Modeling Population Growth

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Growth of bacteria

- Population starts with 50 bacteria
- Each bacteria splits into 2 every hour
- How many are left after 3 hours?

TIME (t)	POPULATION
start: $t = 0$	50
after 1 hour: $t = 1$	$50 \cdot 2 = 100$
after 2 hours: $t = 2$	$50 \cdot 2 \cdot 2 = 50 \cdot 2^2 = 200$
after 3 hours: $t = 3$	$50 \cdot 2 \cdot 2 \cdot 2 = 50 \cdot 2^3 = 400$

Look familiar?

after t hours: $50 \cdot 2^t$

t ← number of splits

50 ↑ initial population

2 ↑ split factor

Can the popualtion grow forever?

- Not in biological systems!
- What limits growth?

Modeling populations

The population in 1 hour depends on the population this hour:

Pop (in 1 hour) = Pop(now)

+ births

- deaths

Example population model

$$P(t+1) = \frac{P(t)}{P(t)+a} \qquad a>0$$

Some definitions

Iteration: one application of the function on the right hand side

$$P(t+1) = \frac{P(t)}{P(t)+a}$$

Some definitions

Iterate: size of the population after one iteration

$$P(t+1) = \frac{P(t)}{P(t)+a}$$

More definitions

Fixed point: population size that does not change after one iteration

Look for where $P(t+1) = P(t)$

Can you find a fixed point?

Fixed point: population size that does not change after one iteration

$$P = \frac{P}{P + a}$$

More definitions

Accumulating point: population size that occurs repeatedly after iterations

May not return to an accumulating point after one iteration

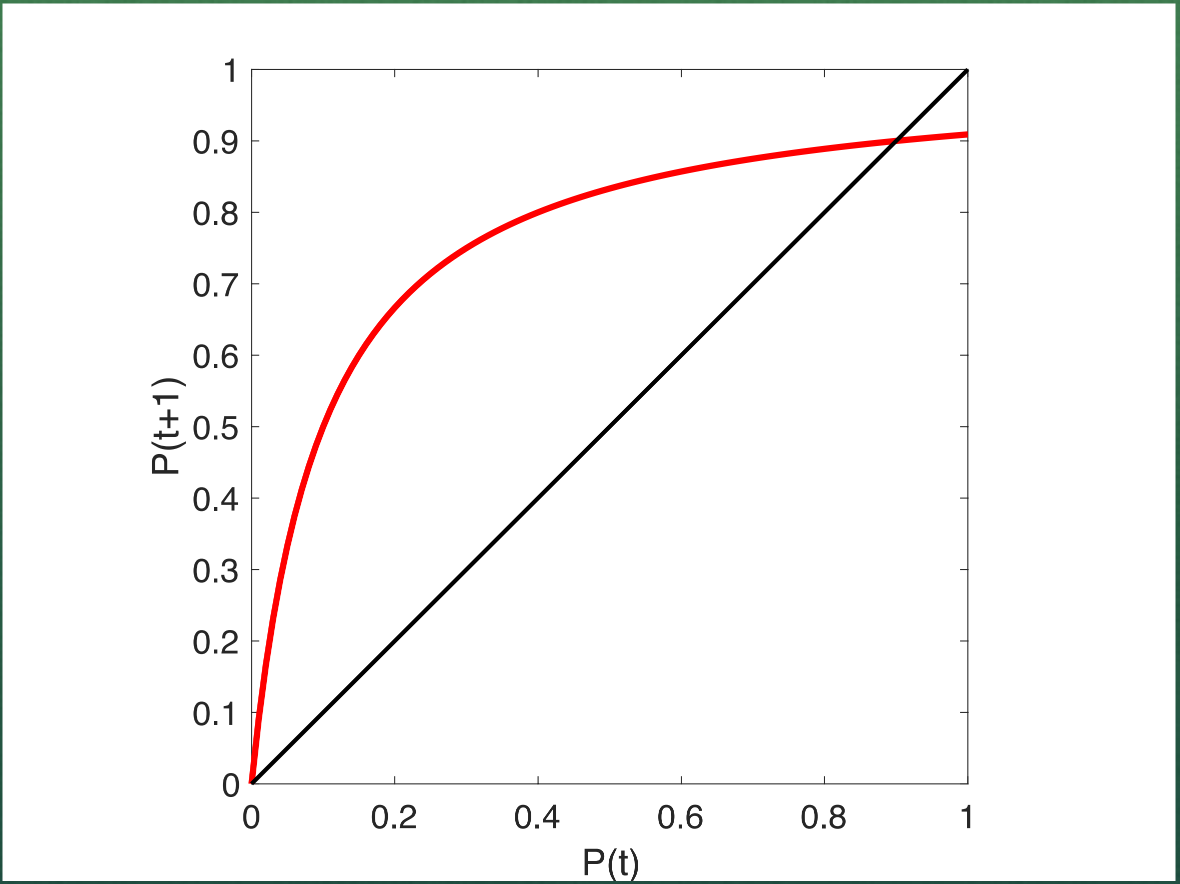
Can you find an accumulating point?

Accumulating point: population size that occurs repeatedly after iterations

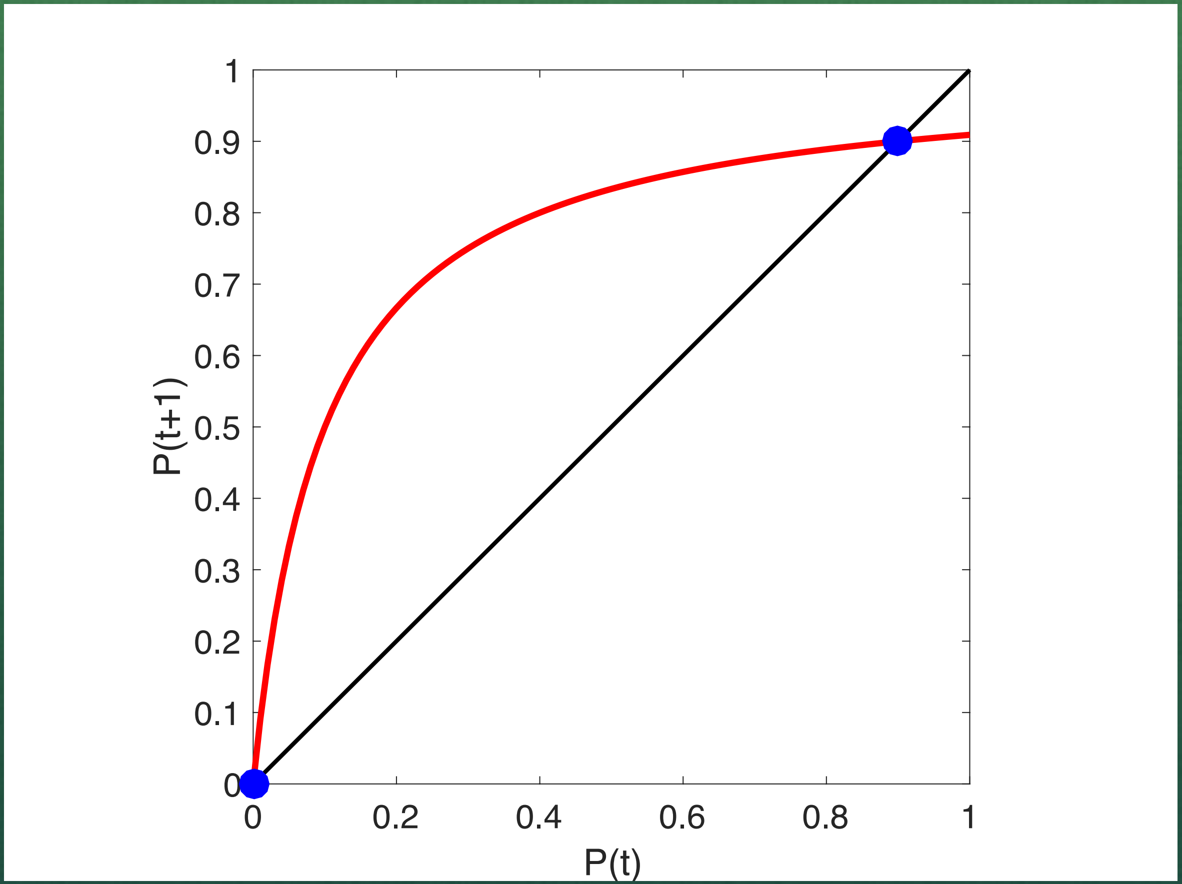
$$P(t+1) = \frac{P(t)}{P(t)+a}$$

$a = 0.1$

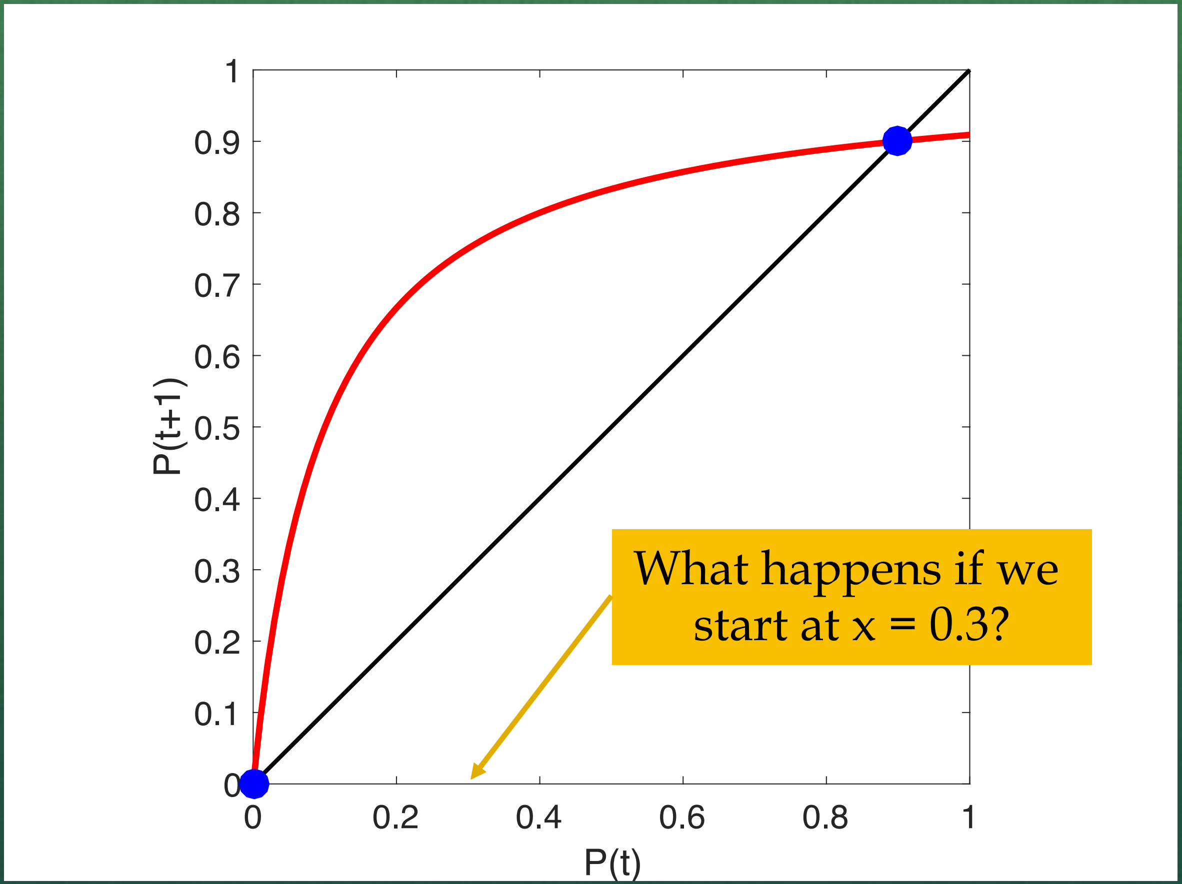
Graphical solutions



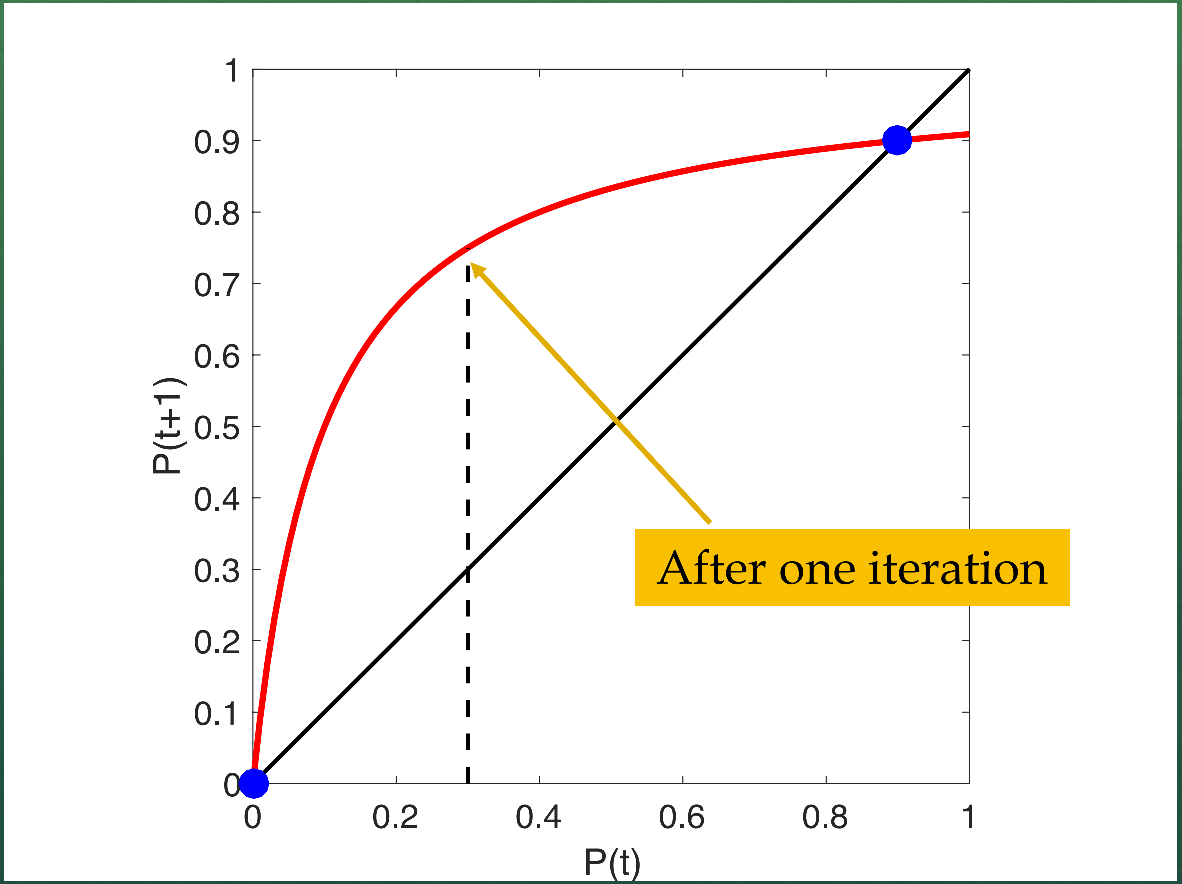
Graphical solutions



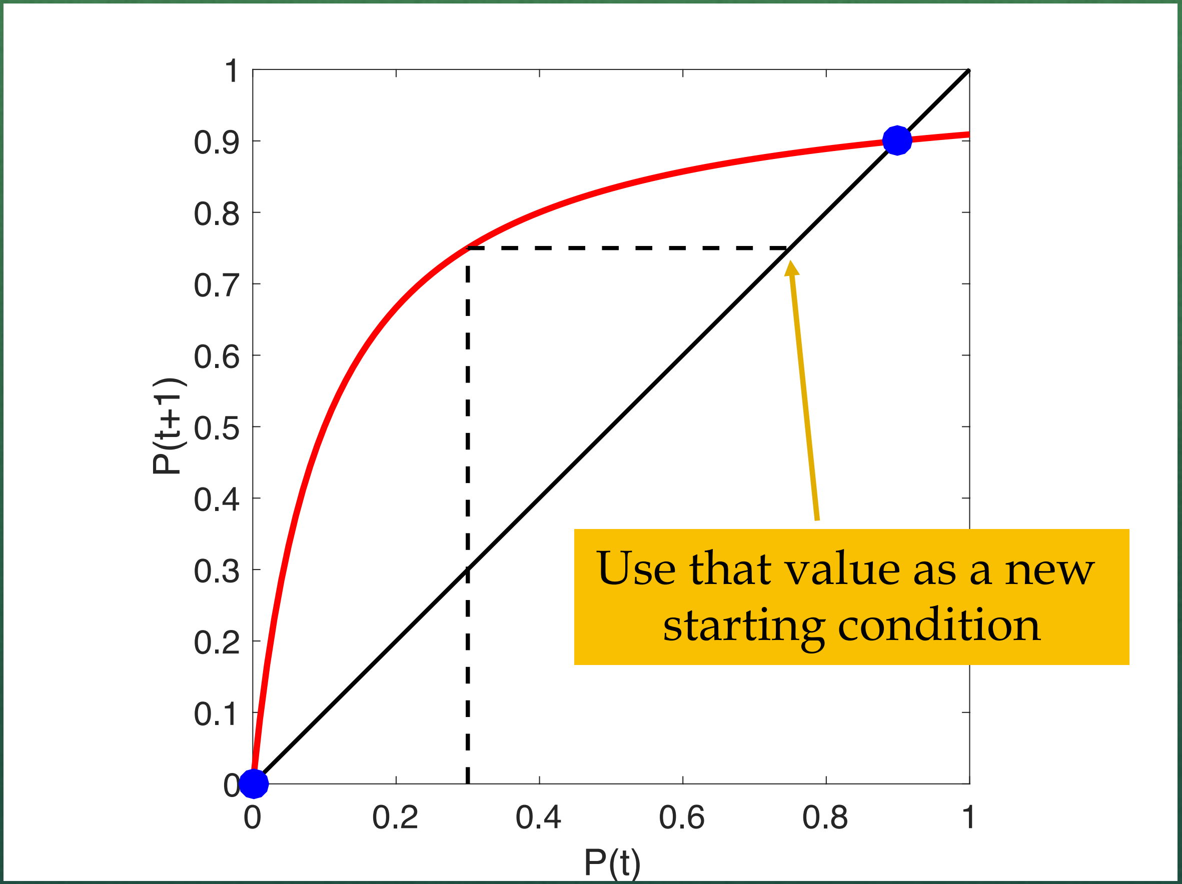
Getting to the fixed points:
Cobweb diagrams



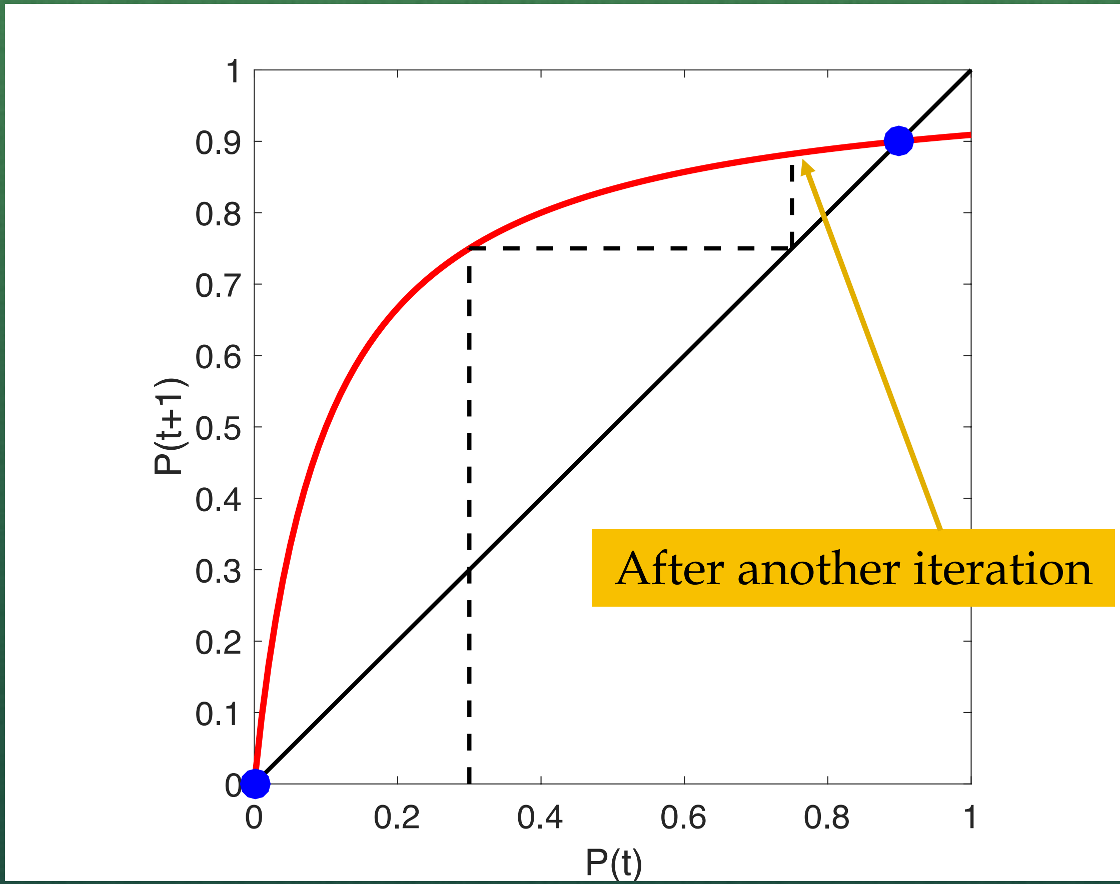
Getting to the fixed points:
Cobweb diagrams



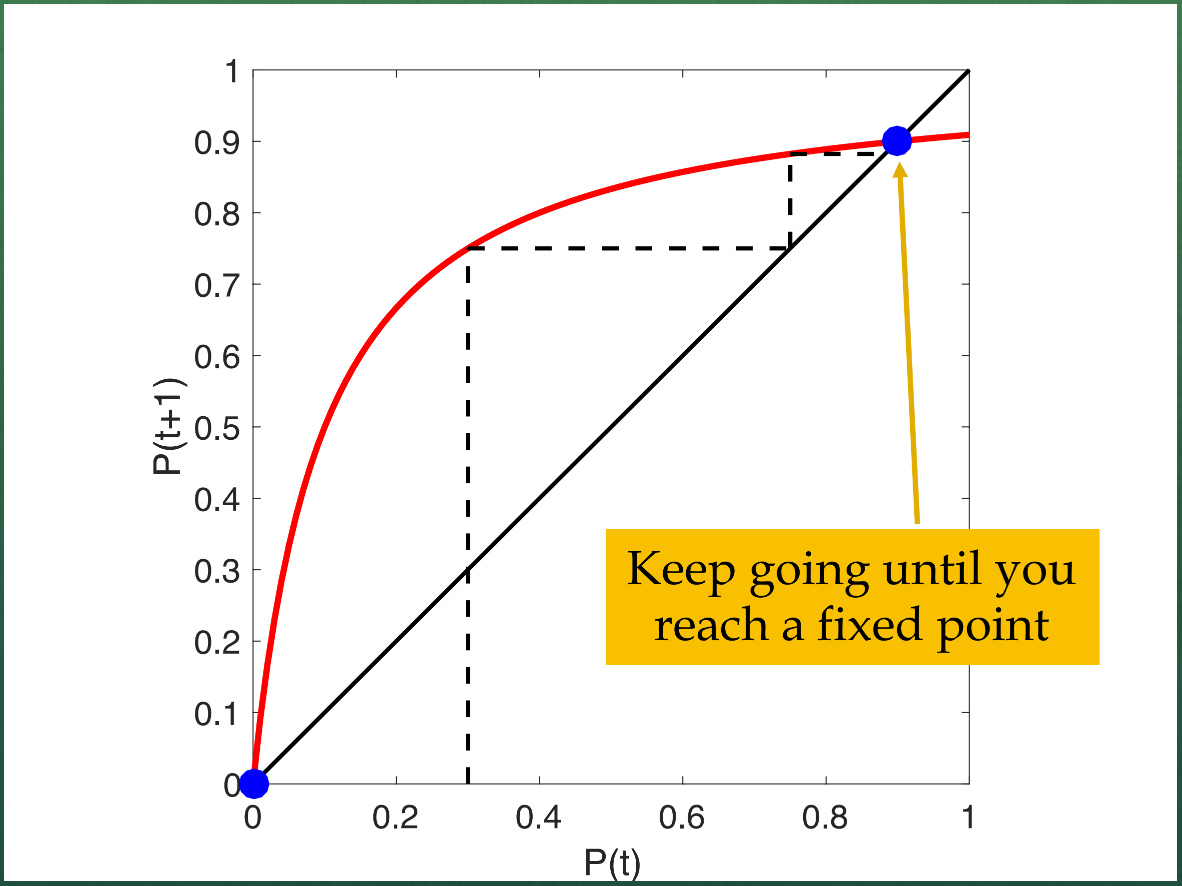
Getting to the fixed points:
Cobweb diagrams



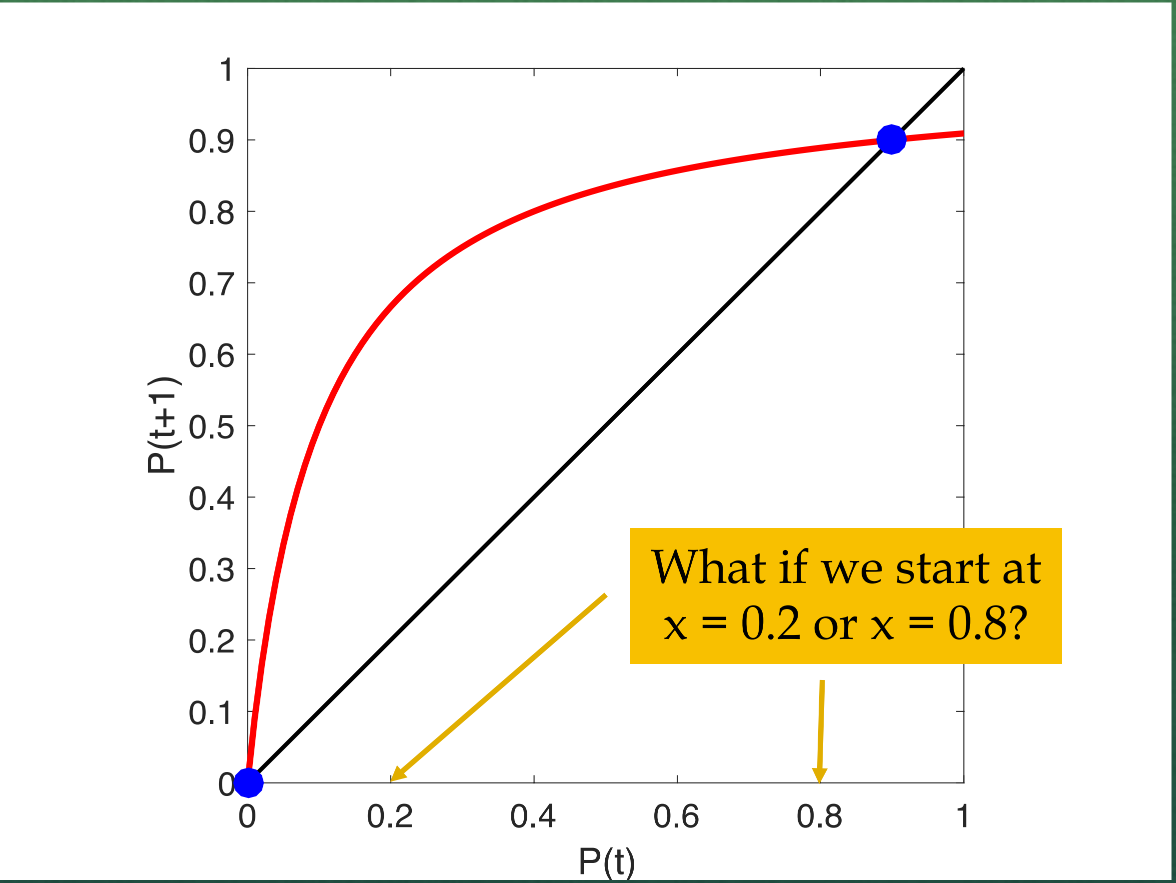
Getting to the fixed points:
Cobweb diagrams



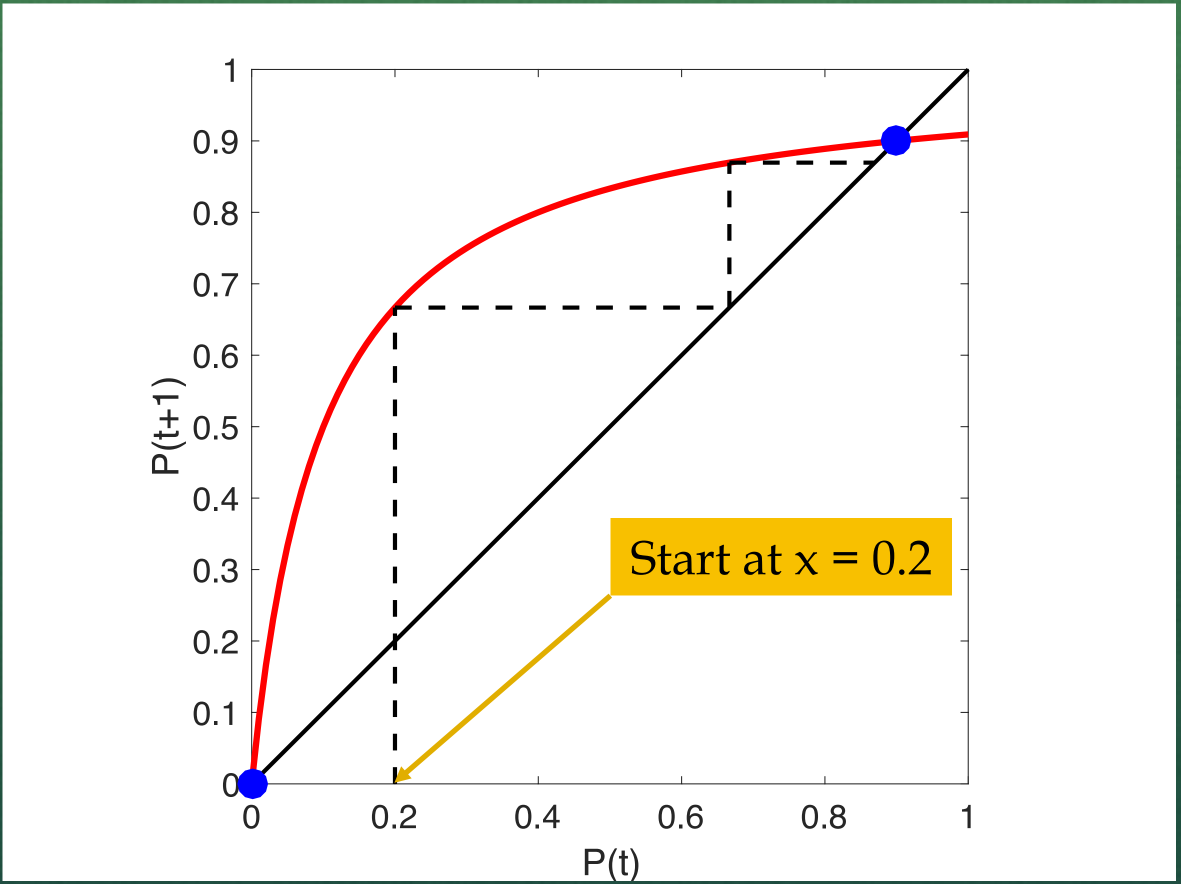
Getting to the fixed points:
Cobweb diagrams



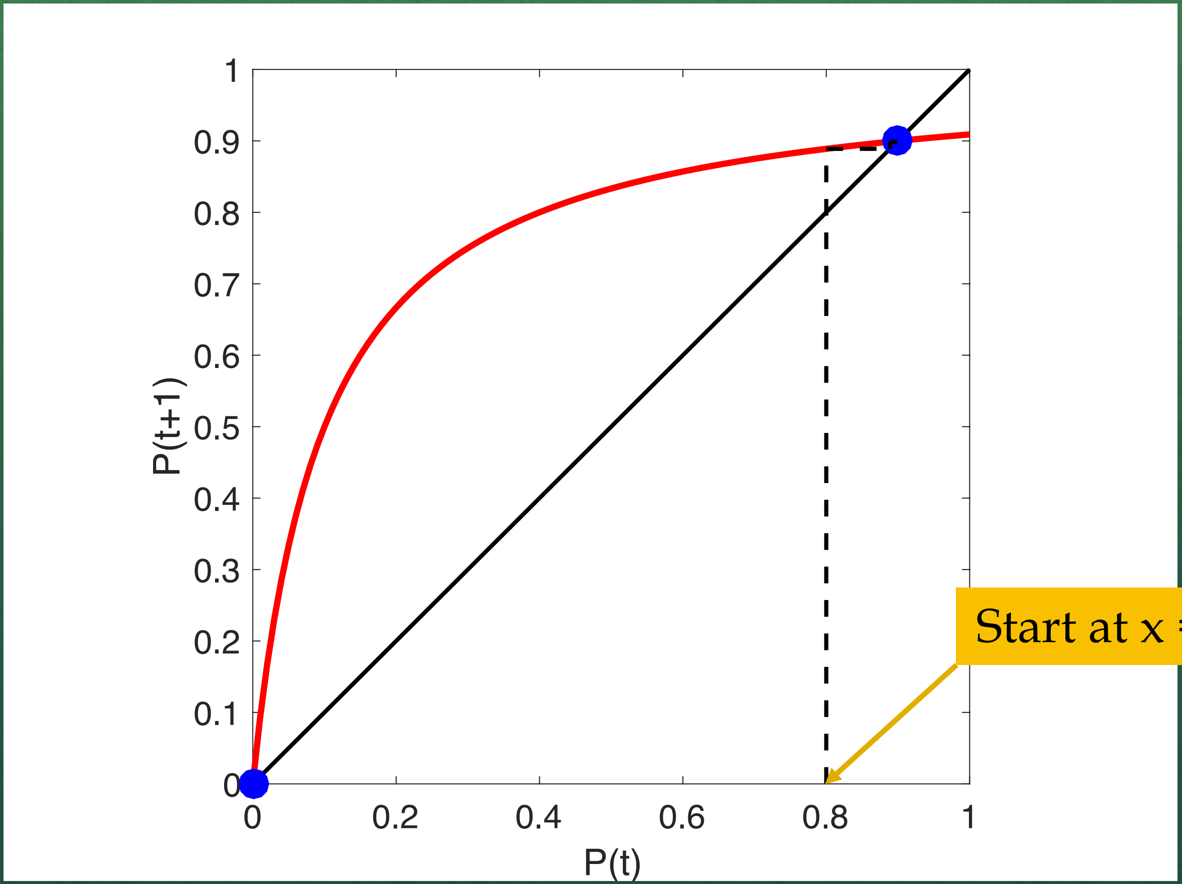
Getting to the fixed points:
Cobweb diagrams



Getting to the fixed points:
Cobweb digrams

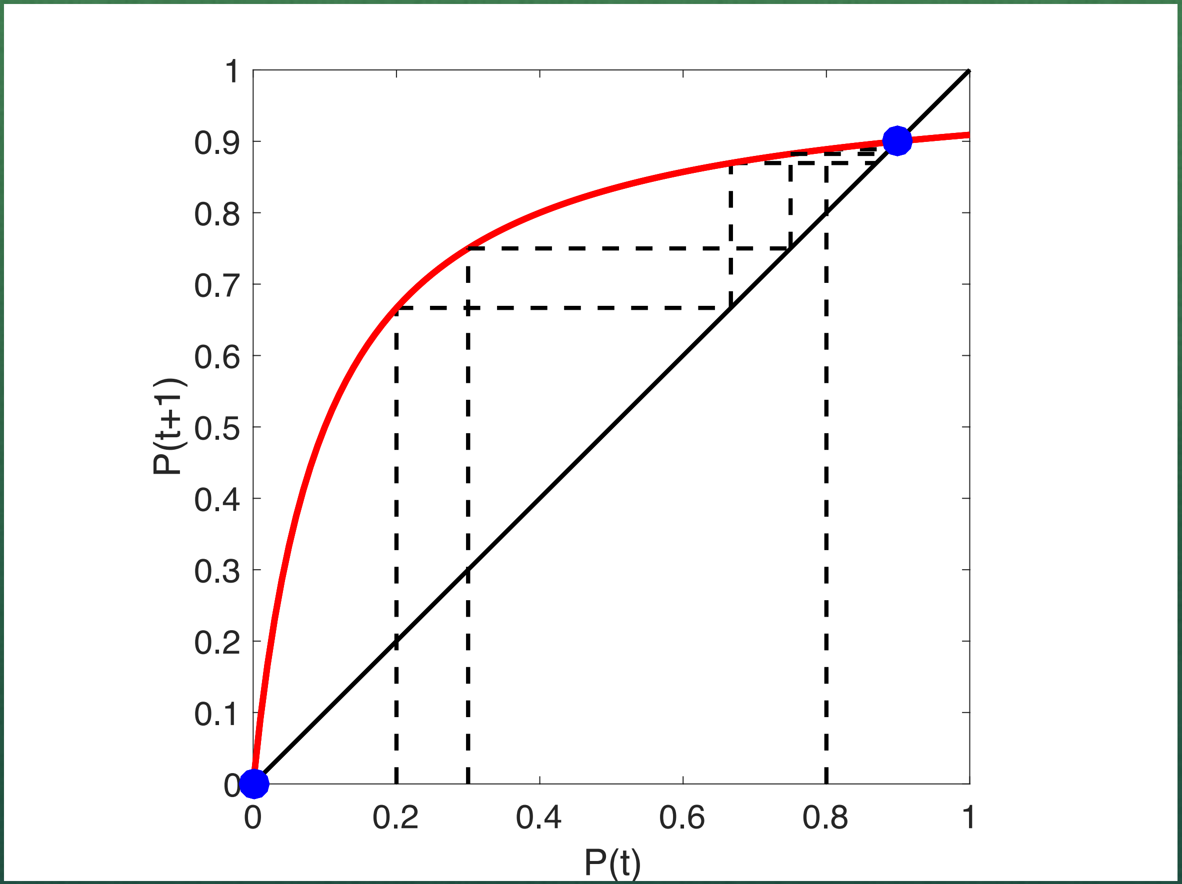


Getting to the fixed points:
Cobweb diagrams

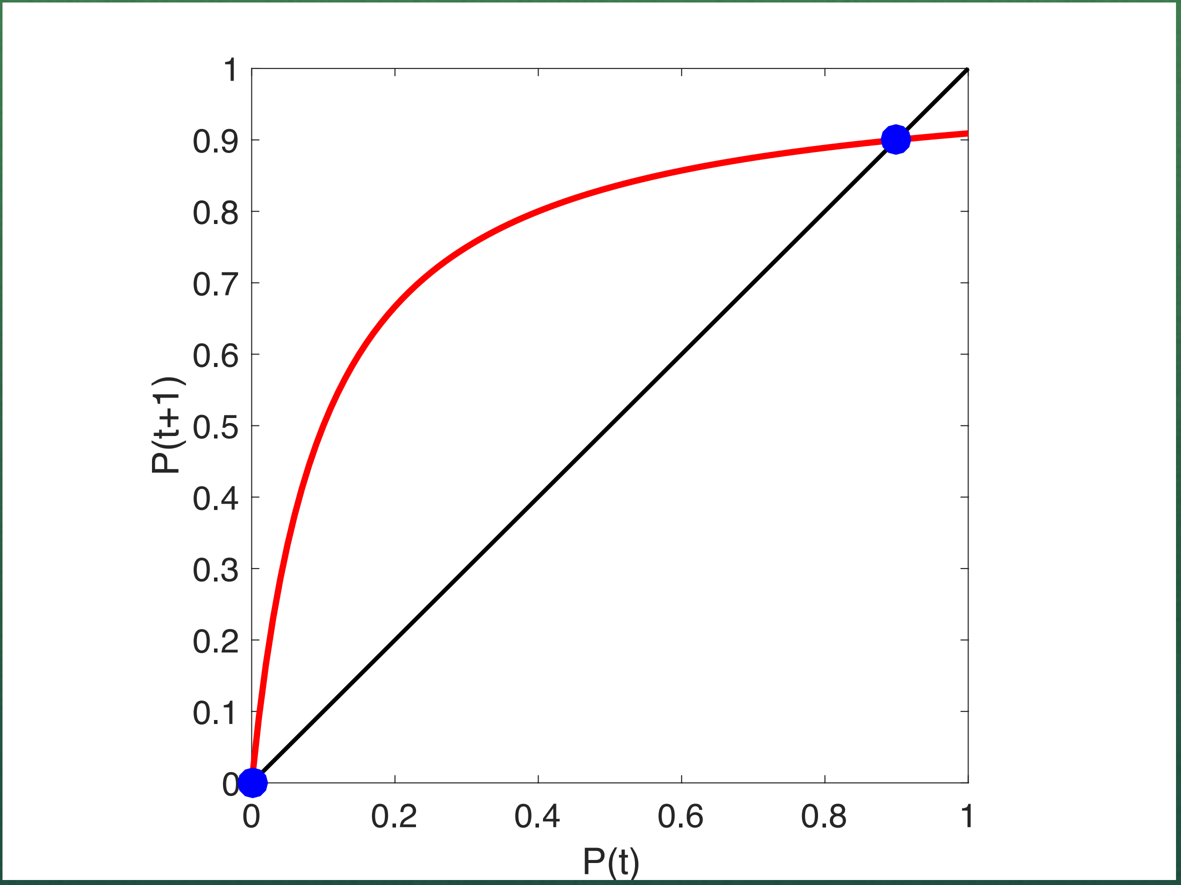


Start at x = 0.8

Getting to the fixed points:
Cobweb diagrams



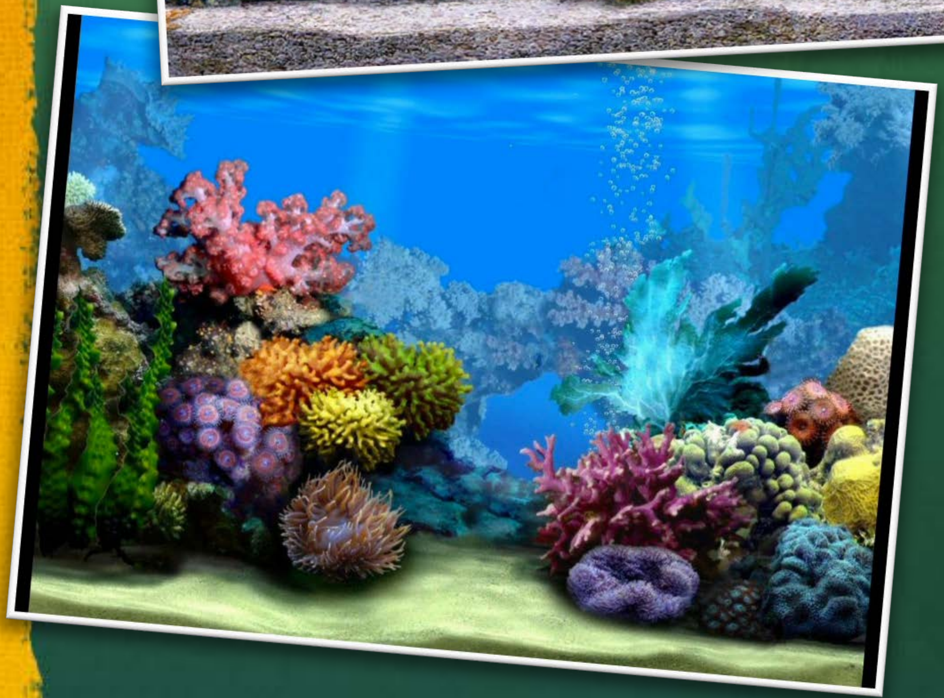
Can we ever get to zero?



Now, imagine a fish tank

How many fish are there?

Only so many fish can fit before overcrowding becomes a problem!



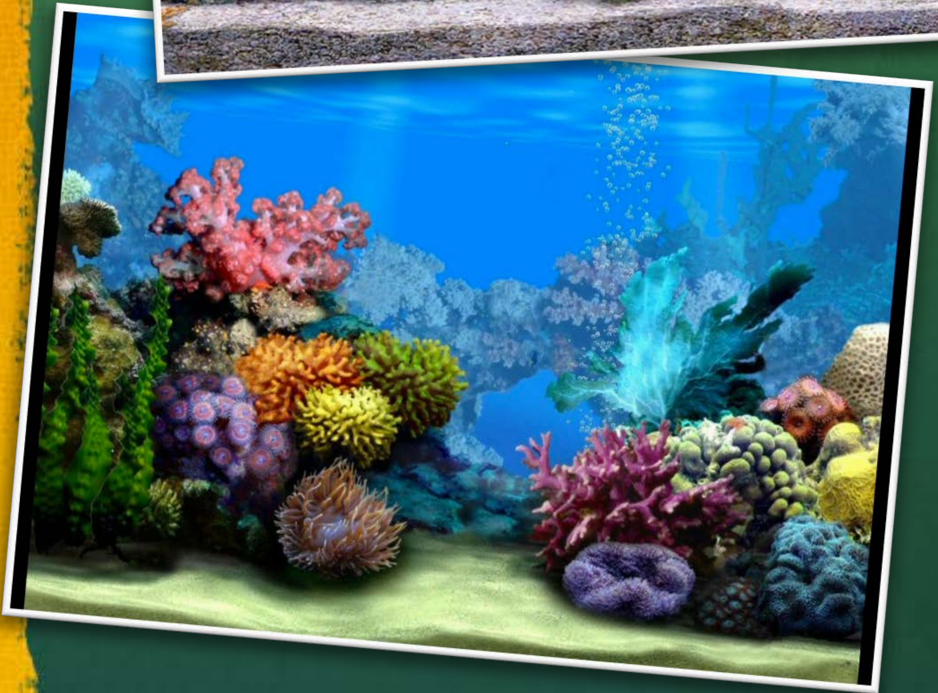
We can model this!

- Let x(t) be the number of fish in the tank this week
- Let r be the amount of food you feed them each week
- Fish population grows slowly when there are more fish
- After one week:

$$x(t+1) = r * x(t) * (1 - x(t))$$

Does it matter how many fish start in the fish tank?

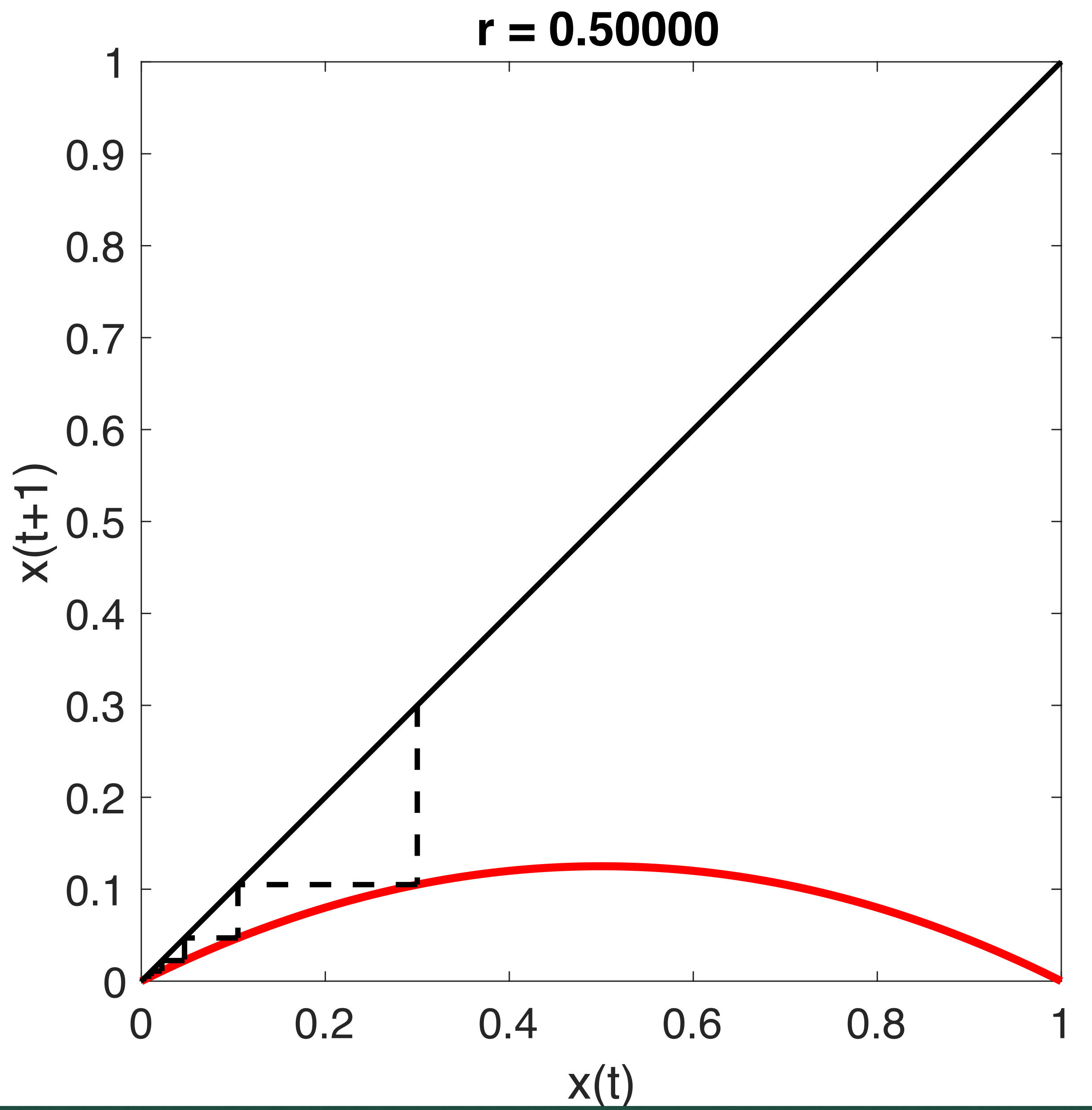
Does it matter how much you feed the fish?

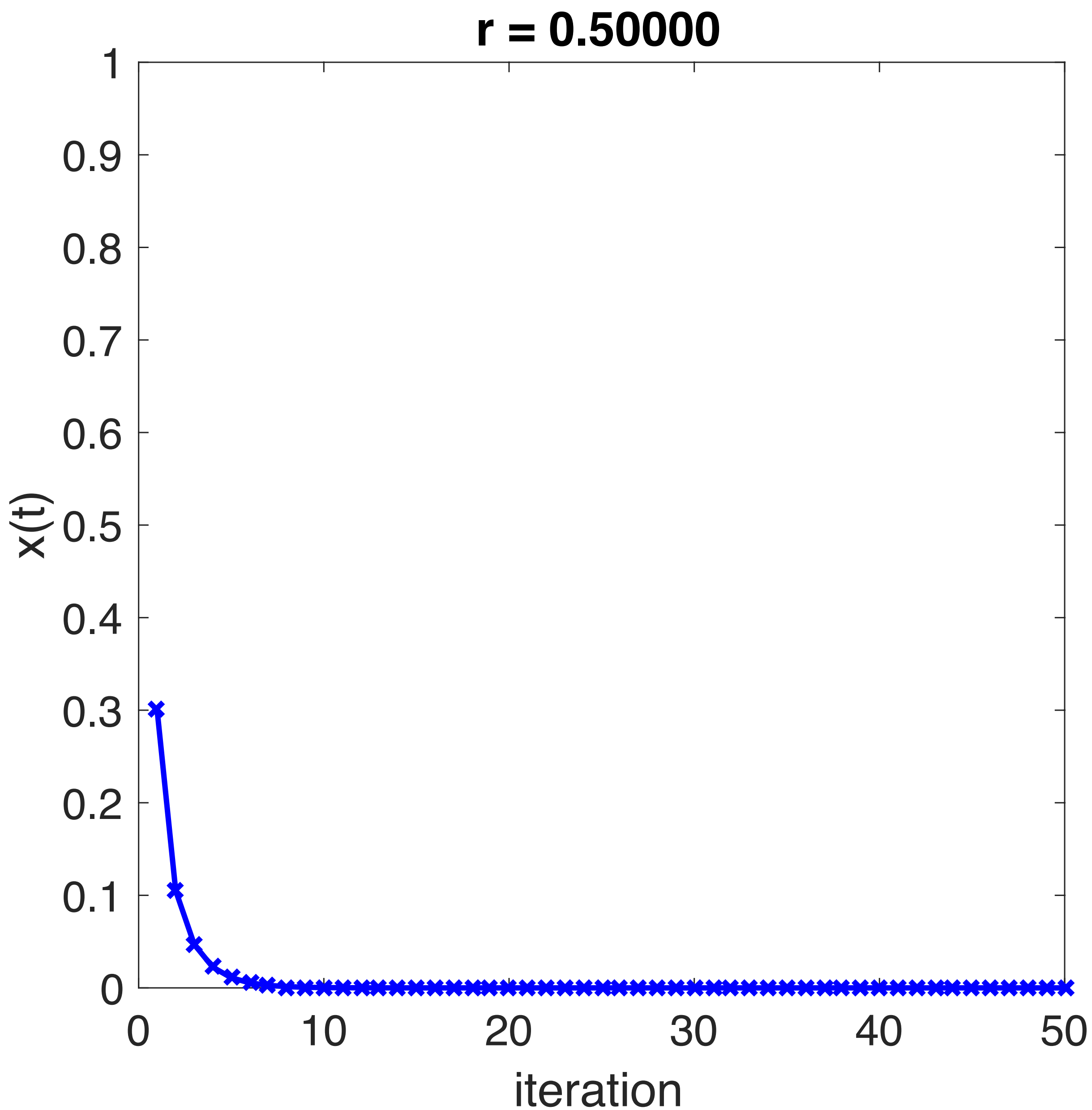


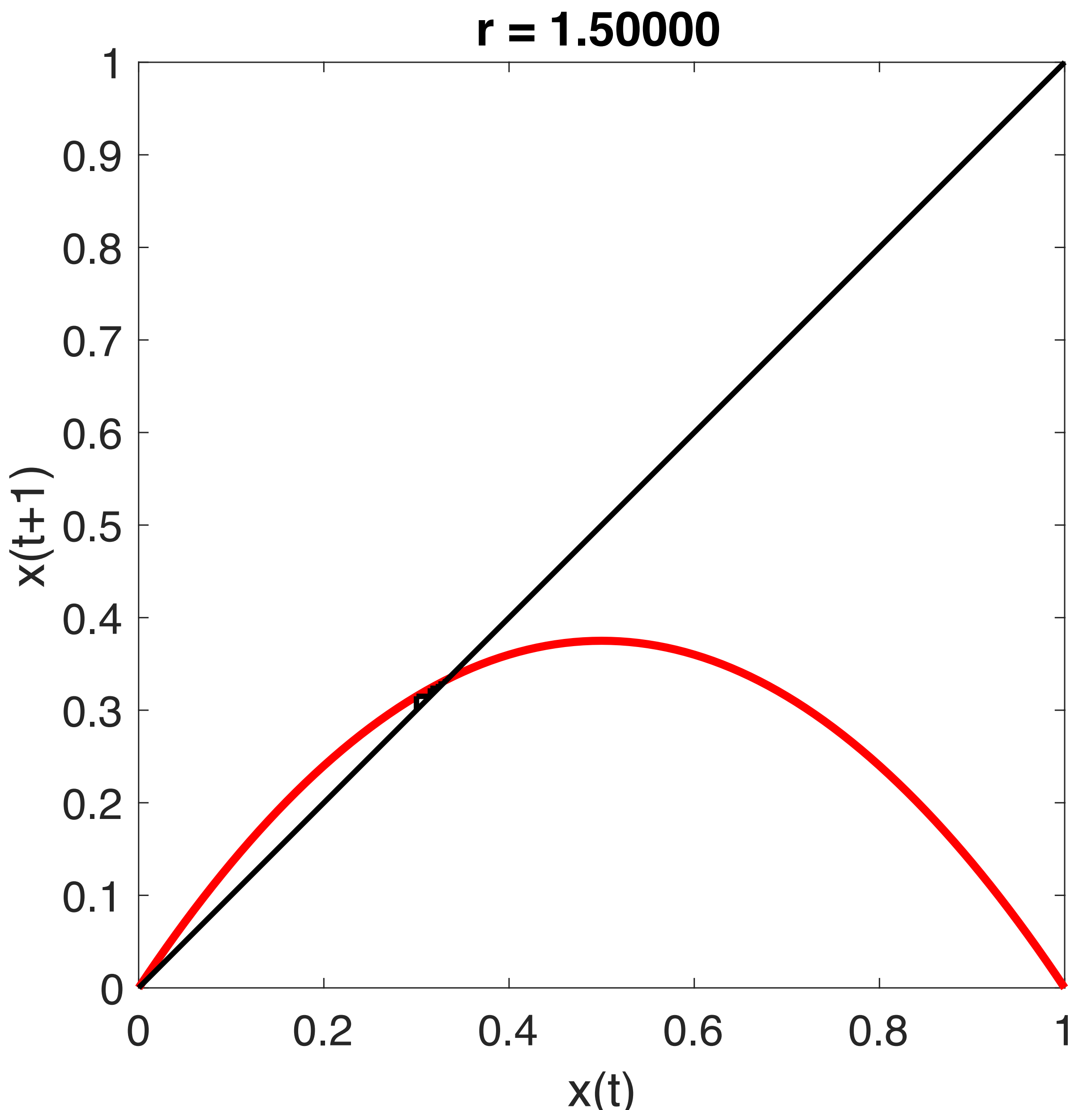
Try cobwebbing solutions

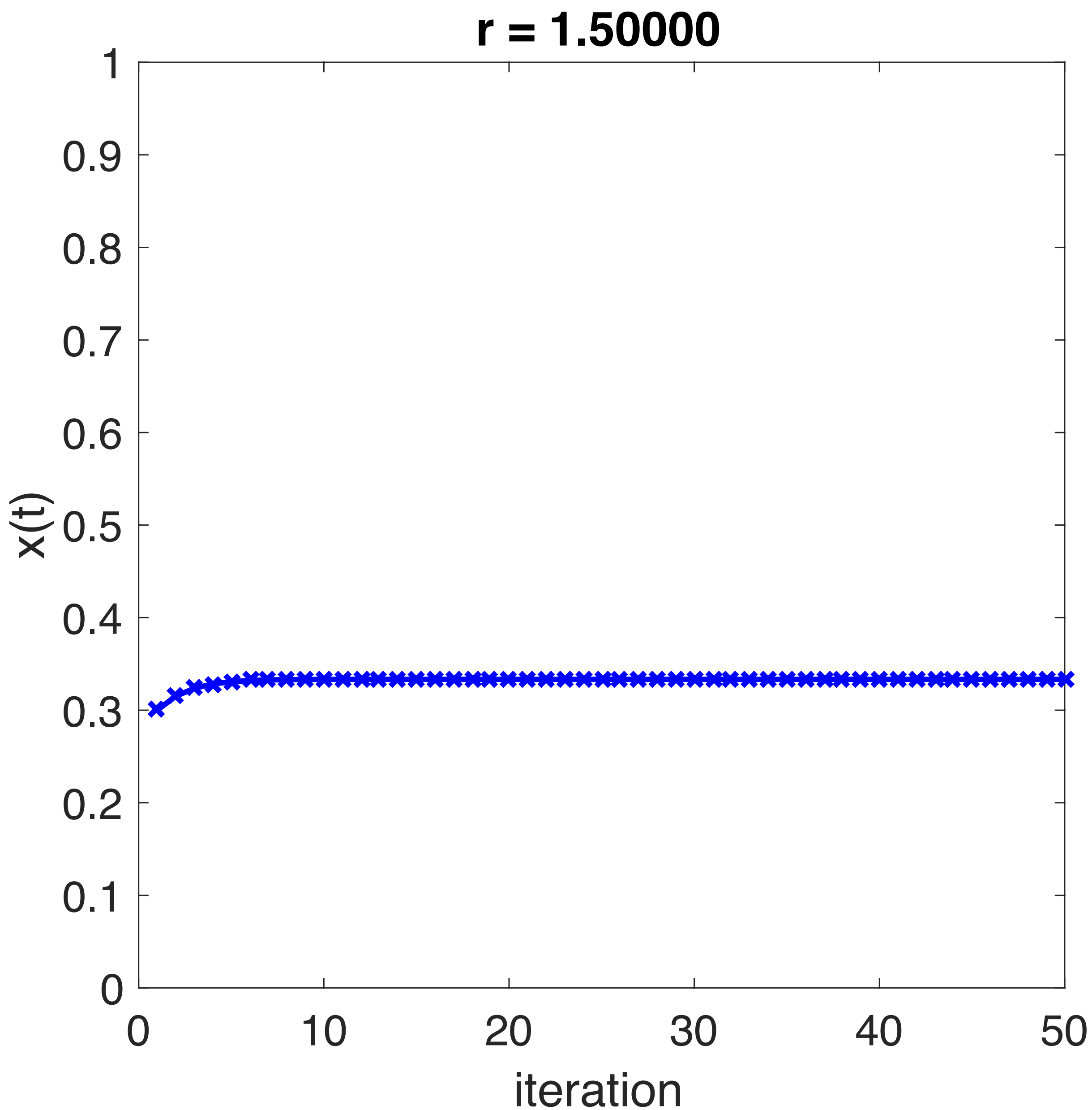
[Handouts]

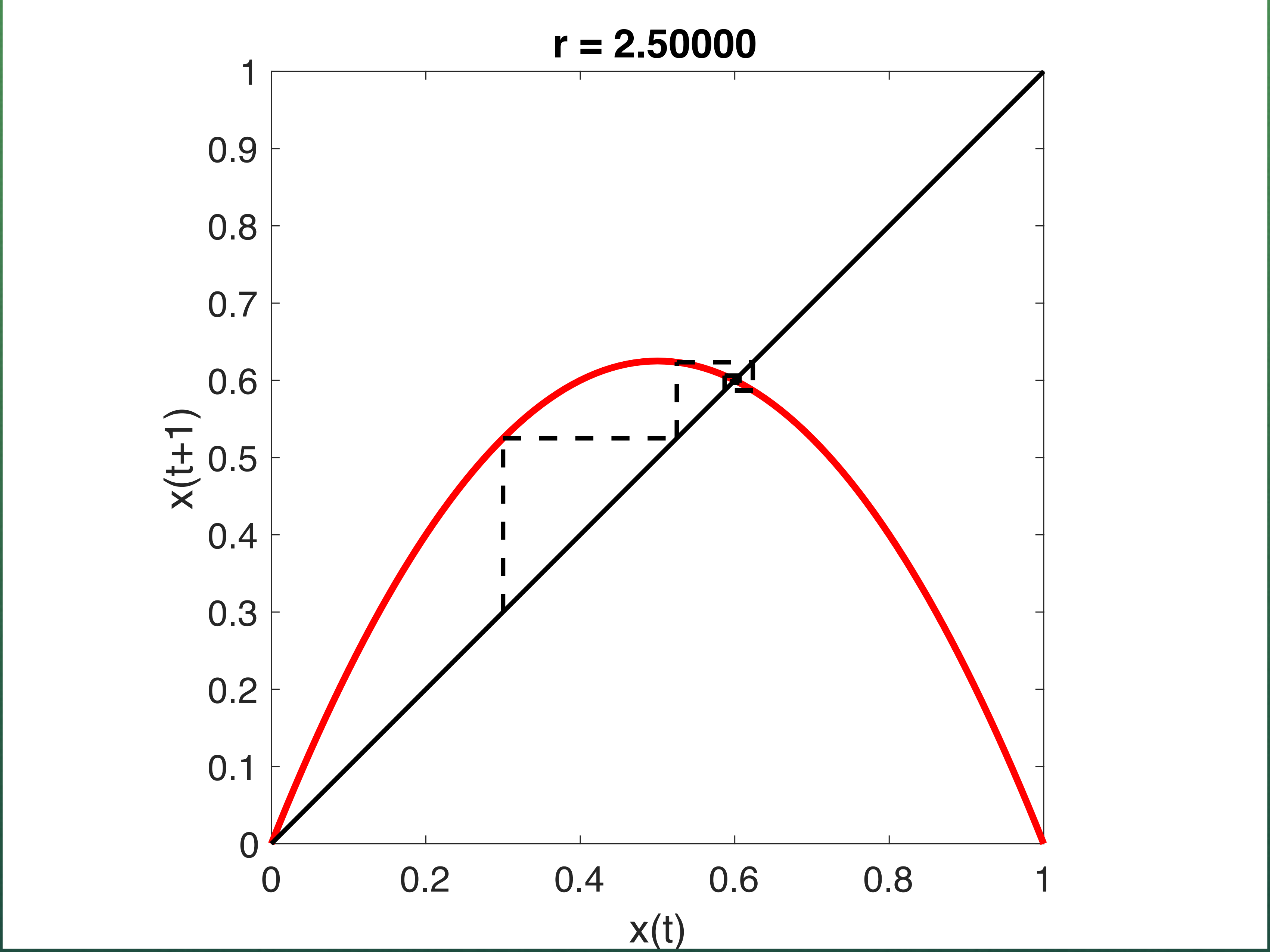
Let’s look at cobweb digrams

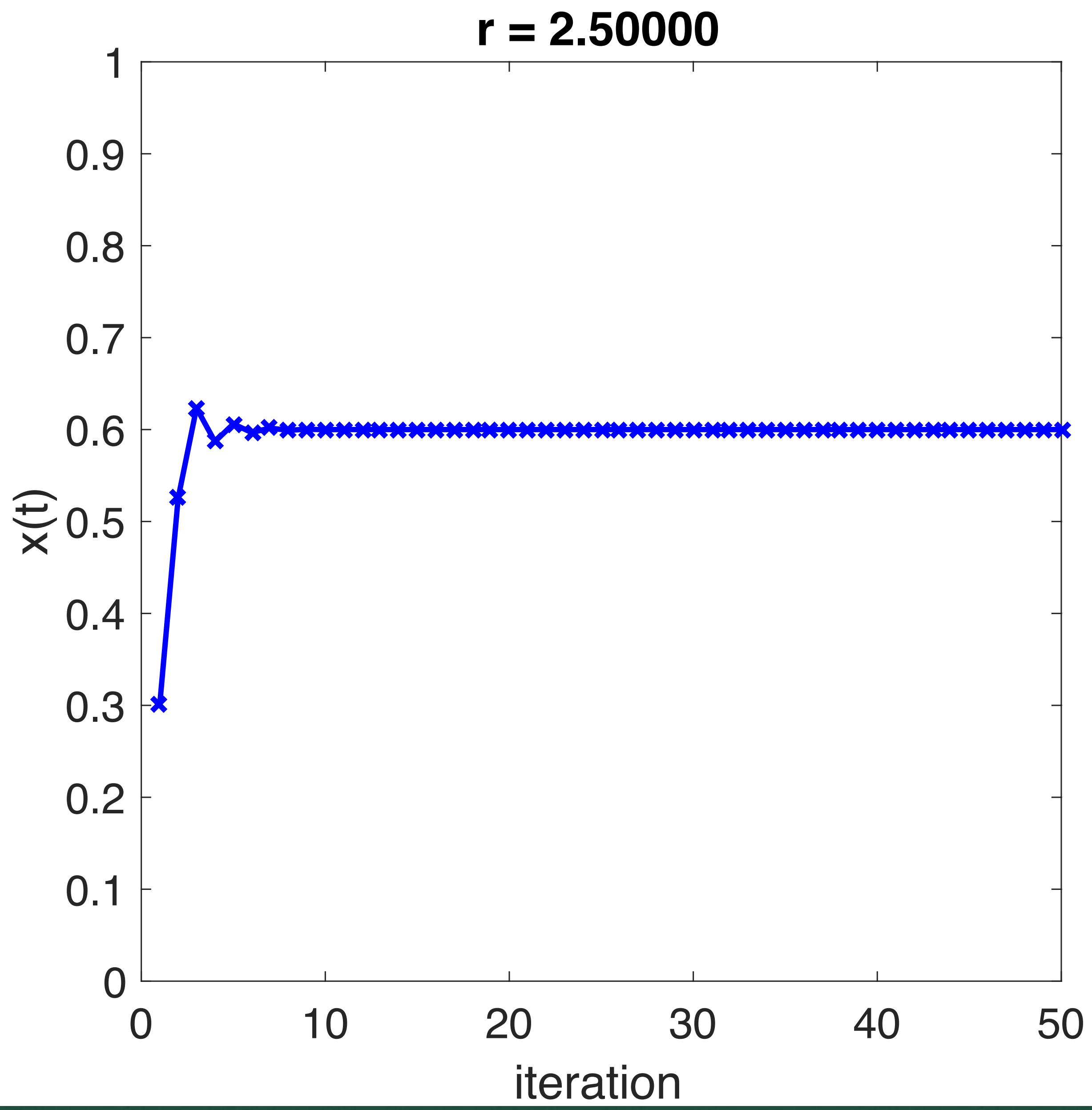

r = 0.50000
x(t+1)
x(t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1


r = 0.50000
x(t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
iteration

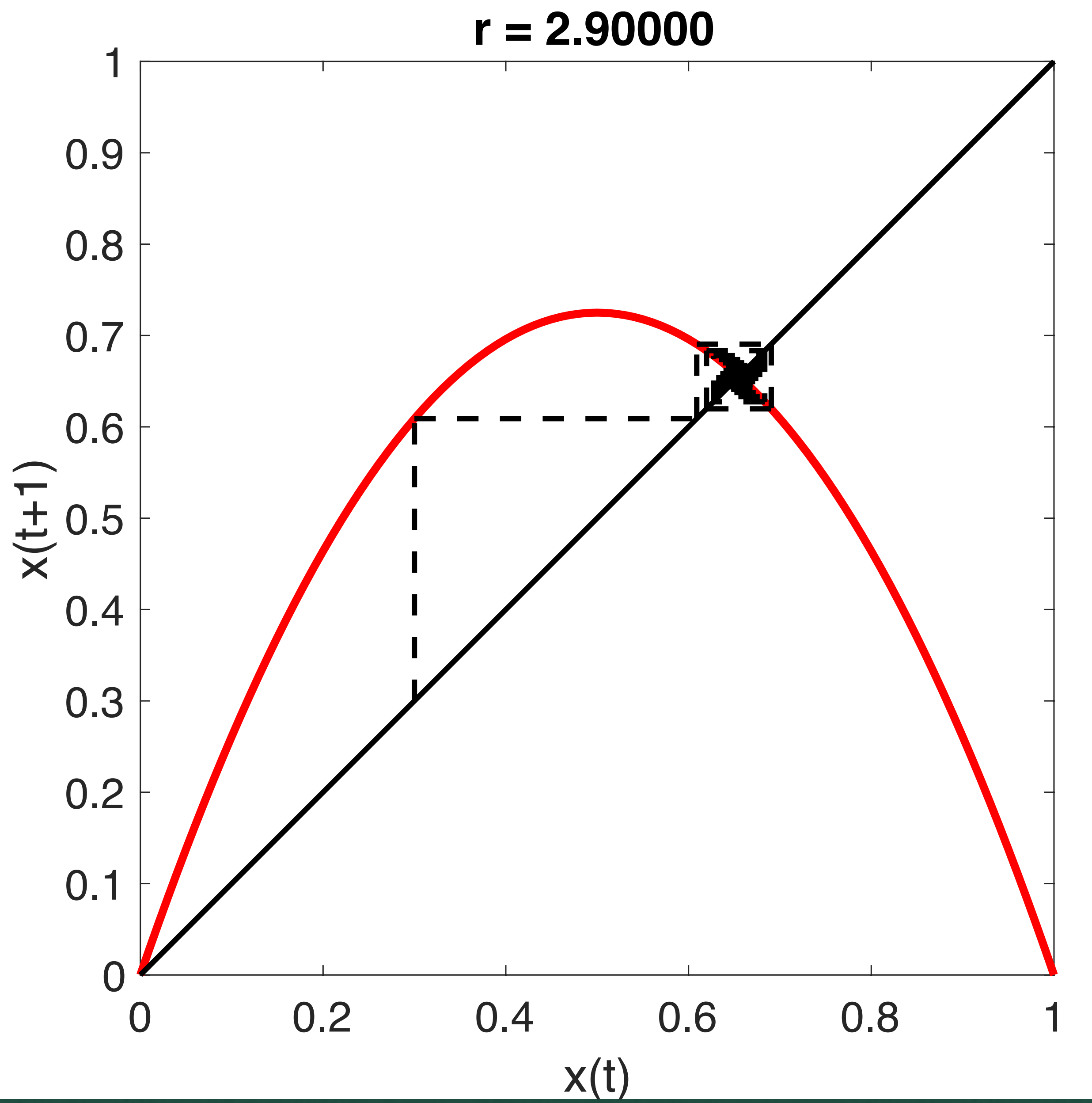

r = 1.50000
x(t+1)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
x(t)

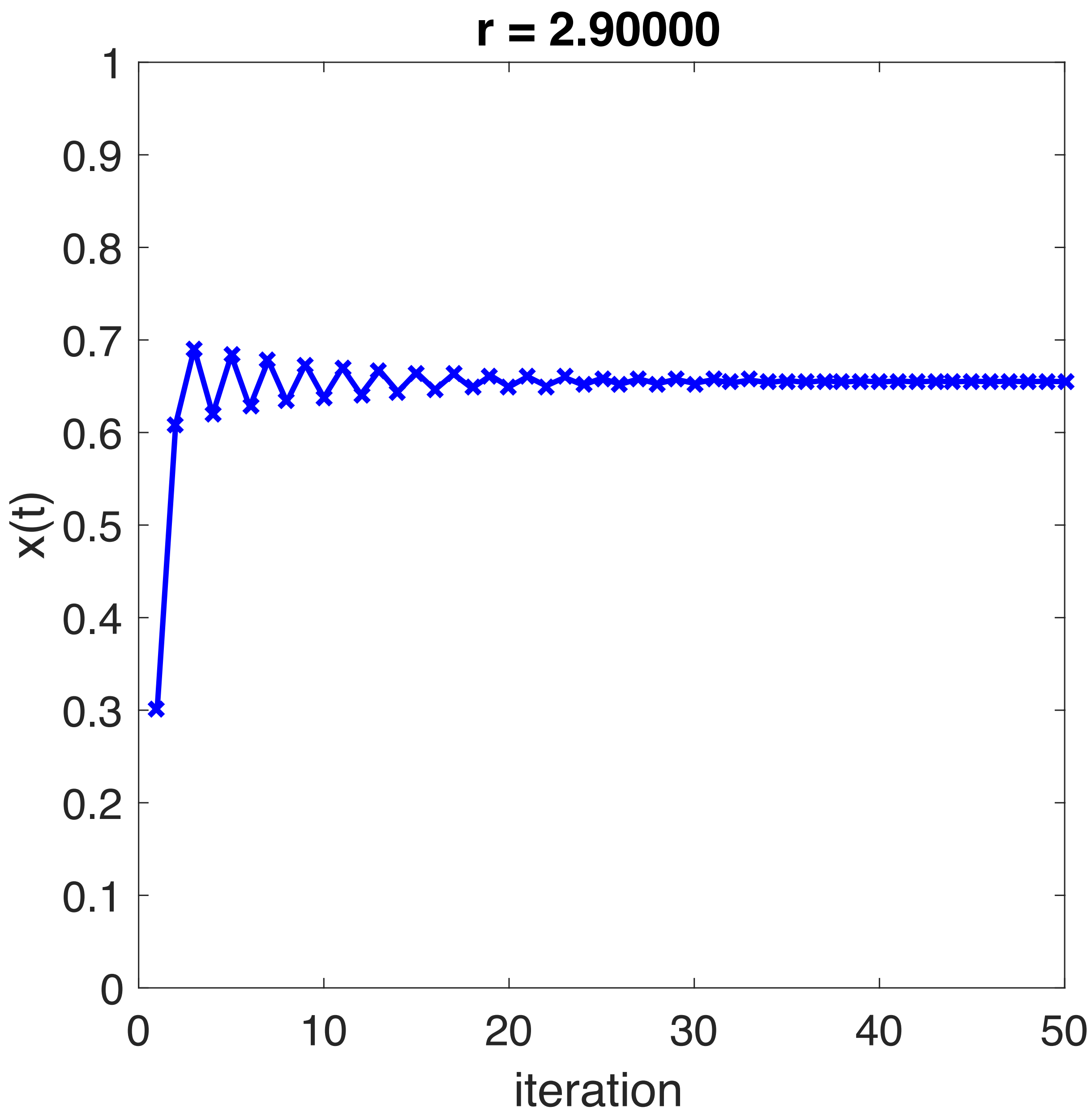

r = 1.50000
x(t)
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50
iteration


r = 2.50000
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x(t+1)
0
0.2
0.4
0.6
0.8
1
x(t)

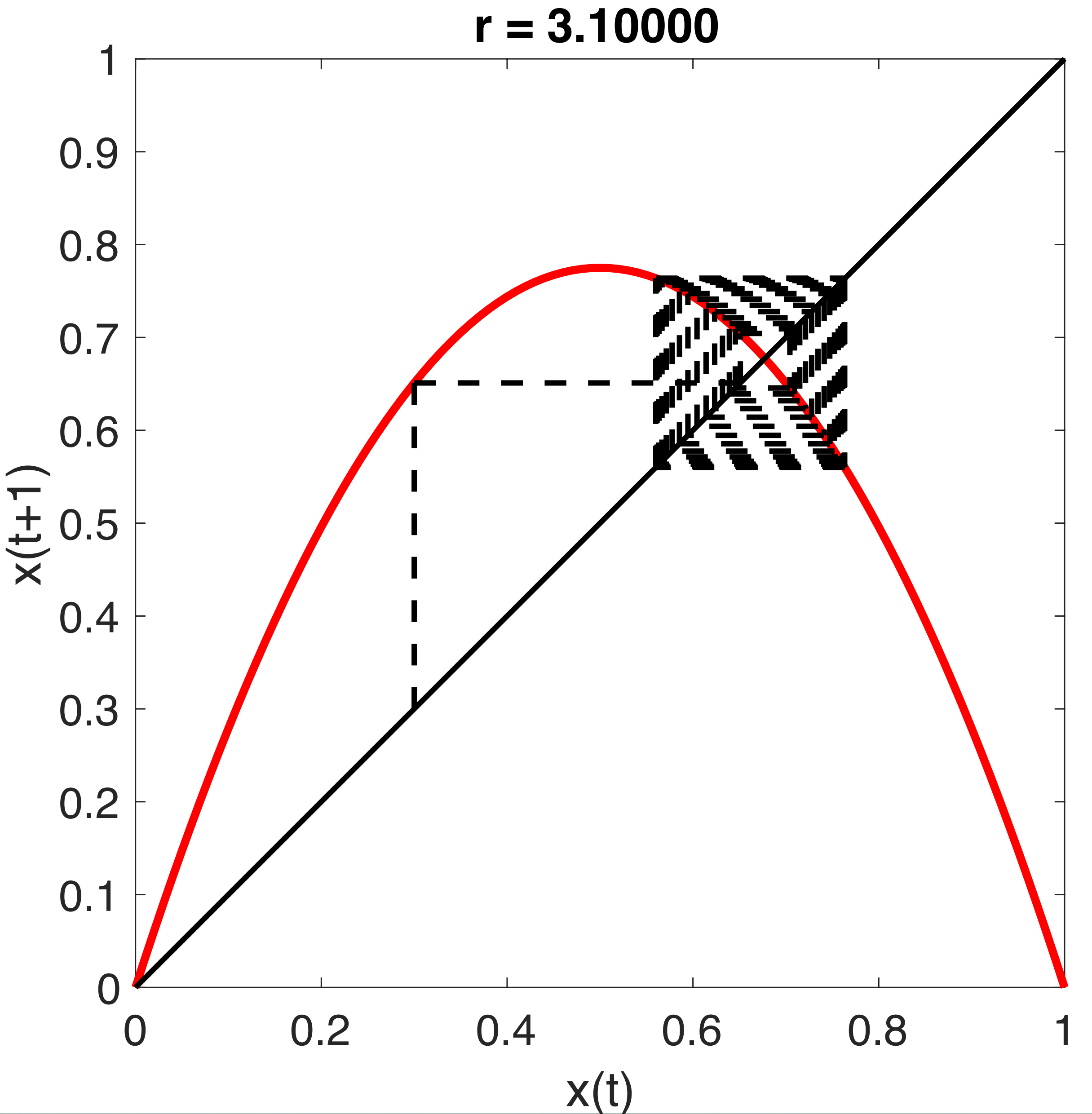


r = 2.50000
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50

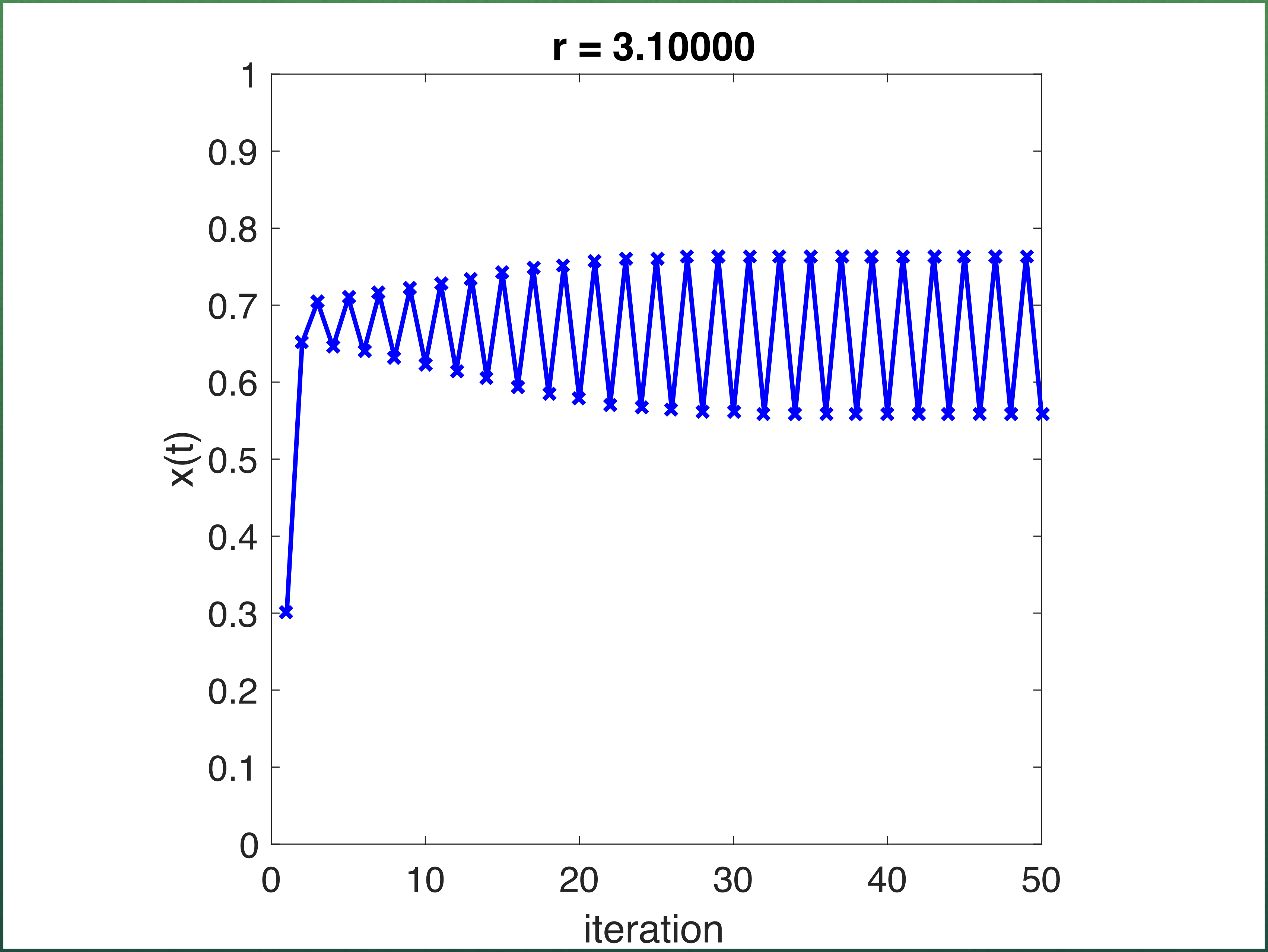

r = 2.90000
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x(t+1)
0
0.2
0.4
0.6
0.8
1
x(t)

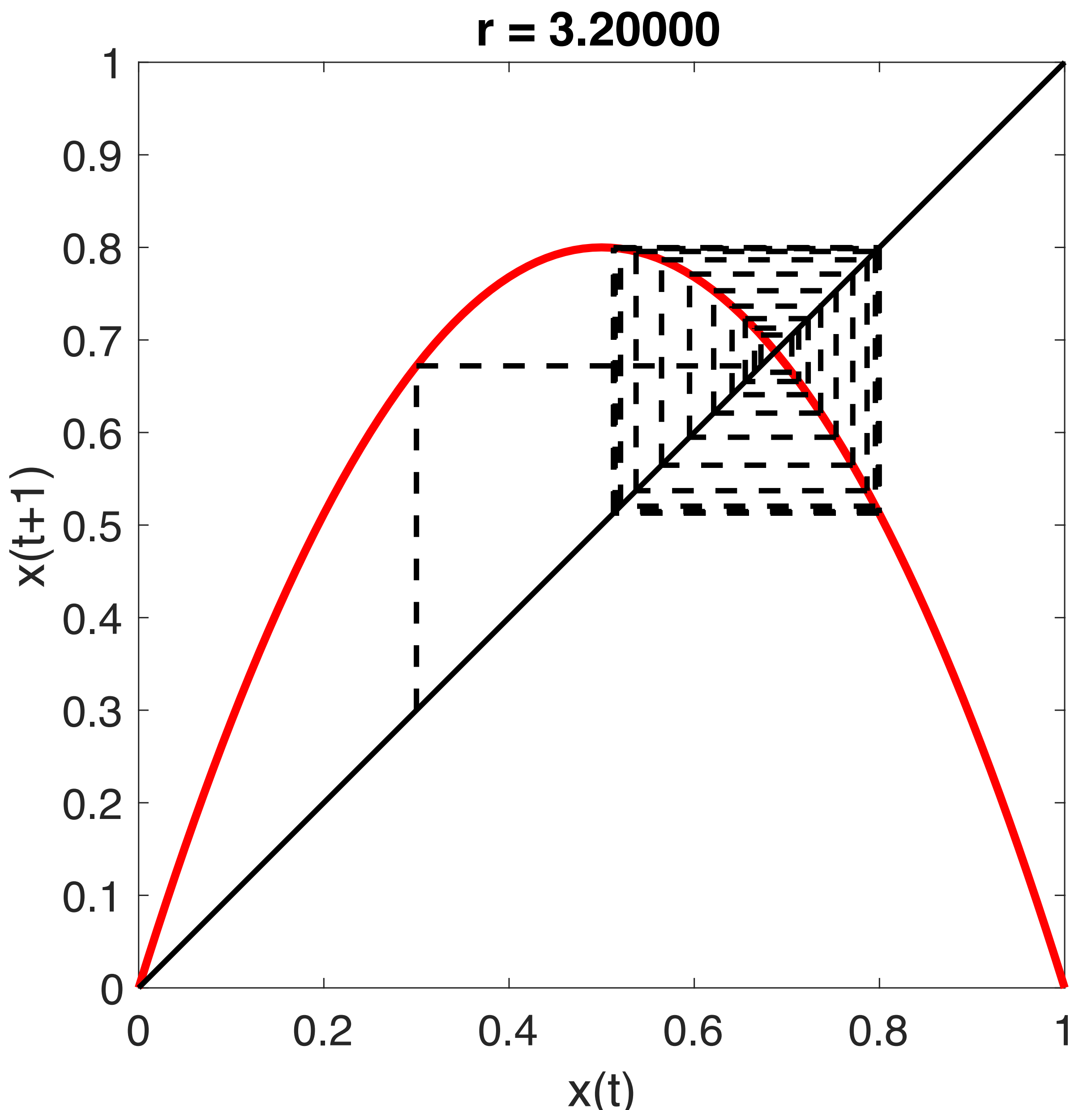


r = 2.90000
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x(t)
0
10
20
30
40
50
iteration

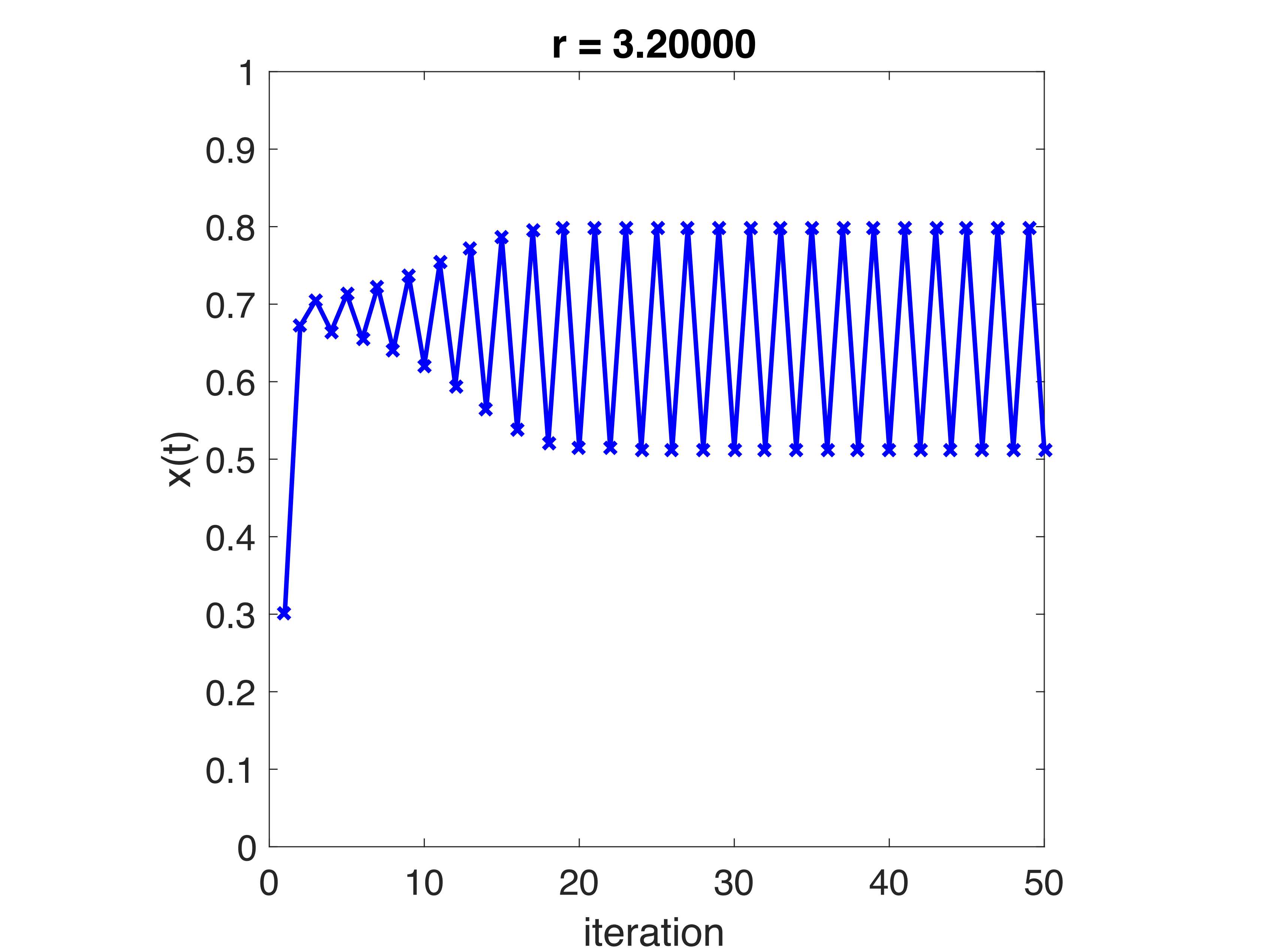


r = 3.10000
x(t+1)
x(t)
0
0.2
0.4
0.6
0.8
1
0.1
0.3
0.5
0.7
0.9

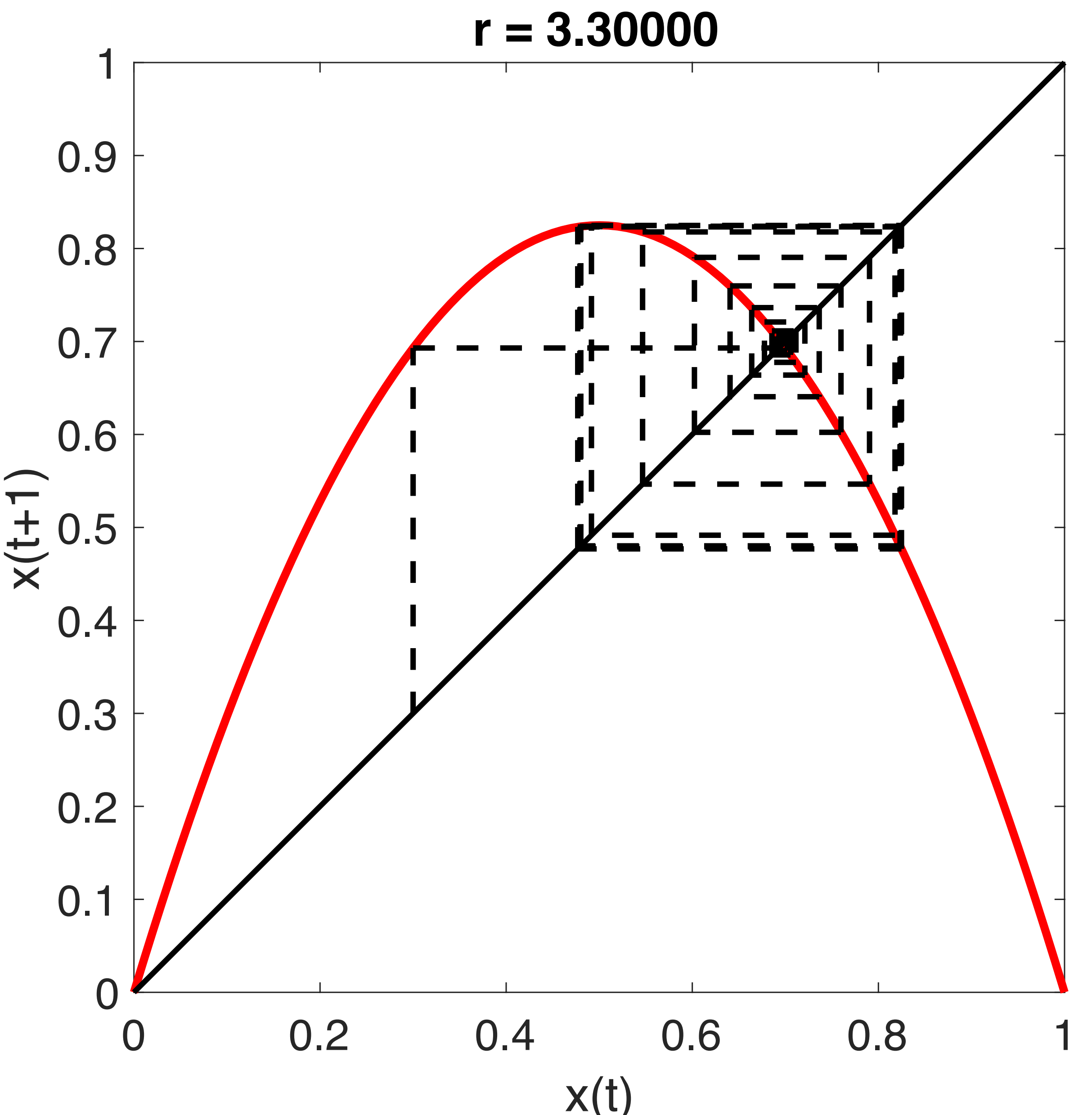

r = 3.10000
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50



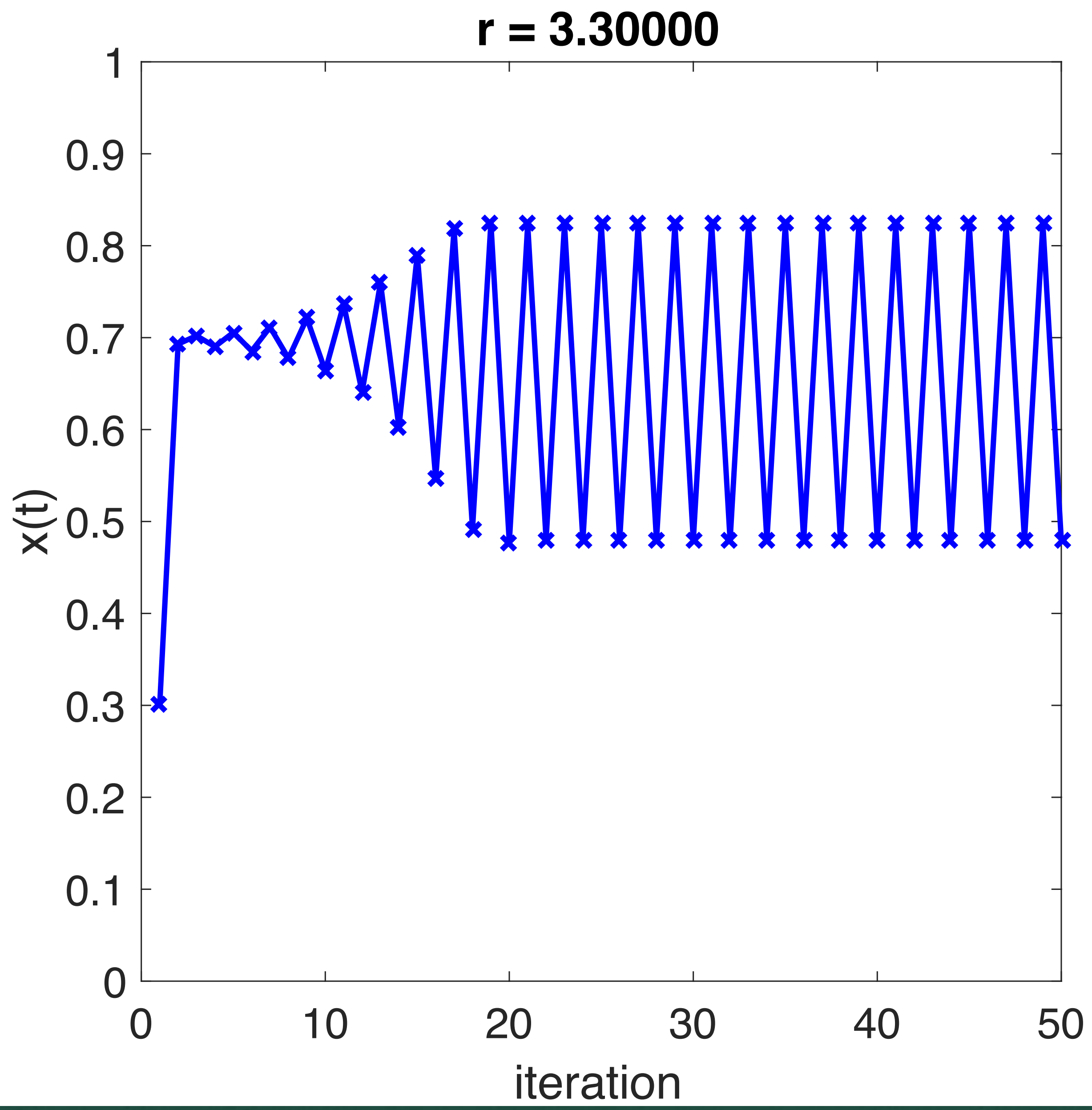
r = 3.20000
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1



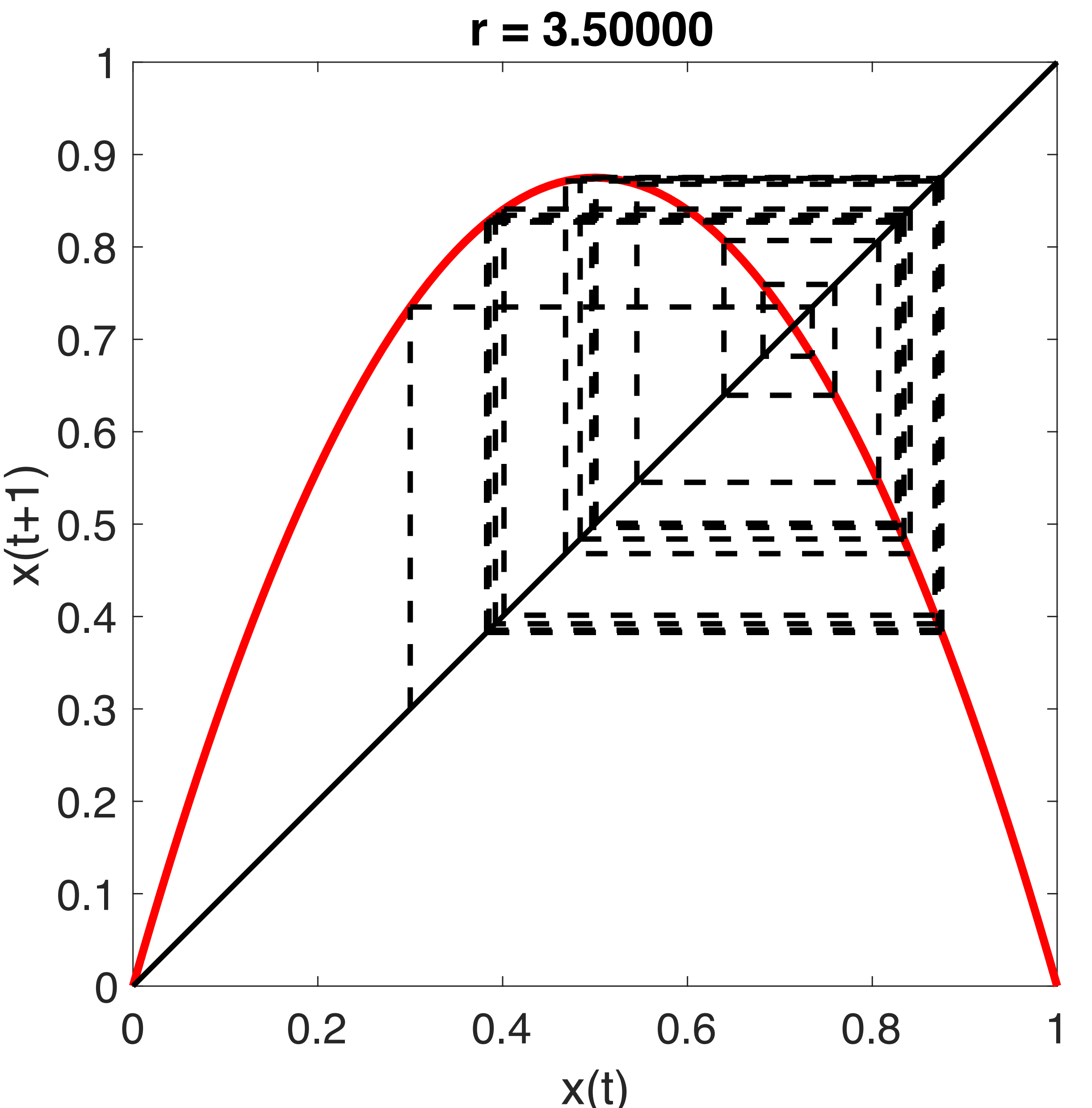
r = 3.20000
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50



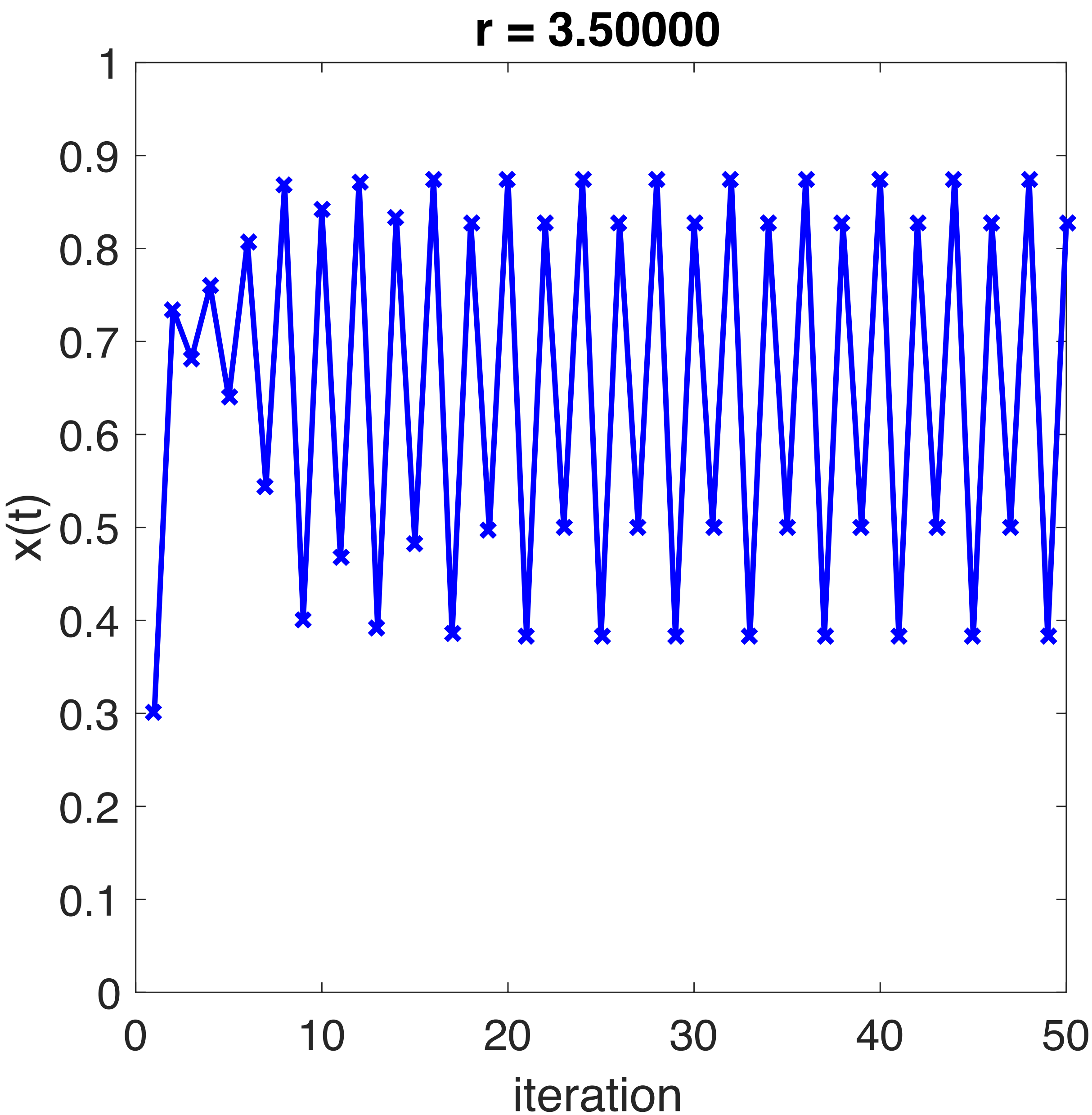
r = 3.30000
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1



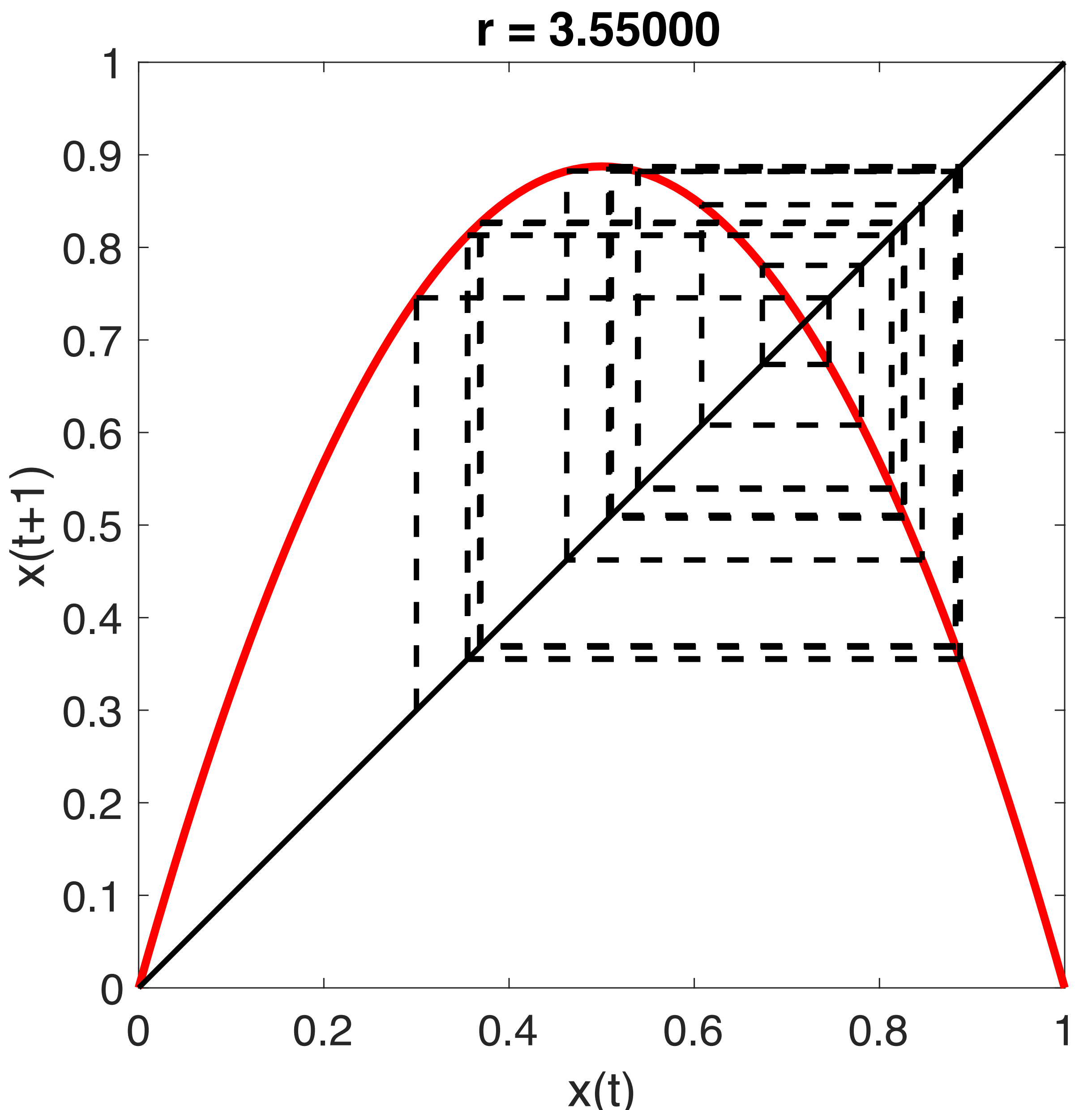
r = 3.30000
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
10
20
30
40
50

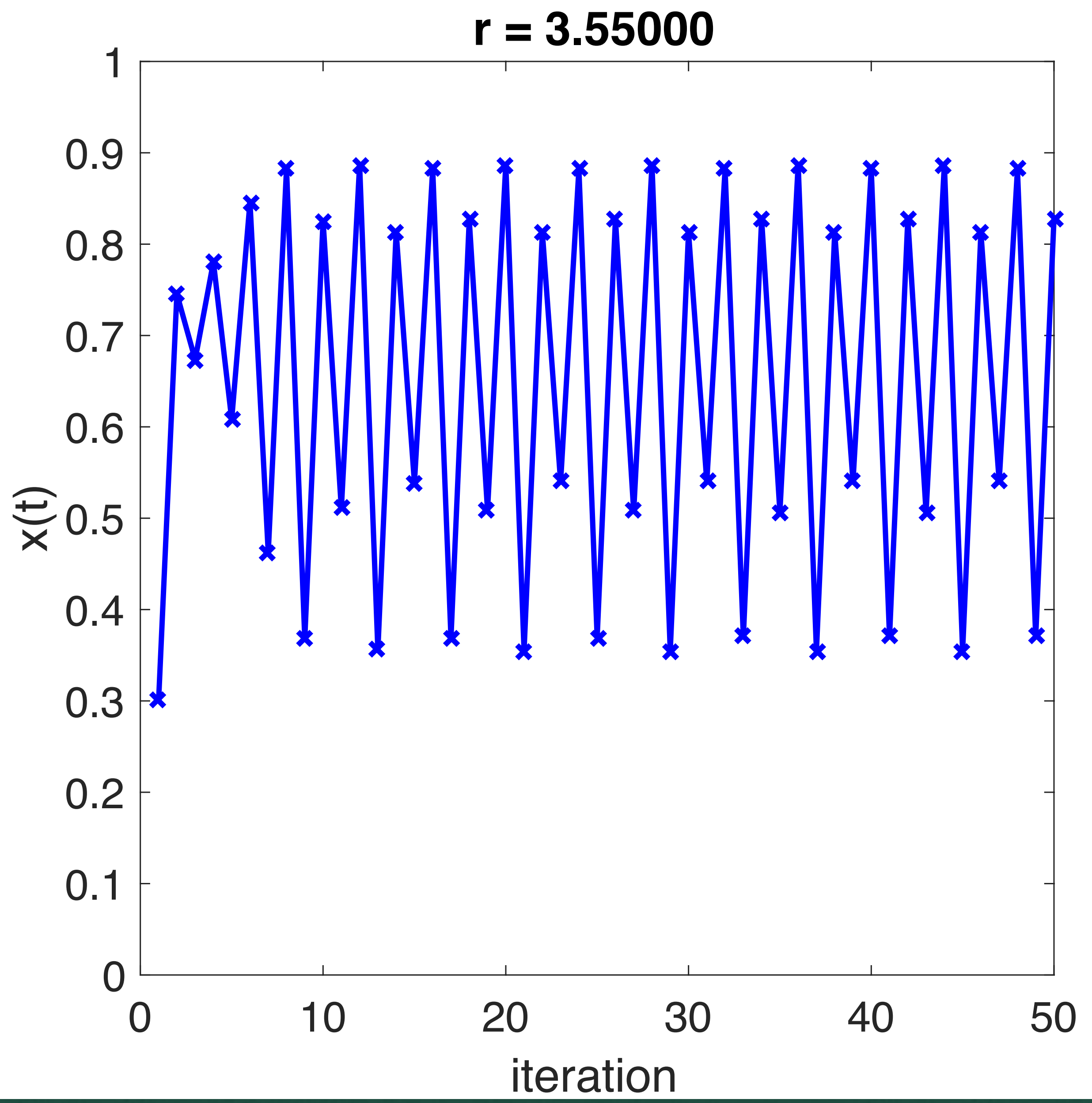


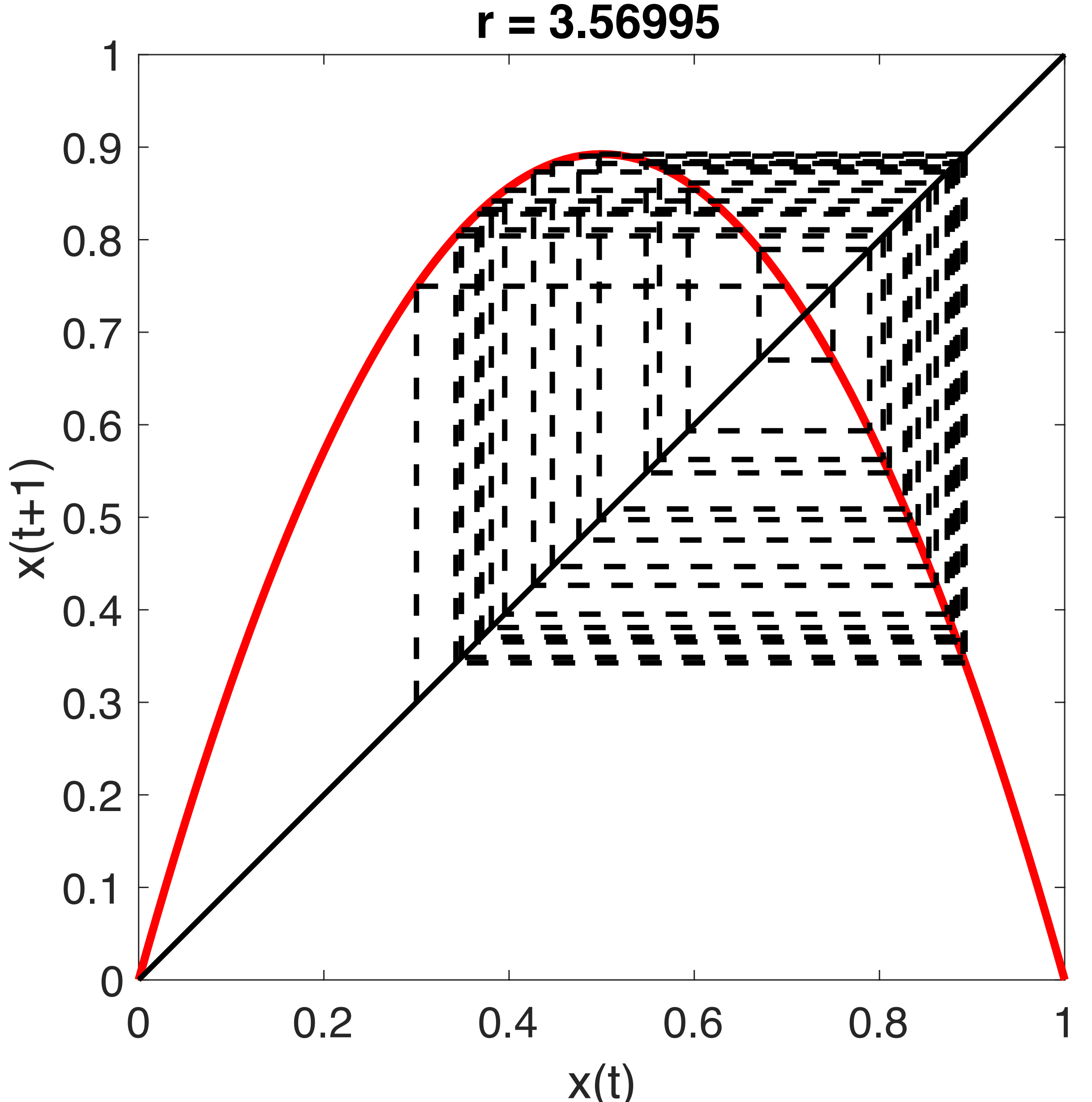
r = 3.50000
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1



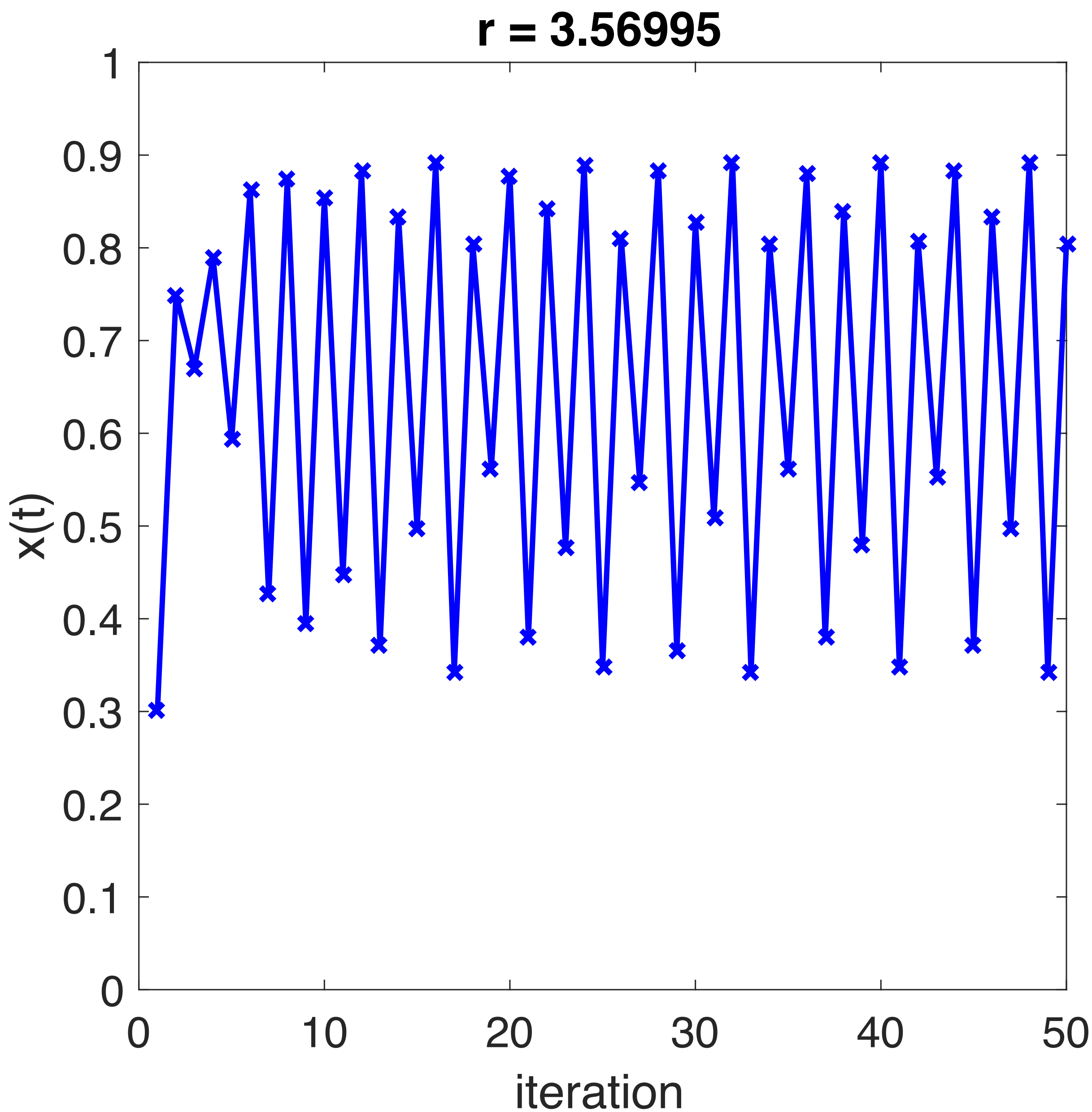
r = 3.50000
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
x(t)
0
10
20
30
40
50
iteration


r = 3.55000
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

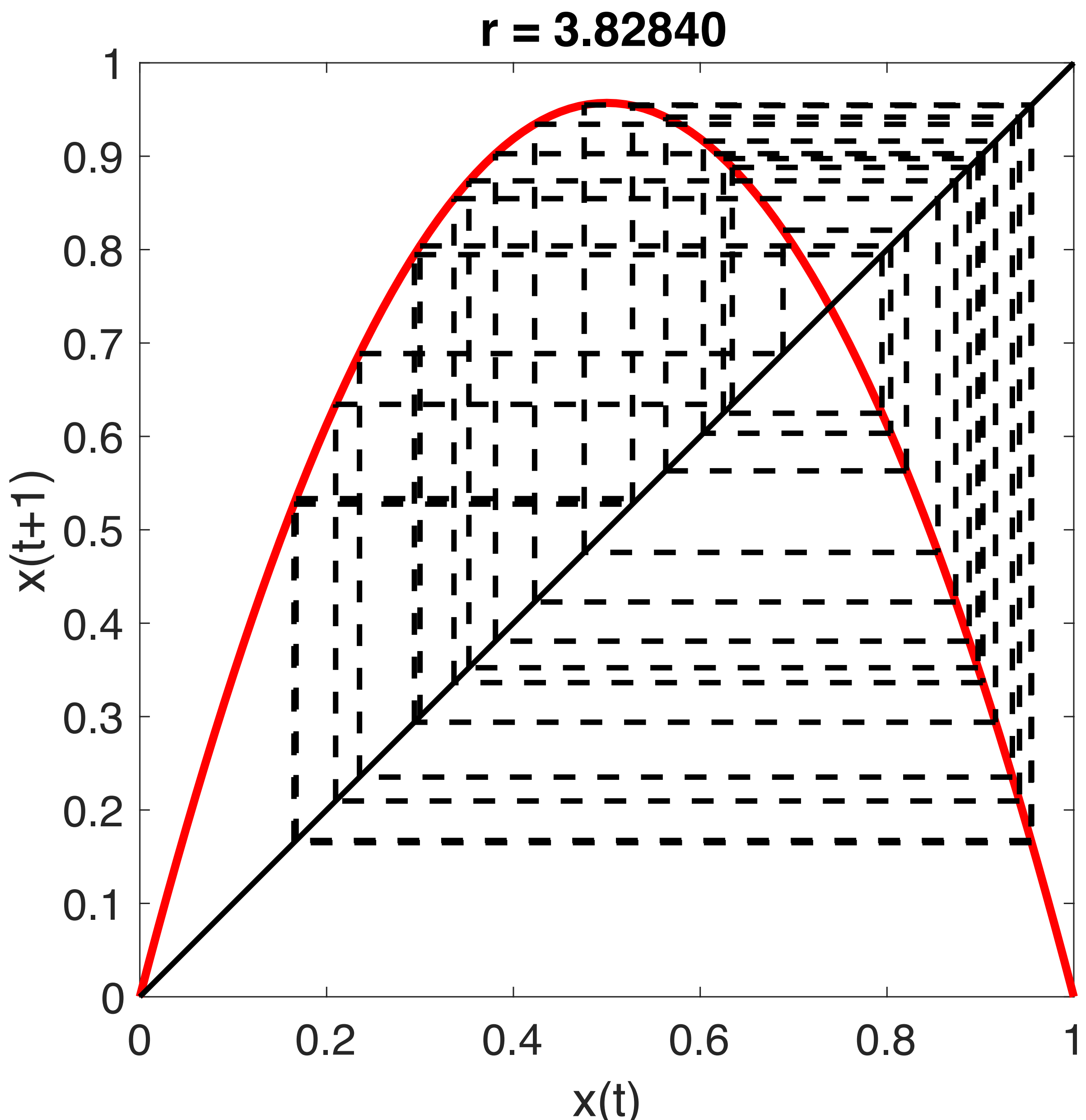

r = 3.55000
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50



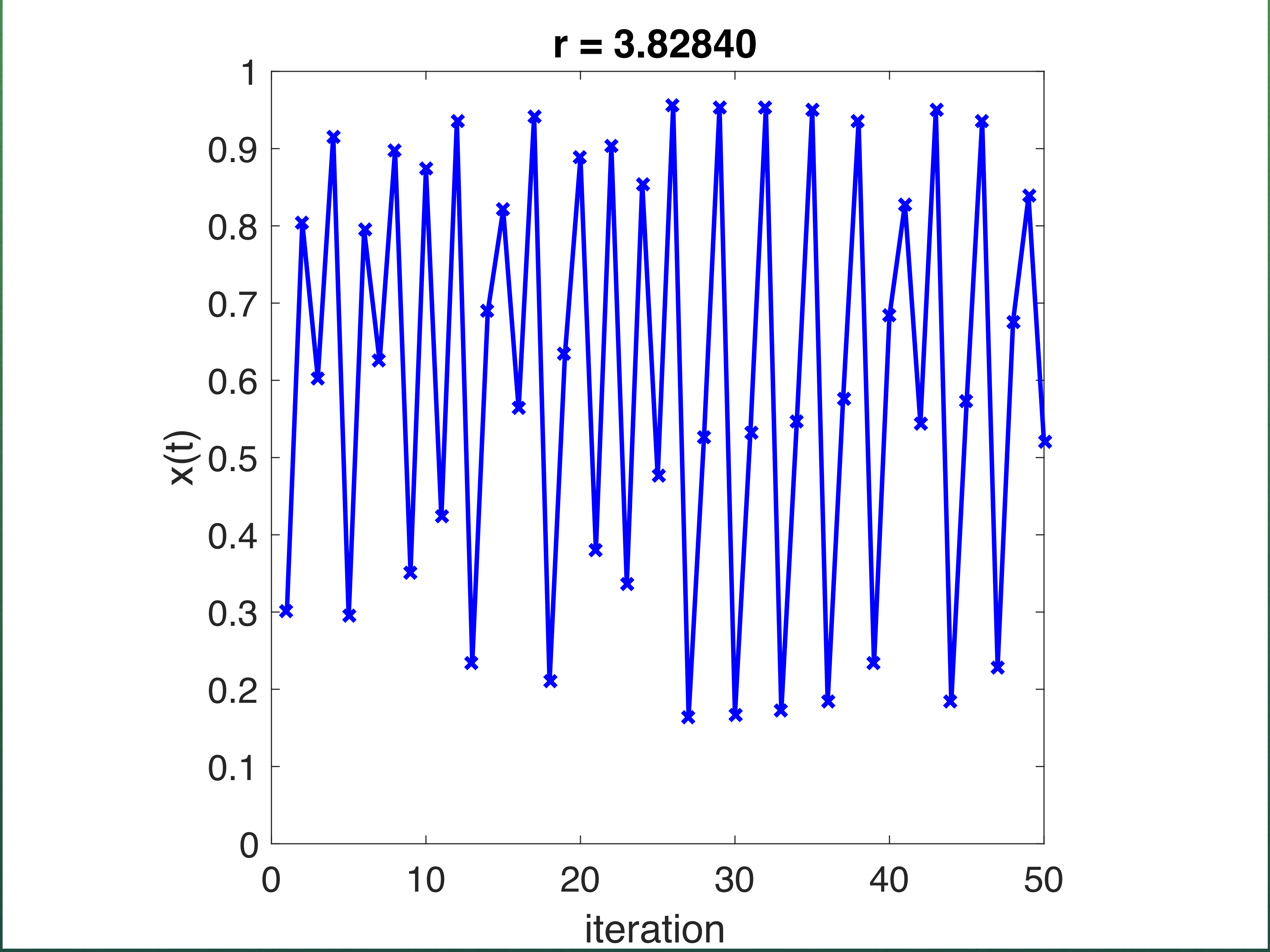
r = 3.56995
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

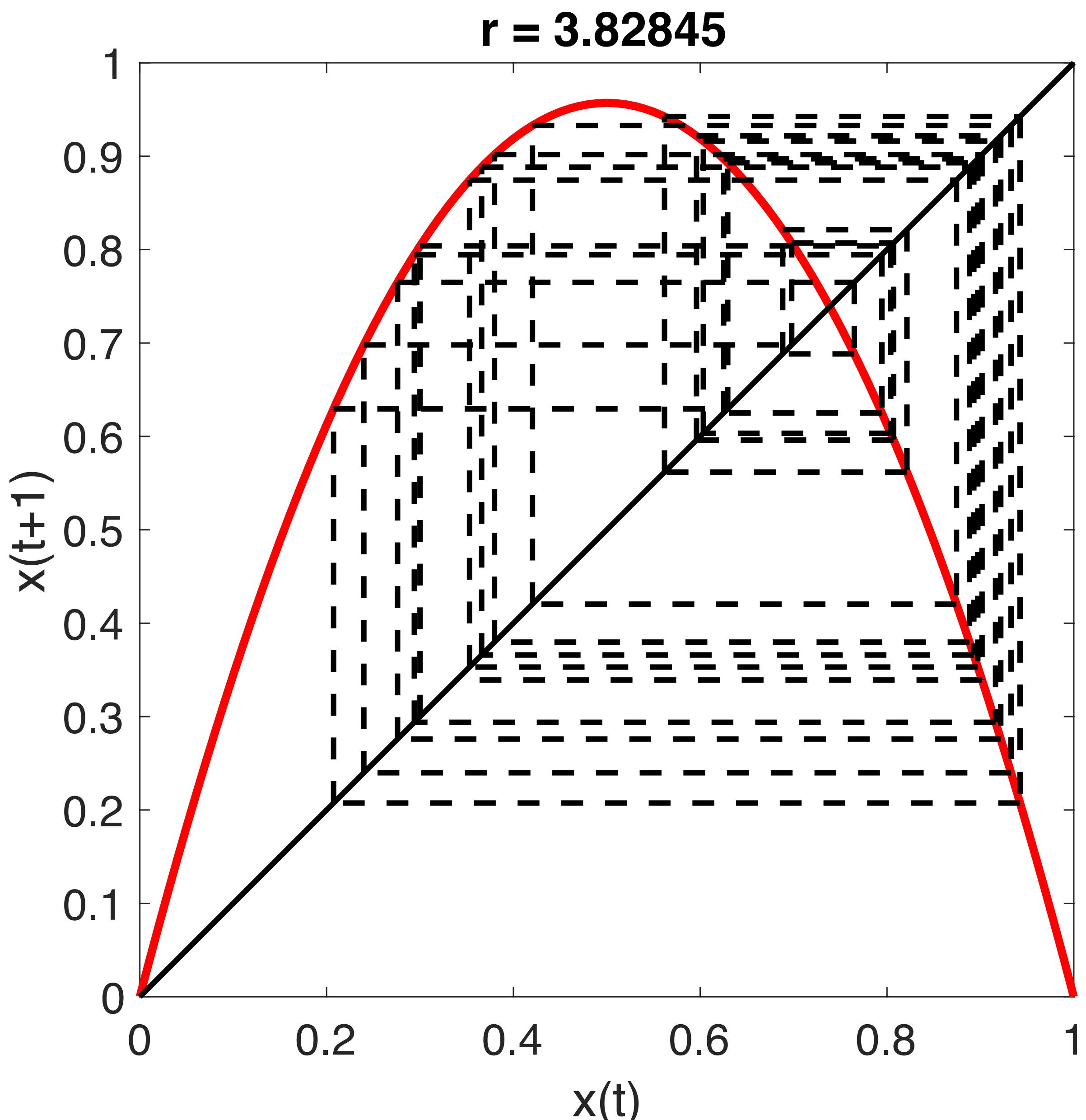


r = 3.56995
x(t)
iteration
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
10
20
30
40
50

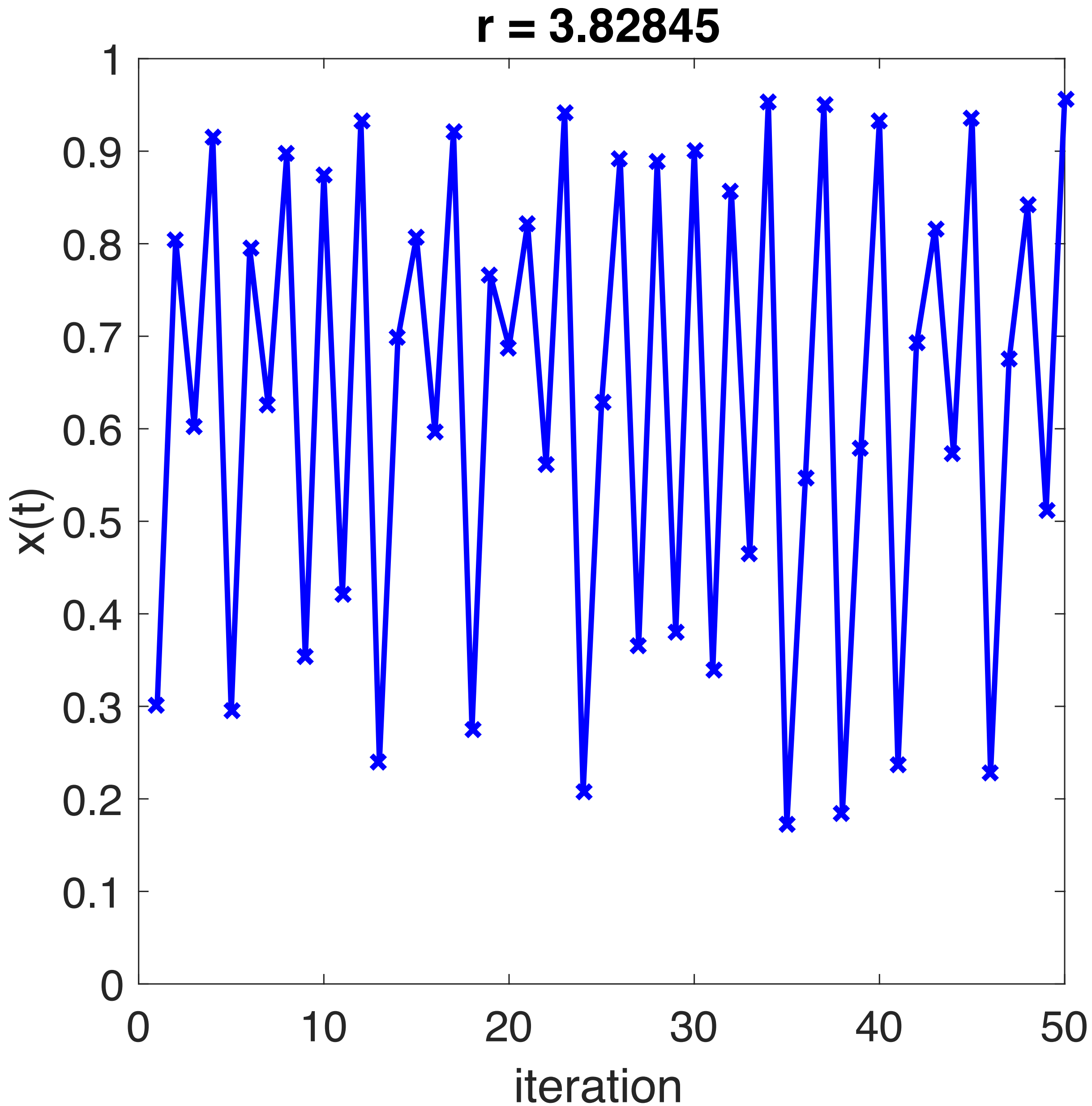


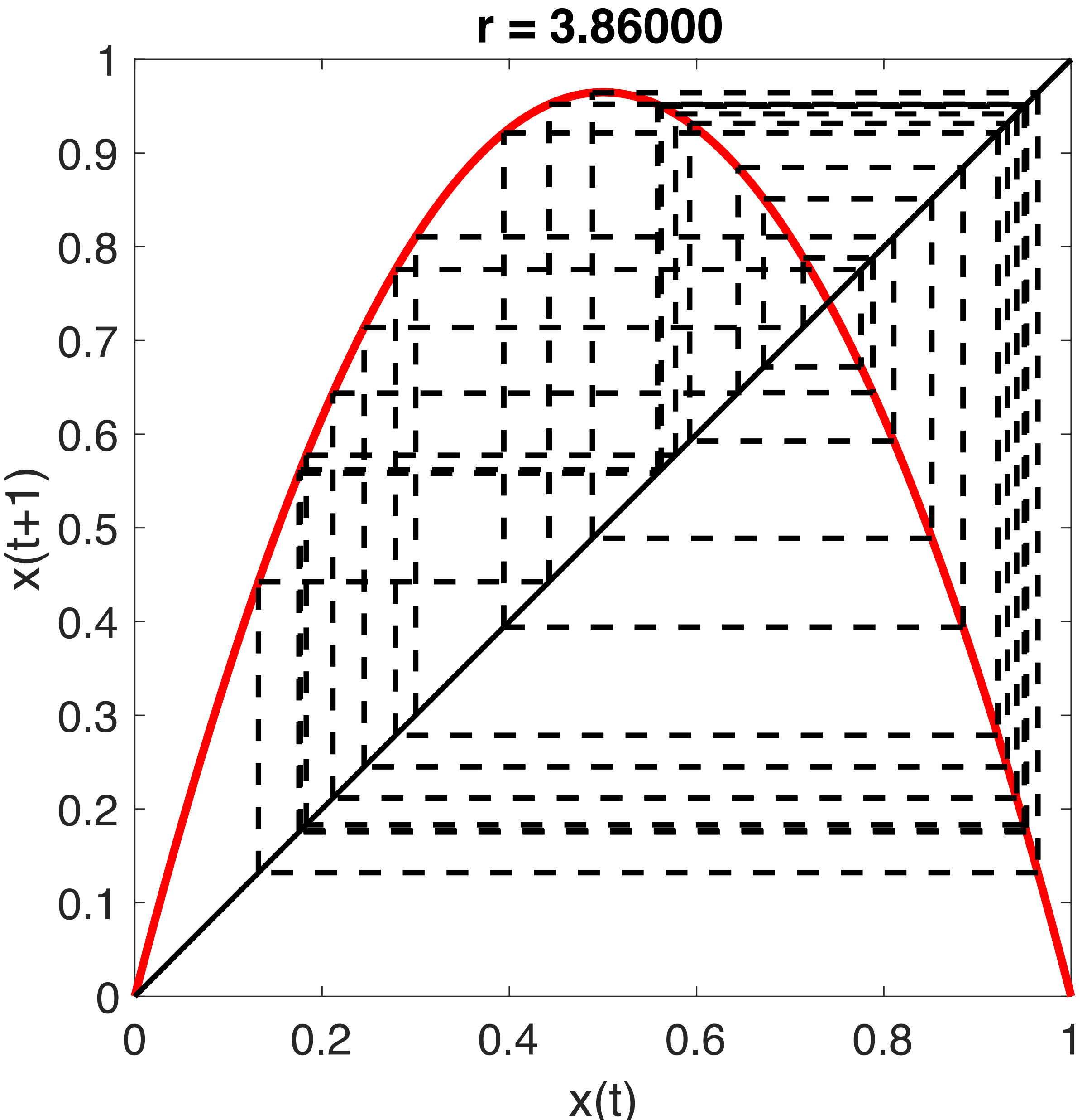
r = 3.82840
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

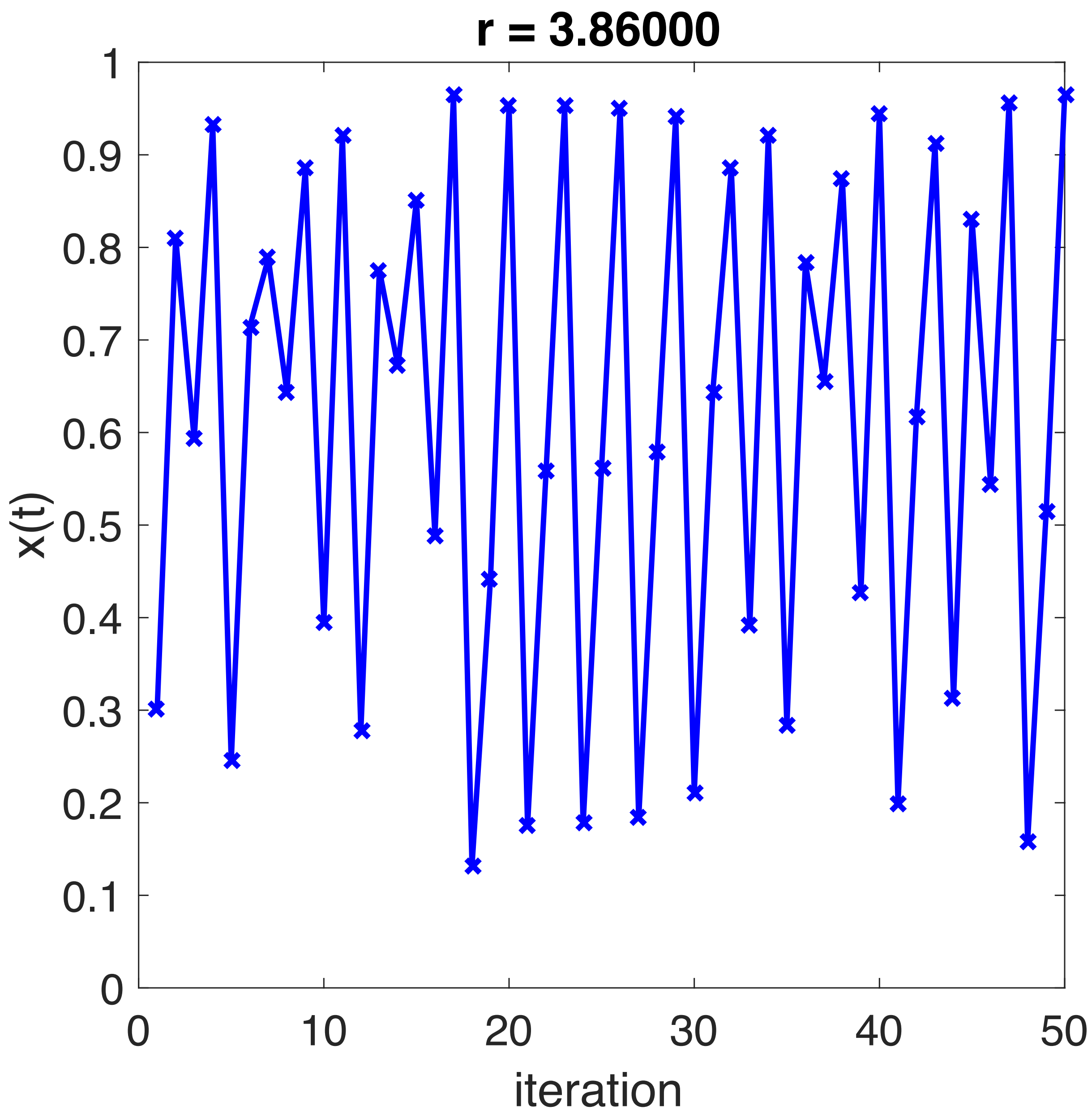

r = 3.82840
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50



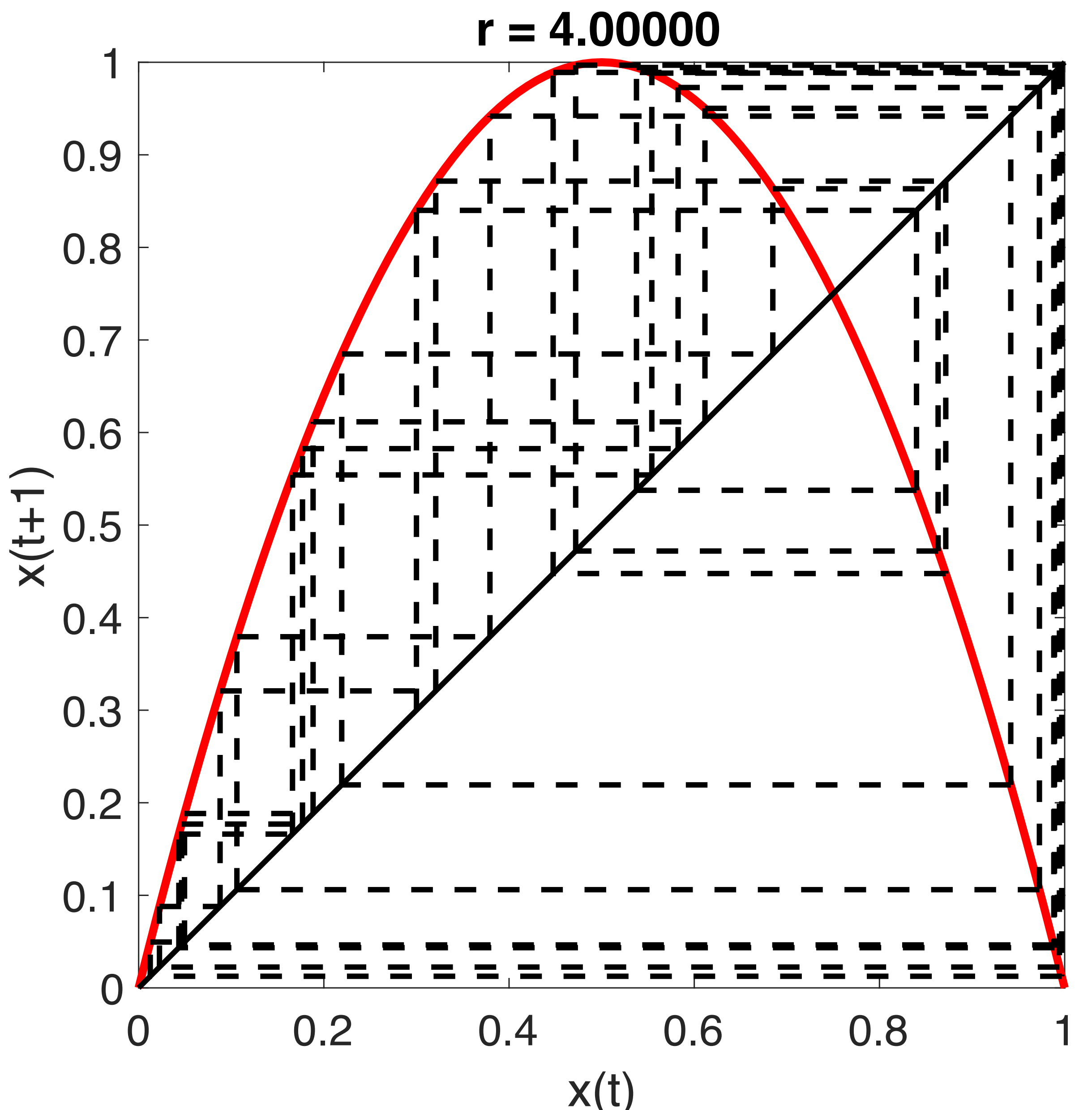
r = 3.82845
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1

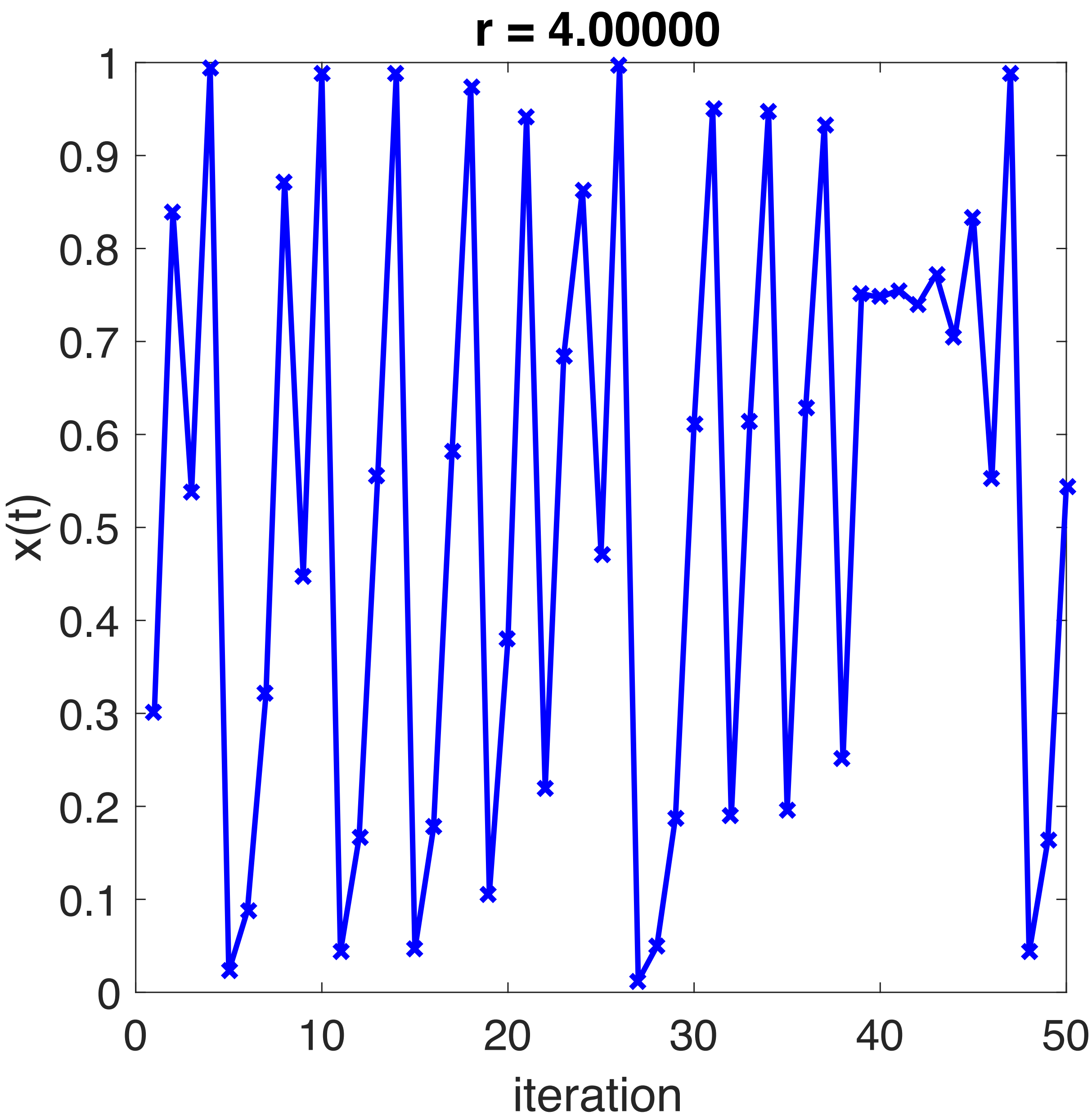

r = 3.82845
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50


r = 3.86000
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1

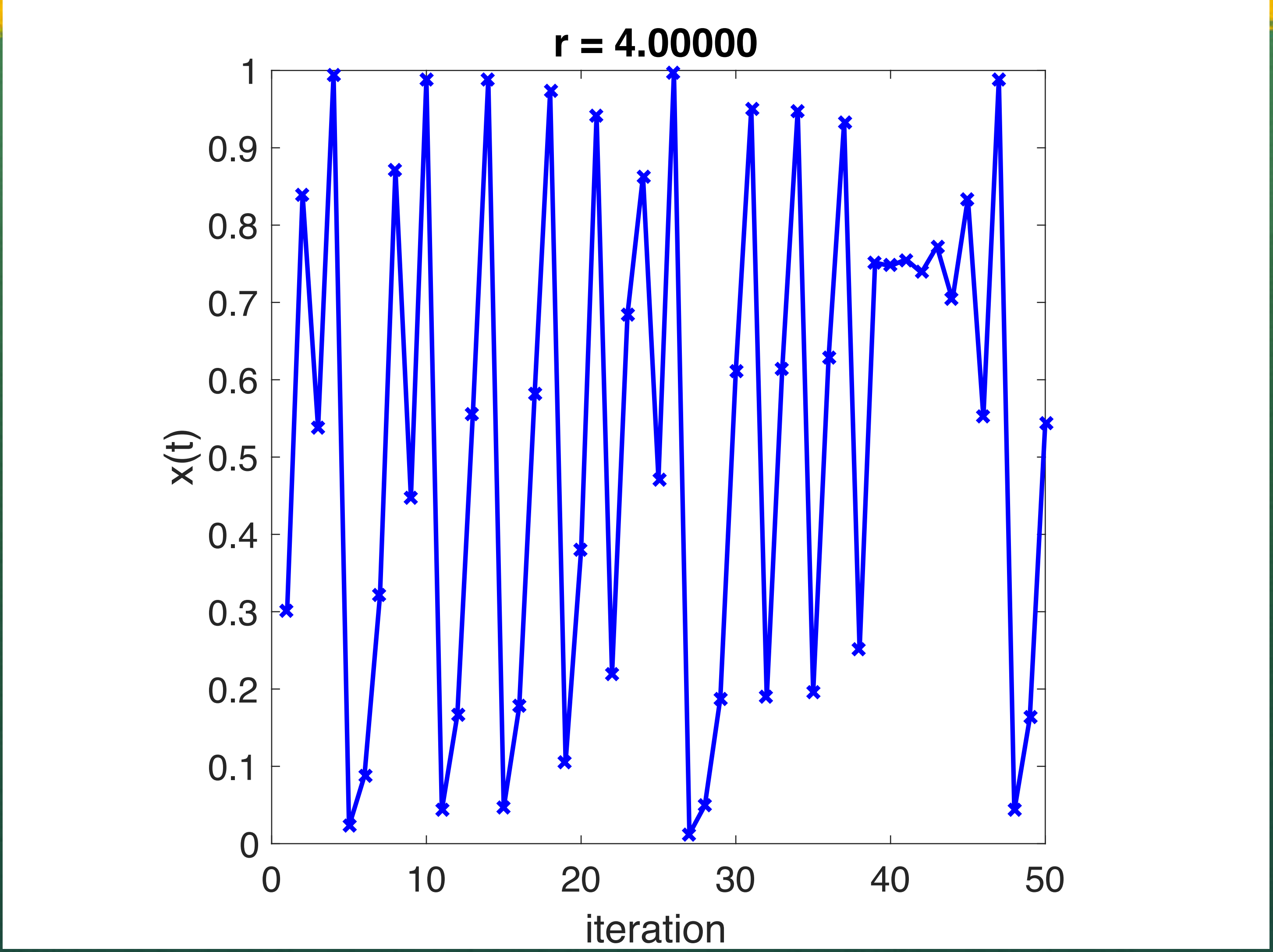


r = 3.86000
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50

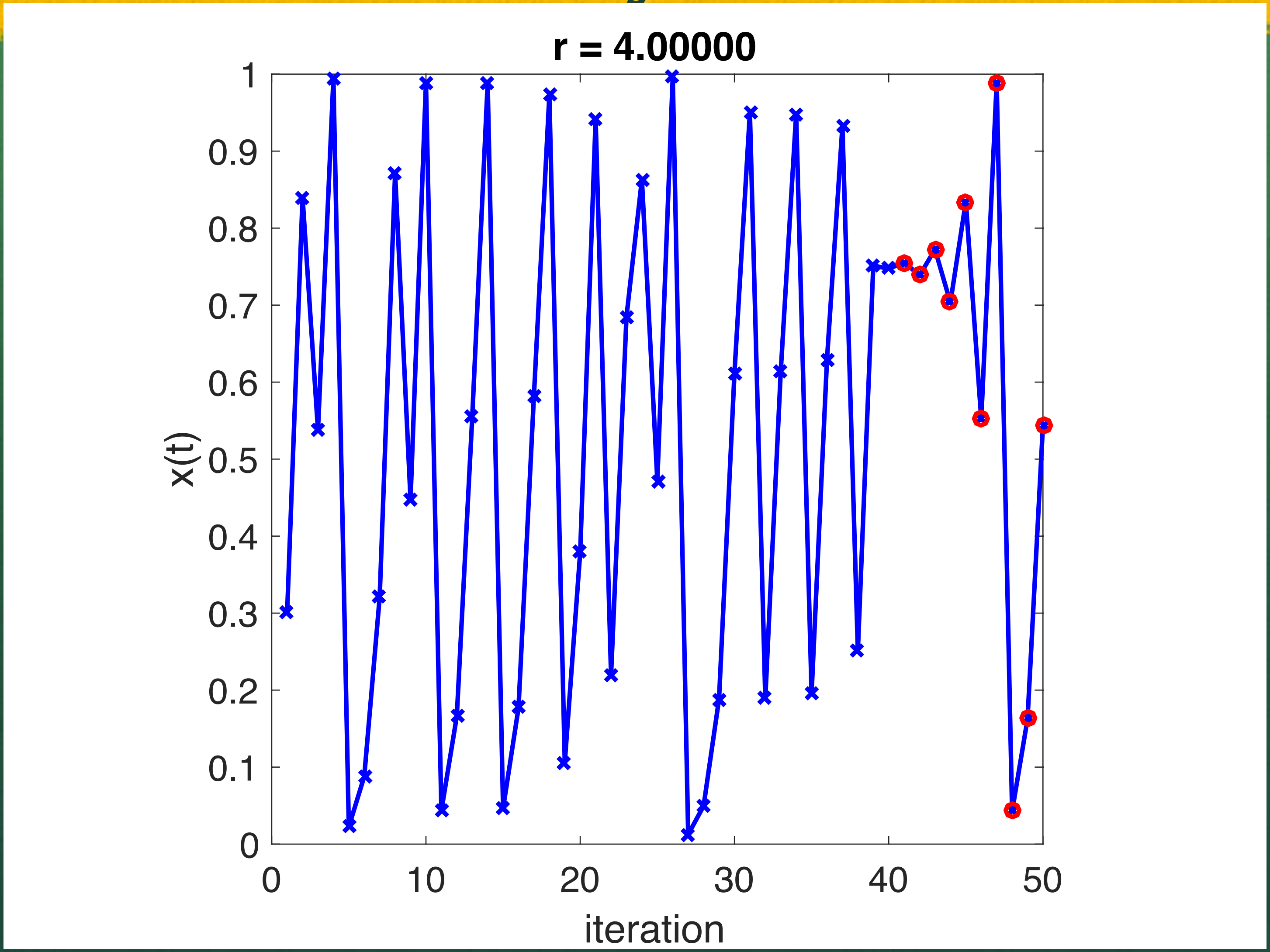

r = 4.00000
x(t+1)
x(t)
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
0.2
0.4
0.6
0.8
1


r = 4.00000
x(t)
iteration
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1
0
10
20
30
40
50

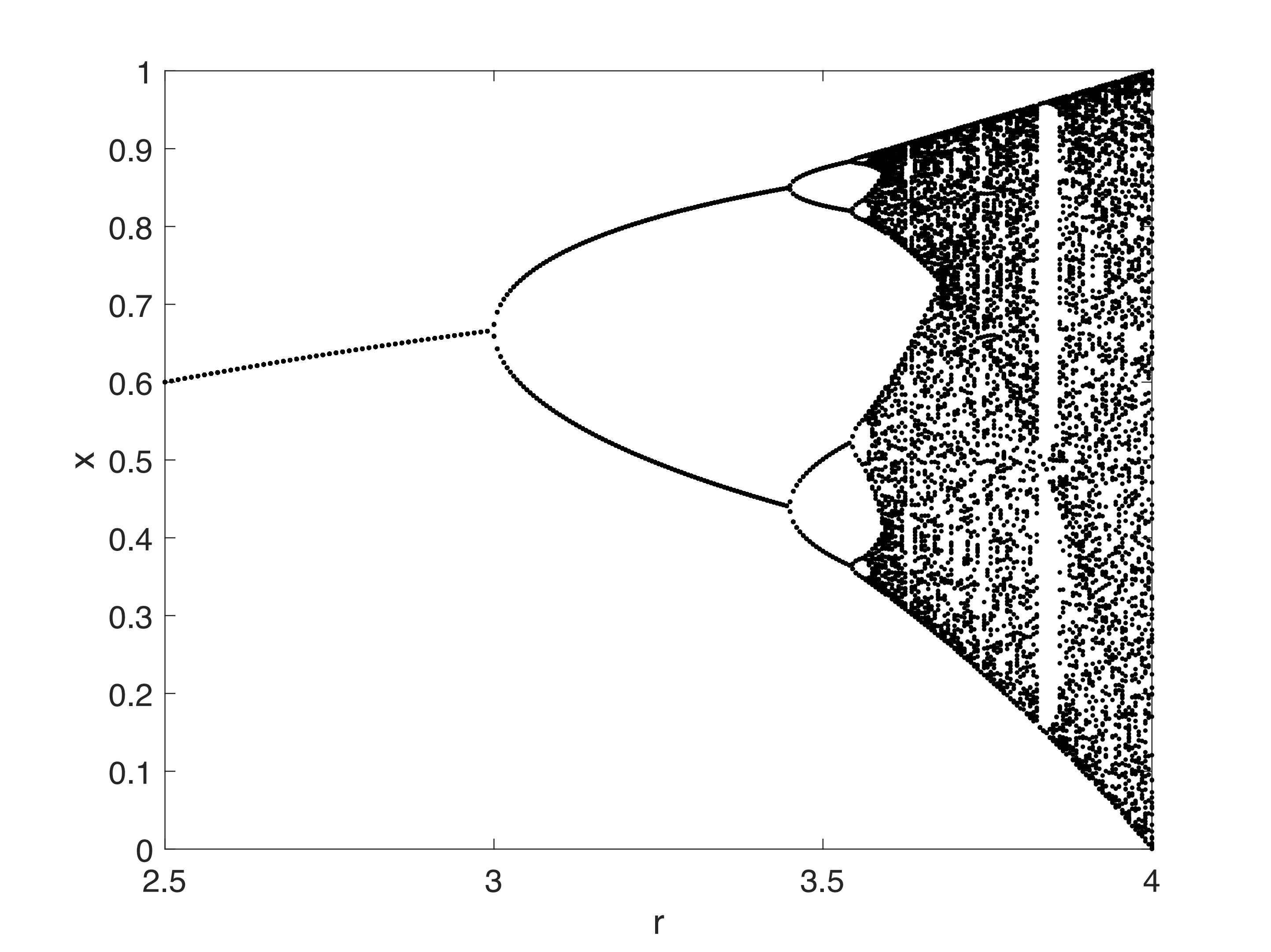
Can we summarize this?

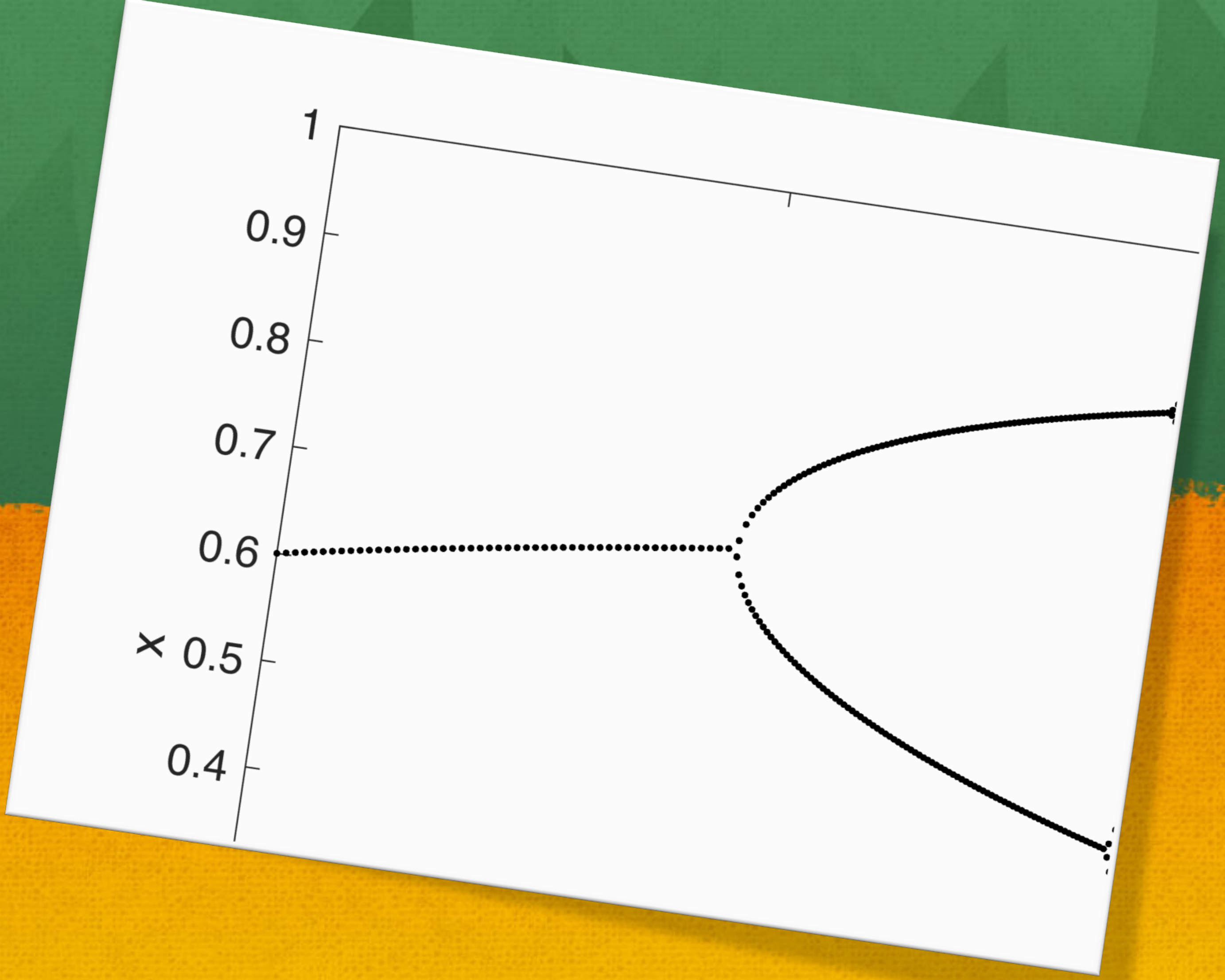


Look at points in the trajectory after many iterations



How dynamics of x change with r



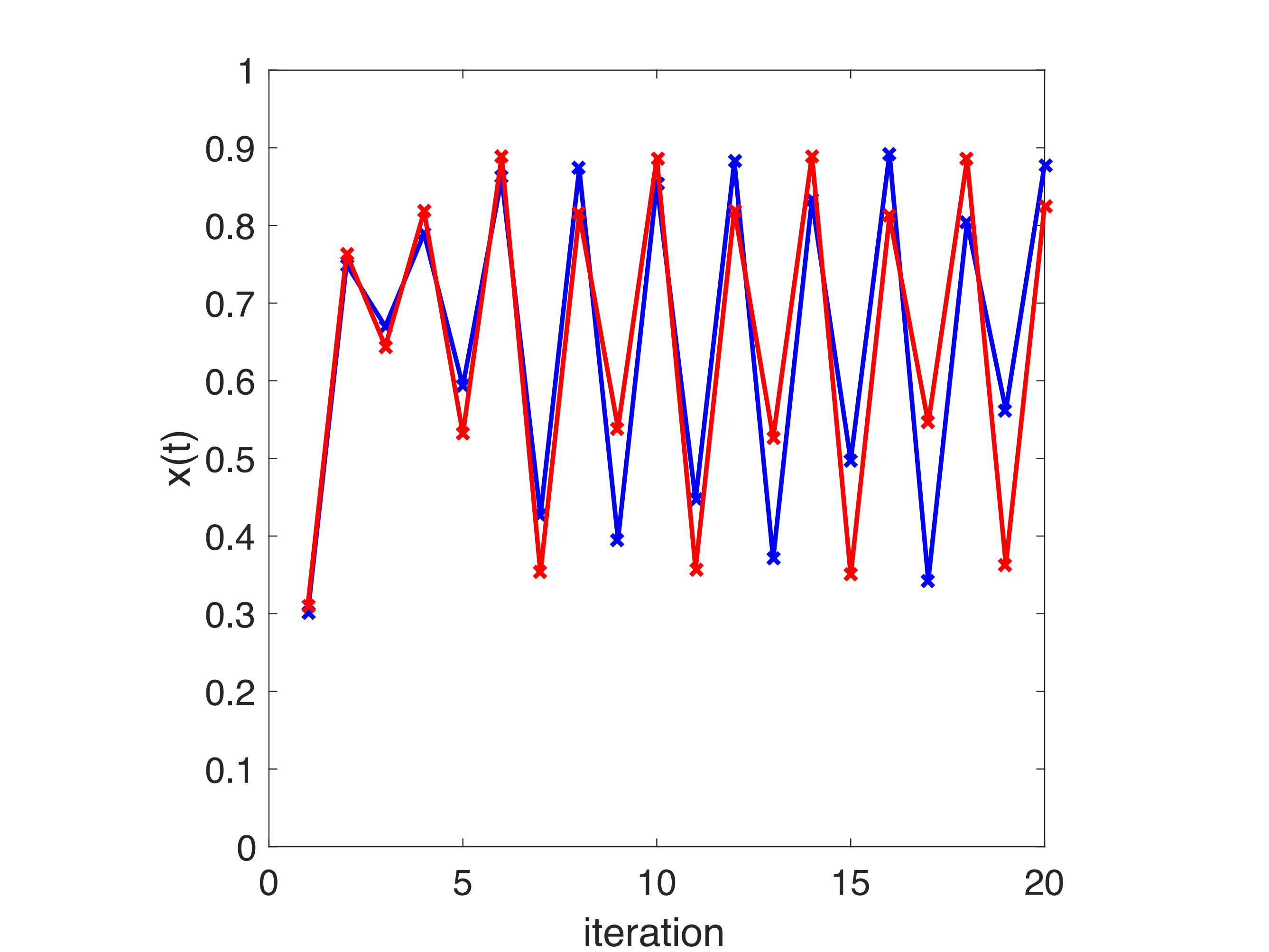


[Movie of bifurcation diagram]

Regions of chaos

- No fixed points
- No accumulating points
- Trajectory never repeats
- Sensitive dependence on starting point

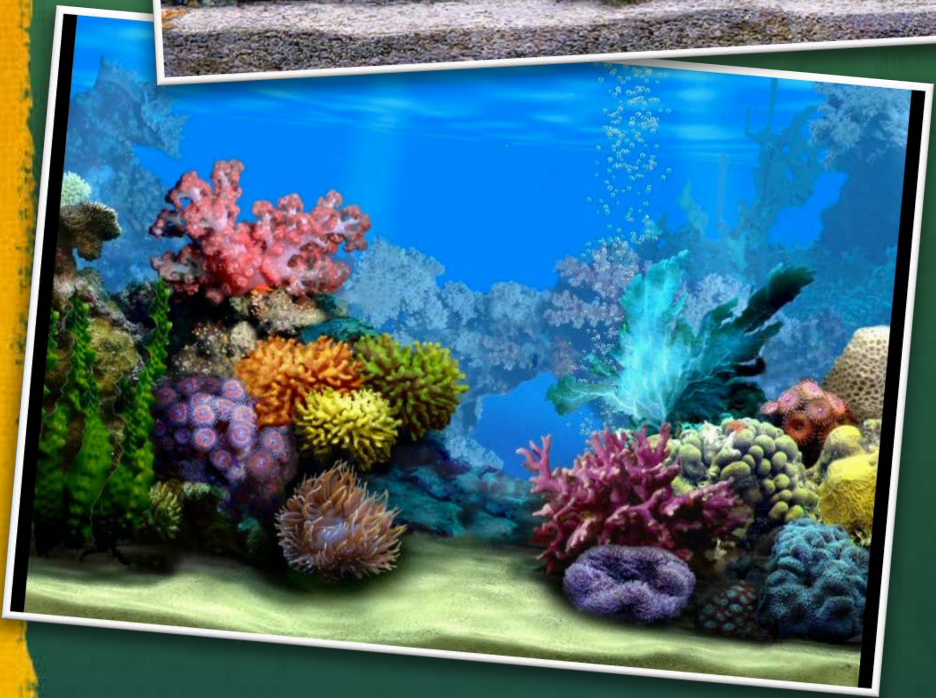
Sensitive dependence on starting point



Chaos can appear even in simple models!

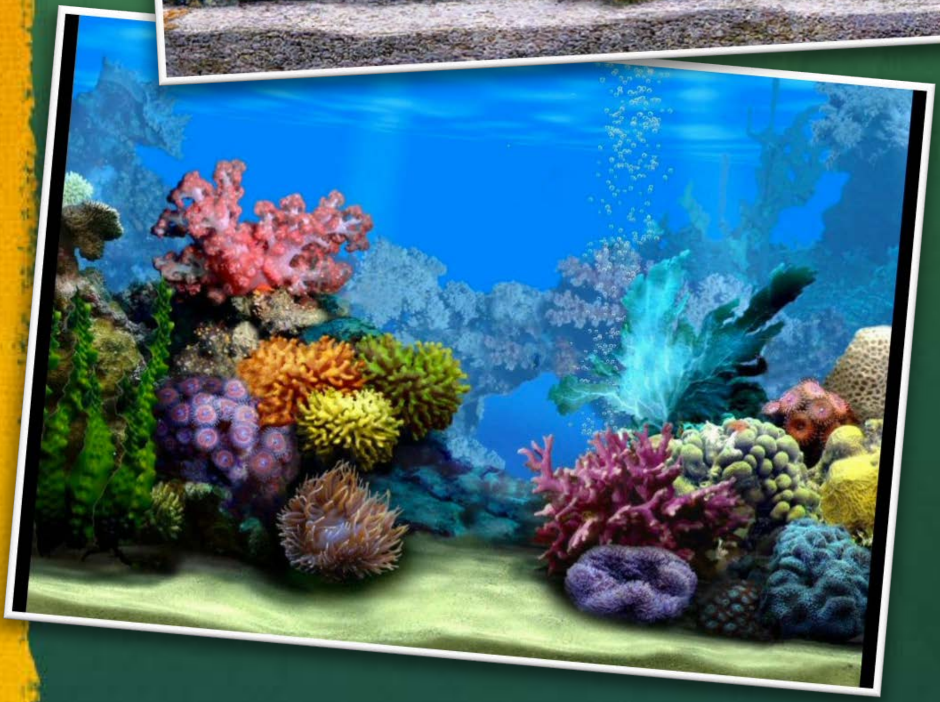
Does it matter how many fish start in the fish tank?

No!



Does it matter how much you feed the fish?

Yes!



THANK YOU!