NUMBER BASES, GRADES 6-7, SESSION 1

BLACKSBURG MATH CIRCLE

The Decimal System

We learn to count in only one of many possible number systems: the decimal system. The decimal system is our default number system. The word decimal is derived from the Latin word *decem* which means ten. Strictly speaking, the decimal system is a number system of base 10. Don't worry about all the fancy language right now. Just follow along with the example below to find out what is so special about the number 10 in the decimal system.

Problem 1. (a) Write the following numbers in expanded form (as sums of powers of 10):

$$102079; \quad 70000 - 6999.$$

(b) Write the following number in standard form (the usual digit representation):

$$(9)(10^5) + (2)(10^3) + (8)(10^1) + (1)(10^0) + (4)(10^0).$$

Binary Numbers

Another number system that is used every day is the binary system. It is the number system that computers use. It can be described as follows: The binary system is a number system with base 2. Every place value corresponds to a power of 2 and a digit may only be either 0 or 1. Some examples of binary numbers are 11_2, 10001_2, 110011_2, and 10101011_2. Note that these are not read as eleven, ten-thousand and one, etc., but as one-one, one-zero-zero-zero-one, and so on (to avoid confusion). Just like the decimal system used powers of 10 to create place values, the binary system uses powers of 2 to do the same. Look at the expanded form of the binary number 110110_2 and the way it is written in the place value chart:

$$110110_2 \to (1)(2^5) + (1)(2^4) + (0)(2^3) + (1)(2^2) + (1)(2^1) + (0)(2^0)$$

Place Value	2^5	2^4	2^3	2^2	2^1	2^0
Place Value	32	16	8	4	2	1
Digit	1	1	0	1	1	0

Just as we did with the decimal numbers, we can now write 110110_2 as a sum of powers, in this case powers of 2. So the binary number 110110_2 is the decimal number 54. Notice that all we actually need to do is add up the numbers in the Place Value row that have a 1 below in the Digits row.

Problem 2. Write the decimal number 238_{10} as a binary number; write the binary number 110001_2 as a decimal number.

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Number Systems of Arbitrary Base

So far we have seen two different number systems of different bases: decimal (base 10) and binary (base 2). Now let's think about the general idea of a number system with base B. What was special about the numbers 10 and 2 in the two number systems we studied? There were a couple of important things:

- The place values were powers of the base.
- The digits were allowed to be integers from 0 to the base minus 1.

So just by changing the base of a number system, we can change the way in which numbers are represented completely. The base characterizes the number system. This motivates the following definition: *A base B number system is a number system in which every place value corresponds to a power of B and a digit may only be an integer from 0 to $B-1$.*

For example, consider the base 6 number system. Place values correspond to powers of 6 and digits may be integers from 0 to $6-1=5$.

Problem 3. The Mayans used a base 20 number system. How do you think they learned to count?

Problem 4. What is the largest 3-digit number in the base system 6 is equal to? What is the largest 3-digit number in the base system n is equal to?

Problem 5. Can we use the base 1 system?

Problem 6. Let's add binary numbers! Do the following binary calculations by first converting the numbers to base 10, performing the calculations, and then converting the answer back to binary:

$$0+0;\ 0+1;\ 1+0;\ 1+1;\ 1+1+1.$$

Problem 7. Use the results from the previous question to do the following binary calculations without converting to base 10. Check your answers by converting the numbers to base 10 and then performing the calculations.

$$10_2+1_2;\quad 1001_2+110_2;\quad 111_2+11_2.$$

Hint: Think of place value and carrying (as with base 10 addition).

Problem 8. A palindrome is a positive integer whose digits are the same when read forwards or backwards. For example, 2002 is a palindrome. How many more 3-digit palindromes exist in the decimal number system than in the binary number system?

Problem 7. What is the minimum number of weights which enables us to weigh any integer number of grams of gold from 1 to 100 on a standard balance with two pans? Weights may be placed only on the left pan.

Problem 8. The same question as in the previous problem, but the weights can be placed on either pan of the balance.

Problem 9.

(a) How many different 5-digit binary numbers are there?
(b) How many different 5-digit binary numbers are there that have 1 as the last digit?
(c) How many different 5-digit base B numbers are there for a given B? Assume $2 \leq B \leq 10$.