Math Circle 9/10/2016 Meeting: Challenge of the week

Make sure that your solution is correct, complete, and clearly written. You should not expect much credit if your proof refers to a false statement, or even if all your statements are true but you forgot to tell us "why?" It is one of the purposes of the Circle to help you improve your "essay-proof" writing style as well as your logical skills.

Problem 1. In a school Ping-Pong tournament, every registered participant played exactly four games. No two of contestants share the same birthday. Prove that there exists at least one player that either won at least 2 games against opponents older than this player, or won at least 2 games against opponents younger this player.

Solution: Suppose that P students participated in the tournament. We first count the total number N of games played by observing that each contestant played 4 games, giving $4P$. We must however divide it by 2 since each game was counted twice (a game A-B between players A and B was counted once when we counted the player A games and again when we counted the player B games), so $N=4P/2=2P$. The proof will now proceed by contradiction. Assume that there is no player who either won at least 2 games against opponents older than this player, or won at least 2 games against opponents younger than this player. It means that each player won at most two games (against one younger and one older player). Moreover, the youngest player can win at most one game (and the same is true for the oldest player). Let us denote by w_k the number of wins of the k-th player. Then $w_k \le 2$ for each k, and for two values of k we have $w_k < 2$, so

$$w_1 + w_2 + \ldots + w_P < 2P.$$

But in N games without draws there are exactly N wins, so $w_1 + w_2 + \ldots + w_P$ is in fact equal to $2P$, a contradiction. That means there should be at least one student who won at least 2 games.