$. Set $h(z) = -\sqrt{2}z + 1$. One verifies that $h(e_1) = E_1'$, $h(e_2) = E_1$, $h(e_3) = e_1$ and $h(\infty) = \infty$. Thus $f = h \circ g$ is a rational function which is a NET map whose Teichmüller map is constant.

Rather than explicitly calculating f, we explicitly calculate an analytic conjugate of f which is simpler. For this, we observe that there exists a unique orientation-preserving isometry k of the hyperbolic plane $\mathbb{H}$ which rotates about i through the

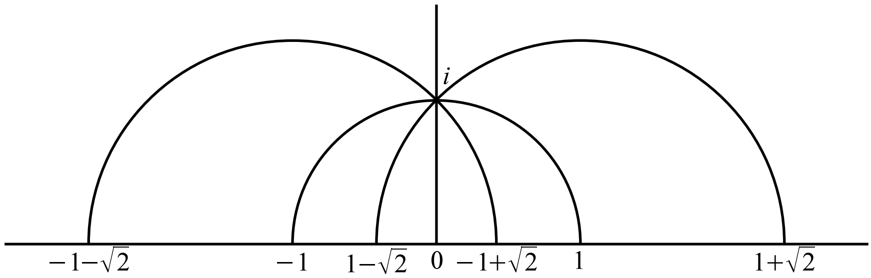

FIGURE 22. Understanding the map k.

angle $5\pi/4$; see Figure 22, where eight hyperbolic sectors with angle $\pi/4$ are drawn at i. The map k has order 8 and stabilizes the set $\{0, \pm 1, \pm 1 \pm \sqrt{2}, \infty\}$. It acts on these points as follows:

$$0 \mapsto -1-\sqrt{2} \mapsto 1 \mapsto 1-\sqrt{2} \mapsto \infty \mapsto -1+\sqrt{2} \mapsto -1 \mapsto 1+\sqrt{2} \mapsto 0.$$

Set $F = k \circ f \circ k^{-1}$. One easily verifies that F is a rational function with degree 4 such that 0, ± 1, ∞ are all critical points of F with $F(0) = F(\infty) = \infty$ and $F(1) = F(-1) = 0$. Thus $F(z) = K(z - z^{-1})^2$ for some complex number K. To determine K, we note that $F(\sqrt{2}-1) = \sqrt{2}-1$. Hence $K(\sqrt{2}-1-(\sqrt{2}+1))^2 = \sqrt{2}-1$, and so $K = \frac{\sqrt{2}-1}{4}$. We conclude that $F(z) = \frac{\sqrt{2}-1}{4}(z - z^{-1})^2$ is a NET map whose Teichmüller map is constant.

In this paragraph we find a functional equation which proves that $\Sigma_F(z) = \Sigma_f(z) = i$ for every $z \in \mathbb{H}$. This not only gives another proof that the Teichmüller map of F is constant but also determines the value of the constant. We choose a basis (λ_2, μ_2) of Λ_2 in the straightforward way relative to the fundamental domain in Figure 21, so that $\wp(\lambda_2) = E_1$ and $\wp(\mu_2) = E_2$. The six $2\Lambda_1$-cosets of $q_1^{-1}(p_1^{-1}(P_2)) = \wp^{-1}(P_2)$ in Λ_2 are represented by 0, λ_2, $2\lambda_2$, $3\lambda_2$, $\lambda_2 + 2\mu_2$ and $3\lambda_2 + 2\mu_2$. Let $\delta \colon \mathbb{C} \to \mathbb{C}$ be the map defined by $\delta(z) = z + 2\lambda_2$. Then $\delta \in \mathrm{Aff}(f)$. Since $\delta \in \Gamma_2$, the map δ_2 from Section 9 is the identity map, and so Σ_{δ_2} is the identity map. Similarly, δ_1^2 and $\Sigma_{\delta_1}^2$ are identity maps. To better understand δ_1, we note that the closed interval $[e_2, e_3] = [-1, 0]$ is a core arc for an essential simple closed curve in $\widehat{\mathbb{C}} \setminus \{0, \pm 1, \infty\}$ with slope 0. Hence $h([e_2, e_3]) = [e_1, E_1] = [1, 1+\sqrt{2}]$ is a core arc for an essential simple closed curve in $\widehat{\mathbb{C}} \setminus h(\{0, \pm 1, \infty\})$ with slope 0. Similarly, $[\infty, e_2] = [-\infty, -1]$ is a core arc for an essential simple closed curve in $\widehat{\mathbb{C}} \setminus \{0, \pm 1, \infty\}$ with slope ∞, and $h([\infty, e_2]) = [E_1, \infty] = [1+\sqrt{2}, \infty]$ is a core arc for an essential simple closed curve in $\widehat{\mathbb{C}} \setminus h(\{0, \pm 1, \infty\})$ with slope ∞. From this we see that the map which δ_1 induces on slopes relative to $\widehat{\mathbb{C}} \setminus h(\{0, \pm 1, \infty\})$ interchanges 0 and ∞. So Σ_{δ_1} is an involution which interchanges 0 and ∞. Theorem 9.1 implies that Σ_{δ_1} comes from $\mathrm{PSL}(2, \mathbb{Z})$. Hence $\Sigma_{\delta_1}(z) = -\frac{1}{z}$. Because $\Sigma_f \circ \Sigma_{\delta_2} = \Sigma_{\delta_1} \circ \Sigma_f$, we obtain that $\Sigma_f(z) = -\frac{1}{\Sigma_f(z)}$ for every $z \in \mathbb{H}$. Hence $\Sigma_f(z) = i$ for every $z \in \mathbb{H}$. Similarly, $\Sigma_F(z) = i$ for every $z \in \mathbb{H}$.

This example illustrates the discussion at the beginning of Section 3 concerning subdivision maps of finite subdivision rules. What follows is a brief description of this. Let $G = k \circ g \circ k^{-1}$ and $H = k \circ h \circ k^{-1}$ (not to be confused with the nonseparating subset of A). Then G is a Euclidean Thurston map, H is an

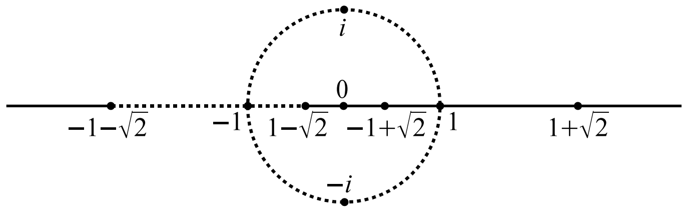

FIGURE 23. The first subdivision of the subdivision complexes for F and G.

orientation-preserving homeomorphism and $F = H \circ G$. The critical points of F and G are 0, ± 1, $\pm i$ and ∞. The postcritical set of G is the image under k of the postcritical set of g, namely, $k(\{e_1, e_2, e_3, \infty\}) = \{\pm 1 \pm \sqrt{2}\}$. Including the action of k, the image in $\widehat{\mathbb{C}}$ of the boundary of the fundamental domain in Figure 21 is $(-\infty, -1-\sqrt{2}) \cup (1-\sqrt{2}, \infty) \cup \{\infty\}$. The map G maps this set into itself. Thus this set together with vertices at $\{\pm 1 \pm \sqrt{2}\}$ make $\widehat{\mathbb{C}}$ into a 2-complex for which G is the subdivision map of a finite subdivision rule. The 1-skeleton of the first subdivision of $\widehat{\mathbb{C}}$ is the union of the real line, the unit circle and $\{\infty\}$. Figure 23 shows the 1-skeleton of the original complex drawn with solid line segments and the remaining 1-skeleton of the subdivision drawn with dashes. Combinatorially, the tiles of this finite subdivision rule are essentially squares which are subdivided into four squares in the straightforward way. The map H maps $(\infty, -1-\sqrt{2}) \cup (1-\sqrt{2}, \infty) \cup \{\infty\}$ to $(1-\sqrt{2}, \infty) \cup \{\infty\}$. Furthermore $H(1-\sqrt{2}) = \infty$, $H(1+\sqrt{2}) = 0$, $H(-1-\sqrt{2}) = 1-\sqrt{2}$ and $H(-1+\sqrt{2}) = -1+\sqrt{2}$. Thus the set $(1-\sqrt{2}, \infty) \cup \{\infty\}$ together with vertices at $\{0, \pm(1-\sqrt{2}), \infty\}$ make $\widehat{\mathbb{C}}$ into a 2-complex for which F is the subdivision map of a finite subdivision rule. The 1-skeleton of its first subdivision is the same as that for G. Although the first subdivisions of these finite subdivision rules are identical, these finite subdivision rules are quite different. The first has bounded valence, while the second does not.

This concludes Example 10.7.

We next prove a general existence theorem.

Theorem 10.8. *If d is an integer with $d > 2$ such that d is divisible by either 2 or 9, then there exists a NET map with degree d whose Teichmüller map is constant.*

Proof. Let d be an integer such that $d > 2$ and d is divisible by either 2 or 9. We prove that there exists a NET map with degree d and constant Teichmüller map. Example 10.7 provides a degree 4 NET map with a constant Teichmüller map. Hence we may assume that $d > 4$.

First suppose that $d > 4$ and $2|d$. Let $A = \mathbb{Z}/2d\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Then A contains a subgroup A' isomorphic to $\mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. Example 10.3 shows that A' has a nonseparating subset. Hence Lemma 10.6 implies that A has a nonseparating subset. As in the discussion following Theorem 10.2, we construct a degree d Thurston map f which is nearly Euclidean with a constant Teichmüller map if its postcritical set contains at least four points. But since $d > 4$, the discussion in Section 2 shows that f does have at least four postcritical points. This proves Theorem 10.8 if $2|d$.

If $9|d$, then we argue in the same way using Example 10.4 with $A = \mathbb{Z}/2d'\mathbb{Z} \oplus \mathbb{Z}/6\mathbb{Z}$, where $d' = d/3$.

This proves Theorem 10.8. □

Remark 10.9. In this remark we present another view of Theorem 10.8. We begin with some general observations.

Instead of having two lattices as usual, suppose that we have three lattices $\Lambda_1 \subseteq \Lambda_2 \subseteq \Lambda_3$. We also have corresponding maps p_i and q_i, and S^2 is identified with $p_i \circ q_i(\mathbb{R}^2)$ so that the four-element sets $p_i \circ q_i(\Lambda_i)$ are equal. Suppose also that $h\colon S^2 \to S^2$ is an orientation-preserving homeomorphism which maps this four-element subset of S^2 to $p_1 \circ q_1(\Lambda_2)$.

Let $f\colon S^2 \to S^2$ be the map induced by $p_1 \circ q_1$ and $p_2 \circ q_2$, so that $f \circ p_1 \circ q_1 = p_2 \circ q_2$. Similarly, let $g\colon S^2 \to S^2$ be the map for which $g \circ p_2 \circ q_2 = p_3 \circ q_3$. Then f and g are Euclidean Thurston maps. The map $F = h \circ f$ is a Thurston map, and it is nearly Euclidean if it has at least four postcritical points. The map $G = h \circ g \circ h^{-1}$ is a Euclidean Thurston map, being topologically conjugate to the Euclidean Thurston map g. The map $E = h \circ g \circ f = G \circ F$ is a NET map if it has at least four postcritical points. The postcritical set of F is contained in $h(P)$. The postcritical set of G is exactly $h(P)$. It follows that if Σ_E, Σ_F and Σ_G are the Teichmüller maps of E, F and G, then $\Sigma_E = \Sigma_F \circ \Sigma_G$. This shows that if Σ_F is constant, then Σ_E is constant. Moreover, since G is a Euclidean Thurston map, Σ_G is a linear fractional transformation. Thus Σ_E and Σ_F differ only by a linear fractional transformation.

Now we return to Theorem 10.8. As in the discussion following Theorem 10.2, we construct lattices Λ_1 and Λ_2 so that $\Lambda_2/\Lambda_1 \cong \mathbb{Z}/2\mathbb{Z}$. Choosing an identification map from $\mathbb{R}^2/\Gamma_2$ to $\mathbb{R}^2/\Gamma_1$ obtains a Euclidean Thurston map f. We choose h so that $h(P_f) = p_1(H)$, where H is the nonseparating subset of $\Lambda_2/2\Lambda_1 \cong \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ as in Example 10.3. The map $F = h \circ f$ is a Thurston map and Σ_F is constant, but F is not nearly Euclidean because it has only three postcritical points (consistent with Theorem 10.10). Let Λ_3 be any lattice containing Λ_2. Choosing an identification map from $\mathbb{R}^2/\Gamma_3$ to $\mathbb{R}^2/\Gamma_2$ gives rise to a Euclidean Thurston map g. Let $G = h \circ g \circ h^{-1}$, and let $E = h \circ g \circ f = G \circ F$. Then E is a NET map if it has at least four postcritical points. The discussion earlier in this remark shows that Σ_E is constant. Because the index of Λ_2 in Λ_3 is arbitrary, this gives another proof of Theorem 10.8 if $2|d$. This way of showing that Σ_E is constant by expressing E as a composition $G \circ F$ is in the spirit of Proposition 5.1 of [1]. A major difference between the treatment here and that in Proposition 5.1 of [1] is that here Σ_E factors with one of the factors a constant function, while there Σ_E factors through a trivial Teichmüller space. Note, in particular, that the rational functions in Example 10.7 with constant pullback maps on Teichmüller space factor as in the previous paragraph.

When 2 is replaced by 9, the earlier observations in this remark again show that once we obtain an example with degree 9, then we obtain an example for every degree divisible by 9.

For our first nonexistence result, we prove that there does not exist a NET map with degree 2 whose Teichmüller map is constant.

Theorem 10.10. *There does not exist a NET map with degree* 2 *whose Teichmüller map is constant.*

Proof. Let $f\colon S^2 \to S^2$ be a Euclidean Thurston map with degree 2 whose Teichmüller map is constant. We seek a contradiction.

As in Section 1, there exist lattices $\Lambda_1 \subseteq \Lambda_2 \subseteq \mathbb{R}^2$ such that the canonical group homomorphism $\widetilde{f}\colon \mathbb{R}^2/2\Lambda_1 \to \mathbb{R}^2/2\Lambda_2$ lifts f. Theorem 10.2 implies that $p_1^{-1}(P_2)$ is a nonseparating subset of $\Lambda_2/2\Lambda_1$. Since $\deg(f) = 2$, it follows that $\Lambda_2/2\Lambda_1 \cong A = \mathbb{Z}/4\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. We fix such an isomorphism, and let H be the subset of A corresponding to $p_1^{-1}(P_2)$. Group inversion generates an equivalence relation on A whose equivalence classes have either one or two elements. Here they are.

$$\{(0,0)\} \quad \{\pm(1,0)\} \quad \{(2,0)\} \quad \{(0,1)\} \quad \{\pm(1,1)\} \quad \{(2,1)\}$$

The set H contains exactly four of these six subsets.

In this paragraph we prove by contradiction that H contains the elements of order 4 in A. Suppose that $(1,0) \notin H$. Let $B = \langle(1,0)\rangle = \{(0,0), \pm(1,0), (2,0)\}$, a cyclic subgroup of A such that A/B is cyclic. Let c_1, c_2, c_3, c_4 be the coset numbers for H relative to B and the generator of A/B. Since H contains four of the six subsets in the last display, it contains either $(0,0)$ or $(2,0)$. If H contains both of these elements, then $c_1 = c_2 = 0$ and $c_3 = c_4 = 1$, contrary to the assumption that H is nonseparating. So H contains exactly one of $(0,0)$ and $(2,0)$. Hence it contains $(1,1)$. Now applying the same argument with $B' = \langle(1,1)\rangle$ obtains a contradiction. This contradiction implies that $(1,0) \in H$. By symmetry, $(1,1) \in H$. Thus H contains the elements of order 4 in A.

The elements of order 4 in $\Lambda_2/2\Lambda_1$ are precisely the elements of $\mathbb{R}^2/2\Lambda_1$ whose images under p_1 are critical points of f. Since H contains the elements of order 4 in A, the critical points of f are postcritical. So in order for the postcritical set P_2 of f to have four points, the restriction of f to P_2 must be surjective. This implies that $p_1^{-1}(P_2)$ contains a representative from every coset of $\ker(\widetilde{f}) = 2\Lambda_2/2\Lambda_1$ in $\Lambda_2/2\Lambda_1$. It follows that H contains exactly one of the elements in $2A = \{(0,0),(2,0)\}$. Now we obtain a contradiction as in the previous paragraph using $B = \langle(1,0)\rangle$.

This proves Theorem 10.10. $\square$

Now we prove a general nonexistence theorem.

Theorem 10.11. *Let A be a finite Abelian group such that $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ and $2A$ is a cyclic group with odd order. Then A does not contain a nonseparating subset.*

Proof. We proceed by contradiction. Suppose that A contains a nonseparating subset $H = \{\pm h_1, \pm h_2, \pm h_3, \pm h_4\}$. The assumptions imply that $A \cong 2A \oplus (A/2A)$. Every subgroup of order 2 in A is a cyclic subgroup B such that A/B is cyclic, and every subgroup of order 2 in A is contained in a cyclic subgroup B such that A/B has order 2, and so is cyclic.

We consider the restriction to H of the canonical group homomorphism from A to $A/2A$. Each of the inverse pairs $\{\pm h_1\}$, $\{\pm h_2\}$, $\{\pm h_3\}$, $\{\pm h_4\}$ determines an element of $A/2A$. There are five possible forms for this map: (1) its image contains all four elements of $A/2A$; (2) its image contains three elements of $A/2A$; (3) its image contains two elements of $A/2A$, each the image of two of the inverse pairs; (4) its image contains two elements of $A/2A$, one the image of one inverse pair and the other the image of three inverse pairs; (5) its image consists of one element of $A/2A$.

In this paragraph we assume that the map from H to $A/2A$ has forms either (1), (2) or (3). A case analysis shows that there exists a subgroup of order 2 in $A/2A$ which contains the image of exactly two of the inverse pairs in H. The inverse image in A of this subgroup of order 2 is a cyclic subgroup B of A such that A/B has order 2, hence is cyclic, and B contains exactly two of the inverse pairs of H. Hence the coset numbers for H relative to B and the generator of A/B are $c_1 = c_2 = 0$ and $c_3 = c_4 = 1$. This contradicts the assumption that H is nonseparating.

In this paragraph we assume that the map from H to $A/2A$ has forms either (4) or (5). By translating H by an element of order 2 if necessary as in Lemma 10.5, we may assume that the image of H in $A/2A$ is contained in a subgroup of order 2. The inverse image in A of this subgroup of order 2 is a cyclic subgroup C of A such that A/C has order 2, and C contains H. Let B be a subgroup of order 2 in A not contained in C. Then B is a cyclic subgroup of A such that $A/B \cong C$ is cyclic, and the coset numbers for H relative to B and any generator of C are distinct because H is a subset of C. This contradiction completes the proof of Theorem 10.11.

□

Theorem 10.12. *There does not exist a NET map with degree an odd squarefree integer and constant Teichmüller map.*

Proof. Suppose that $f\colon S^2 \to S^2$ is a NET map, and suppose that the degree d of f is an odd squarefree integer. We maintain the notation of Section 1. Let $A = \Lambda_2/2\Lambda_1$. Then $A/2A \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$, and since $|A| = 4d$, it follows that $|2A| = d$. Since every finite Abelian group with squarefree order is cyclic, $2A$ is cyclic. Theorem 10.11 now implies that A does not contain a nonseparating subset. Theorem 10.2 now implies that the Teichmüller map of f is not constant. This proves Theorem 10.12. □

It should be true that there exists a NET map with degree d and constant Teichmüller map if and only if $d > 2$ and d is divisible by either 2 or 9; see Saenz Maldonado's thesis [9] for further progress in this direction.

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