Math Circle 10/1/2016 Meeting: Challenge of the week

Make sure that your solution is correct, complete, and clearly written. You should not expect much credit if your proof refers to a false statement, or even if all your statements are true but you forgot to tell us "why?" It is one of the purposes of the Circle to help you improve your "essay-proof" writing style as well as your logical skills.

Problem 3. An evil dragon locked six gnomes in the cave and told them "I happen to have seven pointed hats, in the seven rainbow colors. Tomorrow morning I will blindfold you all and will put a hat on each one of you, and will hide the remaining hat. After that I will remove your blindfolds so you will see the hats of other gnomes, but not the one on your own head. I will not allow you to talk but will take each of you to the other room and ask to name the color of the hidden pointy hat. If at least three of you will give the correct answer, I'll let you go. If not – eat you for a lunch." Can six gnomes devise the strategy to avoid being eaten tomorrow?

Solution: Each gnome can see all but two pointy hats: His own one and the hidden one. So each gnome has to decide which one of the 2 missing colors he will name to the dragon. One possible way to make sure that exactly 3 gnomes will name the color of the hidden hat is as follows: Let's enumerate the seven rainbow colors in some order, say

RED 1; ORANGE 2; YELLOW 3; GREEN 4; BLUE 5; INDIGO 6; VIOLET 7.

If a gnome does not see two colors of the same parity (say 2 and 6, or 3 and 5), he names the color whose number is greater of two (so if he does not see say 2 and 6 he will name indigo). If a gnome does not see two colors of different parity he names the color with the smaller number (for example, if he does not see 3 and 4 he will name yellow). We claim that exactly 3 gnomes will name the correct hidden color if all gnomes use this strategy (that they decide upon beforehand). This can be easily checked: For example, if the dragon hides the yellow hat, gnomes wearing the red, green, and indigo hats will name the correct color.

Remark: It can be shown that there exists no strategy to ensure that 4 gnomes always name the correct color.