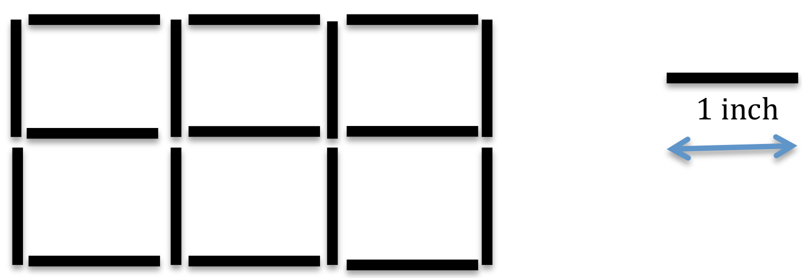
Math Circle 8/17/2016 Meeting: Challenge of the week

Make sure that your solution is correct, complete, and clearly written. You should not expect much credit if your proof refers to a false statement, or even if all your statements are true but you forgot to tell us "why?" It is one of the purposes of the Circle to help you improve your "essay-proof" writing style as well as your logical skills.

Problem 2. Eeyore has 1350 sticks of length 1 inch each, from which he builds a rectangle consisting of 1 inch $\times$ 1 inch squares (see the example of a 3×2 rectangle consisting of 17 sticks in a picture below). What are the dimensions of Eeyore's rectangle?



Solution: If a rectangle has dimensions $m\times n$, it consists of $m+1$ columns and $n+1$ rows of sticks. So there are $(m+1)\times n$ vertical and $(n+1)\times m$ horizontal sticks. We therefore can set up the equation $2mn+m+n=1350$ for our problem. One can try to solve this equation (i.e. find the values of m and n) by the method of trial and error (plugging in different choices for m and n into the equation and trying to match the left hand side with the right hand side). This method however has two major drawbacks: (a) it may take quite some time to find a solution and (b) it does not guarantee that the problem has only one solution found by trial and error (i.e. maybe there are other choices for m and n that also work). Instead, we will use the following trick: We will multiply both sides by 2 to get $4mn+2m+2n+1=2701$.
But $4mn+2m+2n+1=(2m+1)(2n+1)$, as one can be check by opening brackets on the right hand side. This means that we can rewrite our equation as $(2m+1)(2n+1)=2701$. So 2701 can be factored as a product of numbers $2m+1$ and $2n+1$. Now we can find these numbers by factoring 2701 as either $2701=37\times 73$ or as $2701=1\times 2701$ (there are no other factorizations since the numbers 37 and 73 are prime). Hence we either have $2m+1=37,\ 2n+1=73$ (or the other way around) or $2m+1=1,\ 2n+1=2701$ (or the other way around). In the first case, either $m=18,\ n=36$ or $m=36,\ n=18$. In the second case, there is no solution since $m,n>0$. So we see that the problem has only one solution (if we don't care about exchanging m and n), namely the dimensions of the rectangle are 36×18.

Remark: Equations of the type we needed to solve here are called Diophantine. Factorization is a useful tool in their solution.