ENUMERATION OF LATTÈS MAPS

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1. Introduction

These notes provide a simple and effective method for enumerating Lattès maps of small degree. Much of this is contained in [2].

2. Definitions and basic facts for Lattès maps

Following Milnor [4, Remark 3.5] (but not [5]), we define a Lattès map to be a rational function from the Riemann sphere $\widehat{\mathbb{C}}$ to itself such that its local degree at every critical point is 2 and there are exactly four postcritical points, none of which is also critical. Let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Lattès map.

As in Section 3.1 of [4], it follows that there exists an analytic branched cover $\wp\colon \mathbb{C} \to \widehat{\mathbb{C}}$ which is branched exactly over the postcritical points of f and the local degree of $\wp$ at every branch point is 2. (The function $\wp$ is a Weierstrass $\wp$-function up to precomposing and postcomposing with analytic automorphisms.) Let Λ be the set of branch points of $\wp$. It is furthermore true that $\wp$ is a regular branched cover, and its group of deck transformations Γ is generated by the set of all rotations of order 2 about the points of Λ. We refer to Γ as the **orbifold fundamental group** of f. Given rotations $z \mapsto 2\lambda - z$ and $z \mapsto 2\mu - z$ of order 2 about the points $\lambda, \mu \in \Lambda$, their composition, the second followed by the first, is the translation $z \mapsto z + 2(\lambda - \mu)$. We may, and do, normalize so that $0 \in \Lambda$. So Γ contains a subgroup with index 2 consisting of translations of the form $z \mapsto z + 2\lambda$ with $\lambda \in \Lambda$. It follows that Λ is a lattice in $\mathbb{C}$ and that the elements of Γ are the maps of the form $z \mapsto \pm z + 2\lambda$ for some $\lambda \in \Lambda$.

Douady and Hubbard show in [3, Proposition 9.3] that the map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ lifts to a map $\widetilde{f}\colon \mathbb{C} \to \mathbb{C}$ given by $\widetilde{f}(z) = \alpha z + \beta$ for some imaginary quadratic algebraic integer α (possibly an element of $\mathbb{Z}$) such that $\alpha\Lambda \subseteq \Lambda$ and some $\beta \in \Lambda$. The following lemma is devoted to determining to what extent α, β and Λ are determined by the analytic conjugacy class of f.

Lemma 2.1. *Let f_0 be a Lattès map which is analytically conjugate to f. Let $\wp_0\colon \mathbb{C} \to \widehat{\mathbb{C}}$ be a branched cover for f_0 corresponding to $\wp$. Let Λ_0 be the set of branch points of $\wp_0$, and assume that $0 \in \Lambda_0$. Suppose that $f_0\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ lifts to a map $\widetilde{f_0}\colon \mathbb{C} \to \mathbb{C}$ given by $\widetilde{f_0}(z) = \alpha_0 z + \beta_0$. Then the following hold.*

1. *$\alpha_0 = \pm\alpha$.*
2. *$\beta_0 = \gamma\beta + \delta$ where $\gamma \in \mathbb{C}^\times$ and $\delta \in (\alpha + 1)\Lambda_0 + 2\Lambda_0$.*
3. *$\Lambda_0 = \gamma\Lambda$ with γ as in statement 2.*

Conversely, if $\wp_0\colon \mathbb{C} \to \widehat{\mathbb{C}}$ is a branched cover as above, if Λ_0 is the set of branch points of $\wp_0$ with $0 \in \Lambda_0$ and if $\widetilde{f_0}(z) = \alpha_0 z + \beta_0$ such that statements 1, 2 and 3 hold, then $\widetilde{f_0}$ is the lift of a Lattès map which is analytically conjugate to f.

Proof. Let σ be a Möbius transformation such that $f_0 = \sigma \circ f \circ \sigma^{-1}$. Just as for f and f_0, the map $\sigma\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ lifts to a map $\widetilde{\sigma}\colon \mathbb{C} \to \mathbb{C}$ given by $\widetilde{\sigma}(z) = \rho z + \nu$ for some $\rho \in \mathbb{C}^\times$ and $\nu \in \mathbb{C}$. Since σ maps the postcritical set of f to the postcritical set of f_0, $\widetilde{\sigma}$ maps Λ to Λ_0, that is, $\rho\Lambda + \nu = \Lambda_0$. In other words, the coset $\rho\Lambda + \nu$ of the group $\rho\Lambda$ equals the group Λ_0. The only way that a coset can be a group is if it is the trivial coset, and so $\nu \in \Lambda_0$ and $\Lambda_0 = \rho\Lambda$. Since $\widetilde{f_0}$ and $\widetilde{\sigma} \circ \widetilde{f} \circ \widetilde{\sigma}^{-1}$ are both lifts of f_0, they differ by an element of Γ_0, the group of deck transformations of $\wp_0$. In other words, $\pm\widetilde{f_0}(z) + 2\lambda_0 = \widetilde{\sigma} \circ \widetilde{f} \circ \widetilde{\sigma}^{-1}(z)$ for some $\lambda_0 \in \Lambda_0$. So we have the following.

$$\begin{aligned}\pm(\alpha_0 z + \beta_0) + 2\lambda_0 &= \pm\widetilde{f_0}(z) + 2\lambda_0 = \widetilde{\sigma} \circ \widetilde{f} \circ \widetilde{\sigma}^{-1}(z) = \rho\widetilde{f}(\rho^{-1}(z - \nu)) + \nu \\ &= \rho(\alpha\rho^{-1}(z - \nu) + \beta) + \nu = \alpha z + \rho\beta + (1 - \alpha)\nu\end{aligned}$$

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Hence $\alpha_0 = \pm\alpha$, which yields statement 1. Furthermore

$$\pm\beta_0 = \rho\beta + (1-\alpha)\nu - 2\lambda_0.$$

We have seen that $\nu \in \Lambda_0$. So setting $\gamma = \pm\rho$ and $\delta = \pm((\alpha+1)\nu - 2\alpha\nu - 2\lambda_0)$, we have that $\beta_0 = \gamma\beta + \delta$ with $\gamma \in \mathbb{C}^\times$ and $\delta \in (\alpha+1)\Lambda_0 + 2\Lambda_0$. We now have verified statements 1, 2 and 3.

For the converse, it is a straightforward matter to construct $\widetilde{\sigma}$ such that $\widetilde{\sigma}$ conjugates $\widetilde{f}$ to $\widetilde{f}_0$ up to the action of Γ_0. One checks that $\widetilde{\sigma}$ descends to a rational map $\sigma\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ which has local degree 1 at every point of $\widehat{\mathbb{C}}$. So σ is a Möbius transformation. It follows that $\widetilde{f}_0$ is the lift of an analytic conjugate of f.

This completes the proof of Lemma 2.1.

□

Lemma 2.1 implies that with an appropriate modification of β the lattice Λ may be replaced by $\gamma\Lambda$, where γ is any nonzero complex number, without changing the analytic conjugacy class of f. In Section 7 of Chapter 2 of the number theory book [1] by Borevich and Shafarevich, the lattices Λ and $\gamma\Lambda$ are said to be similar. Theorem 9 and the remark following it in Section 7 of Chapter 2 of [1] imply that every lattice in $\mathbb{C}$ is similar to a lattice with a $\mathbb{Z}$-basis consisting of 1 and τ, where τ lies in the standard fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half complex plane. More precisely, τ satisfies the following inequalities.

$$\begin{gathered} \mathrm{Im}(\tau) > 0 \\ -\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2} \\ |\tau| \geq 1 \text{ and if } |\tau| = 1, \text{ then } \mathrm{Re}(\tau) \geq 0 \end{gathered} \tag{2.2}$$

Moreover there is only one such τ which satisfies these inequalities. This and Lemma 2.1 imply that τ is uniquely determined by the analytic conjugacy class of f.

Corollary 2.3. *Let $\widetilde{f}_0(z) = \alpha_0 z + \beta_0$ be a $\mathbb{C}$-linear map such that $\widetilde{f}_0(\Lambda) \subseteq \Lambda$. Then $\widetilde{f}_0$ descends via the branched cover $\wp\colon \mathbb{C} \to \widehat{\mathbb{C}}$ to a rational function which is analytically conjugate to f if and only if $\alpha_0 = \pm\alpha$ and $\beta_0 = \gamma\beta + \delta$ where $\gamma\Lambda = \Lambda$, $\gamma = e^{\pm 2\pi i/n}$ with $n \in \{1,2,3,4,6\}$ and $\delta \in (\alpha+1)\Lambda + 2\Lambda$.*

Proof. In this situation $\Lambda_0 = \Lambda$. So just as for $\widetilde{f}$, the containment $\gamma\Lambda \subseteq \Lambda$ implies that the complex number γ is in fact an imaginary quadratic algebraic integer. Because $\gamma\Lambda = \Lambda$, γ is invertible, that is, it is a unit. But all imaginary quadratic units have the form $e^{\pm 2\pi i/n}$ with $n \in \{1,2,3,4,6\}$. This discussion and Lemma 2.1 prove Corollary 2.3.

□

In this paragraph we consider related effects of complex conjugation. By the complex conjugate of a rational map f, we mean the rational map gotten by applying complex conjugation to the coefficients of f. It is easy to see that the complex conjugate of a Lattès map is also a Lattès map. By applying complex conjugation to the Lattès map f, the branched cover $\wp$ and the lift $\widetilde{f}$ of f, we see that the complex conjugate of $\widetilde{f}$ is a lift of the complex conjugate of f. The behavior of f is essentially the same as the behavior of $\overline{f}$, so when considering $\widetilde{f}(z) = \alpha z + \beta$, we assume that $\mathrm{Im}(\alpha) \geq 0$. Since $\widetilde{f}$ and $-\widetilde{f}$ both lift f, we may also assume that $\mathrm{Re}(\alpha) \geq 0$. This shows that the restrictions put on α in the following lemma are reasonable.

Lemma 2.4. *As above, let Λ be the inverse image in $\mathbb{C}$ of the postcritical set of the Lattès map $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$, and let $\widetilde{f}(z) = \alpha z + \beta$ be a lift of f. Suppose that 1 and τ form a $\mathbb{Z}$-basis of Λ. Multiplication by α determines an endomorphism of Λ. Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be the matrix of this endomorphism with respect to the ordered $\mathbb{Z}$-basis $(1,\tau)$. Suppose that $Im(\alpha) > 0$. Then $Re(\alpha) \geq 0$ and τ lies in the standard fundamental domain for the action of $SL(2,\mathbb{Z})$ on the upper half complex plane if and only if the following inequalities are satisfied.*

$$\begin{gathered} c > 0 \\ a \geq -\frac{c}{2} \\ \max\{a - c + 1, -a\} \leq d \leq a + c \\ b \leq -c \text{ and if } b = -c, \text{ then } d \geq a \end{gathered}$$

Proof. From the first column of the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ it follows that $\alpha = a + c\tau$. So $\tau = \frac{\alpha - a}{c}$. The matrix of the endomorphism determined by $\alpha - a$ is $\left[\begin{smallmatrix} 0 & b \\ c & d-a \end{smallmatrix}\right]$. Because the eigenvalues of this matrix are $\alpha - a$ and $\overline{\alpha - a}$, its trace is twice the real part of $\alpha - a$ and its determinant is the square of the modulus of $\alpha - a$. So $\mathrm{Re}(\alpha - a) = \frac{d-a}{2}$ and $|\alpha - a|^2 = -bc$. Hence $\mathrm{Re}(\tau) = \frac{d-a}{2c}$ and $|\tau|^2 = -\frac{b}{c}$. Similarly $\mathrm{Re}(\alpha) = \frac{a+d}{2}$.

Suppose that $\mathrm{Re}(\alpha) \geq 0$ and that the inequalities in line 2.2 hold. Since $\tau = \frac{\alpha - a}{c}$ and $\mathrm{Im}(\alpha) > 0$, the inequality $\mathrm{Im}(\tau) > 0$ implies that $c > 0$, giving the first inequality in the statement of the lemma. For the second inequality, we combine $\mathrm{Re}(\alpha) \geq 0$ and $\mathrm{Re}(\tau) \leq \frac{1}{2}$ to obtain $a + d \geq 0$ and $d - a \leq c$, hence $a - d \geq -c$. So $a \geq -\frac{c}{2}$, giving the second inequality in the statement of the lemma. Combining $-\frac{1}{2} < \mathrm{Re}(\tau) \leq \frac{1}{2}$ and $\mathrm{Re}(\alpha) \geq 0$ obtains $-c < d - a \leq c$ and $a + d \geq 0$, which easily gives the third inequality in the statement of the lemma. The fourth inequality follows from the fact that $|\tau| \geq 1$ with equality only if $\mathrm{Re}(\tau) \geq 0$.

Proving the converse is straightforward.

This proves Lemma 2.4.

□

Lemma 2.5. *(1) In Corollary 2.3 the case $\gamma = \pm i$ occurs only when $\tau = i$, and the case $\gamma = \pm e^{\pm 2\pi i/3}$ occurs only when $\tau = e^{2\pi i/6}$.*
(2) Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be as in Lemma 2.4, and let M be the reduction of $\left[\begin{smallmatrix} a+1 & b \\ c & d+1 \end{smallmatrix}\right]$ modulo 2. Then a complete list of distinct coset representatives of $(\alpha + 1)\Lambda + 2\Lambda$ in Λ is given by

$$0 \text{ if } \mathrm{rank}(M) = 2$$
$$0, 1, \tau, \tau + 1 \text{ if } \mathrm{rank}(M) = 0$$
$$0, \lambda \text{ if } \mathrm{rank}(M) = 1,$$

where λ is any element of Λ whose image in $\Lambda/2\Lambda$ is not in the column space of M.

Proof. If $\gamma = \pm i$, then $i \in \Lambda$ because $\gamma\Lambda \subseteq \Lambda$. But then i is an integral linear combination of 1 and τ with τ in the standard fundamental domain for the action of $\mathrm{SL}(2, \mathbb{Z})$ on the upper half complex plane. This implies that $\tau = i$. A similar argument applies when $\gamma = \pm e^{\pm 2\pi i/3}$. This proves statement 1.

Statement 2 is clear.

□

Since the elements of the orbifold fundamental group are Euclidean isometries, $\widetilde{f}$ multiplies areas uniformly by the factor $\deg(f)$. Translation by β does not change areas. Multiplication by α multiplies lengths by $|\alpha|$ and areas by $|\alpha|^2 = \alpha\overline{\alpha}$. Multiplication by α also corresponds to multiplication by the matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, and this multiplies areas by its determinant. Therefore $\deg(f) = \alpha\overline{\alpha} = ad - bc$.

Lemma 2.6. *If a, b, c and d satisfy the inequalities of Lemma 2.4, then $ad - bc \geq 3c^2/4$. If equality holds, then $a = -d$ and $b = -c$.*

Proof. The inequalities of Lemma 2.4 imply that $|a - d| \leq c$. Hence $c^2 \geq (a-d)^2 \geq (a-d)^2 - (a+d)^2 = -4ad$. Since $b \leq -c$ and $c > 0$, it follows that $ad - bc \geq -c^2/4 + c^2 = 3c^2/4$. This gives the inequality of the first statement of the lemma. The second statement is now clear.

□

Let f be a Lattès map with lift $\widetilde{f}(z) = \alpha z + \beta$ and lattice Λ as above. We say that f is **flexible** if $\alpha \in \mathbb{R}$, equivalently, $\alpha \in \mathbb{Z}$. We say that f is **rigid** if $\alpha \notin \mathbb{R}$. If f is flexible, then since multiplication by an integer stabilizes every lattice in $\mathbb{C}$, Λ can be arbitrary. There are uncountably many analytic conjugacy classes of Lattès maps for every integer $\alpha \geq 2$. On the other hand, there are only finitely many analytic conjugacy classes of rigid Lattès maps with a given degree. To see why, first note that Lemma 2.6 and the paragraph before it imply that in the rigid case a bound on $\deg(f) = ad - bc$ puts a bound on c. We may assume that $\mathrm{Im}(\alpha) > 0$. The inequalities of Lemma 2.4 imply that a bound on c puts a lower bound on a and d. Because $b \leq -c$, $c > 0$ and $|a - d| \leq c$, a bound on $ad - bc$ puts an upper bound on both a and d. So a bound on $ad - bc$ puts a bound on c, a, d and therefore b. So if $\deg(f)$ is bounded, then there are only finitely many possibilities for a, b, c and d. These values determine α in the upper half plane and τ. Given α and τ, there are always at most four possibilities for β up to equivalence. So if f is rigid and $\deg(f)$ is bounded, then there are only finitely many possibilities for the analytic conjugacy class of f.

deg(f)	a	b	c	d	α	β	τ	real
2	0	-2	1	0	$\sqrt{-2}$	0	$\sqrt{-2}$	yes
2	0	-2	1	1	$\frac{1+\sqrt{-7}}{2}$	0	$\frac{1+\sqrt{-7}}{2}$	no
2	0	-2	1	1	$\frac{1+\sqrt{-7}}{2}$	1	$\frac{1+\sqrt{-7}}{2}$	no
2	1	-1	1	1	$1+\sqrt{-1}$	0	$\sqrt{-1}$	no
3	0	-3	1	0	$\sqrt{-3}$	0	$\sqrt{-3}$	yes
3	0	-3	1	0	$\sqrt{-3}$	1	$\sqrt{-3}$	yes
3	0	-3	1	1	$\frac{1+\sqrt{-11}}{2}$	0	$\frac{1+\sqrt{-11}}{2}$	no
3	1	-2	1	1	$1+\sqrt{-2}$	0	$\sqrt{-2}$	no
3	1	-2	1	1	$1+\sqrt{-2}$	1	$\sqrt{-2}$	no
3	1	-1	1	2	$\frac{3+\sqrt{-3}}{2}$	0	$\frac{1+\sqrt{-3}}{2}$	no
3	-1	-2	2	1	$\sqrt{-3}$	0	$\frac{1+\sqrt{-3}}{2}$	yes
3	-1	-2	2	1	$\sqrt{-3}$	1	$\frac{1+\sqrt{-3}}{2}$	yes

Table 1. All Lattès maps with degrees 2 or 3 up to complex conjugation.

3. Lattès maps with degrees 2 or 3

In this section we enumerate all Lattès maps with degrees 2 or 3 up to analytic conjugacy.

As in Section 2, let $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ be a Lattès map, let Λ be a lift of the postcritical set of f to $\mathbb{C}$ and let $\widetilde{f}(z) = \alpha z + \beta$ be a lift of f. If $\alpha \in \mathbb{Z}$, then the degree of f is α^2, which is not 2 or 3. So as in the paragraph before Lemma 2.4, replacing f by $\overline{f}$ if necessary, we may assume that $\mathrm{Re}(\alpha) \geq 0$ and $\mathrm{Im}(\alpha) > 0$. We have seen that f uniquely determines a complex number τ which lies in the standard fundamental domain for the action of $\mathrm{SL}(2,\mathbb{Z})$ on the upper half complex plane. Multiplication by α determines an endomorphism of Λ. Let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ be the matrix of this endomorphism with respect to the ordered $\mathbb{Z}$-basis $(1,\tau)$ of Λ. Using Lemma 2.4 we see that giving α and τ is equivalent to giving integers a, b, c, d which satisfy the inequalities of Lemma 2.4. The degree of f is $|\alpha|^2 = ad - bc$.

Table 1 enumerates all Lattès maps with degrees 2 or 3 up to complex conjugation and analytic conjugacy. The next two paragraphs show how to determine these values of a, b, c, and d. Having $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$, we obtain α as its eigenvalue in the upper half plane. The equation $\alpha = a + c\tau$ then gives τ. The value of β comes from Lemma 2.5. The case in which $\gamma = \pm i$ in statement 1 of Lemma 2.5 is not needed, and the other case in statement 1 is needed only for the last row of Table 1. The last column in Table 1 deals with whether $f = \overline{f}$. This can be determined using Lemma 2.1 and the paragraph preceding Lemma 2.4. The word "yes" in this column means that $f = \overline{f}$, that is, the coefficients of this rational function are real numbers. The word "no" means that f is not analytically conjugate to a rational function which equals its complex conjugate. Up to analytic conjugacy, there are 7 Lattès maps with degree 2 and 12 Lattès maps with degree 3.

In this paragraph we determine the values of a, b, c and d in Table 1 for $\deg(f) = 2$. Since $\deg(f) = ad - bc$, Lemma 2.6 implies that $c = 1$. Now Lemma 2.4 implies that $0 \leq a \leq d \leq a+1$ and $b \leq -1$. So ad and $-bc$ are nonnegative integers whose sum is 2. One easily checks that the possibilities for (a,b,c,d) are $(0,-2,1,0)$, $(0,-2,1,1)$ and $(1,-1,1,1)$. This determines the values of a, b, c, and d in Table 1 for $\deg(f) = 2$.

Now we proceed as in the previous paragraph for $\deg(f) = 3$. In this case Lemma 2.6 gives that either $c = 1$ or $c = 2$ and $b = -2$. If $c = 1$, then as in the previous paragraph, ad and $-bc$ are nonnegative integers whose sum is 3. One easily checks that in this case the possibilities for (a,b,c,d) are $(0,-3,1,0)$, $(0,-3,1,1)$, $(1,-2,1,1)$ and $(1,-1,1,2)$. If $c = 2$, then since $b = -2$ and $ad - bc = 3$, we have that $ad = -1$. This and the inequality $\max\{a - c + 1, -a\} \leq d$ from Lemma 2.4 imply that $a = -1$ and $d = 1$. This determines the values of a, b, c and d in Table 1 for $\deg(f) = 3$.

4. NET map presentations

In this section we indicate how to transform Lattès map presentations as in Table 1 into Euclidean NET map presentations.

The lattice Λ which appears in Section 2 is the preimage in the plane of the postcritical set of our Lattès map. We identify Λ with Λ_1 in the usual NET map notation. Since our affine map is given by $z \mapsto \alpha z + \beta$ with $\beta \in \Lambda_1$, we take $\Lambda_2 = \alpha^{-1}\Lambda_1$. Our $\mathbb{Z}$-basis of $\Lambda_1 = \Lambda$ consists of $\lambda_1 = 1$ and $\lambda_2 = \tau$, so we take α^{-1} and $\alpha^{-1}\tau$ as our $\mathbb{Z}$-basis of Λ_2. We have the following.

$$\lambda_1 = 1 = \alpha\alpha^{-1} = \alpha^{-1}(\alpha \cdot 1) = \alpha^{-1}(a + c\tau) = a\alpha^{-1} + c\alpha^{-1}\tau$$
$$\lambda_2 = \tau = \alpha(\alpha^{-1}\tau) = \alpha^{-1}(\alpha \cdot \tau) = \alpha^{-1}(b + d\tau) = b\alpha^{-1} + d\alpha^{-1}\tau$$

This shows that the matrix $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ is the matrix of the multiplication map $z \mapsto \alpha z$ with respect to the ordered $\mathbb{Z}$-basis $(\alpha^{-1}, \alpha^{-1}\tau)$ of Λ_2 and that the coordinates of λ_1 form its first column and the coordinates of λ_2 form its second column. Hence A is the usual matrix which appears in the affine map $x \mapsto Ax + b$ for NET map presentations. (This b is not to be confused with the previous b.) The column b consists of the coordinates of β with respect to α^{-1} and $\alpha^{-1}\tau$. For example, if $\beta = 1$, then $b = \left[\begin{smallmatrix} a \\ c \end{smallmatrix}\right]$ because $1 = \lambda_1 = a\alpha^{-1} + c\alpha^{-1}\tau$. Of course, the entries of b are determined only modulo 2. Taking the line segments which appear in NET map presentations to be trivial, we obtain a NET map presentation for our Lattès map.

In this paragraph we discuss NET map presentations for conjugate Lattès maps. If the affine map $z \mapsto \alpha z + \beta$ determines a given Lattès map, then $z \mapsto \overline{\alpha} z + \overline{\beta}$ determines the conjugate Lattès map. To obtain a NET map presentation for the conjugate Lattès map, we use the same lattices and bases. Because the eigenvalues of A are α and $\overline{\alpha}$, by considering the trace of A, we see that $\alpha + \overline{\alpha} = a + d$. So multiplication by $\overline{\alpha}$ is the same as multiplication by $a + d - \alpha$. So our new matrix is $(a+d)I - A = \left[\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right]$. Because the map $z \mapsto -z$ is a deck transformation, we may multiply this matrix by -1 to obtain $\left[\begin{smallmatrix} -d & b \\ c & -a \end{smallmatrix}\right]$. (The case in which α is purely imaginary corresponds to the case in which $a + d = 0$, in which case our new matrix equals our old matrix.) To obtain a NET map presentation for the conjugate Lattès map, all that remains to do is to express $\overline{\beta}$ in terms of α^{-1} and $\alpha^{-1}\tau$.

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