Math Circle 10/29/2016 Meeting: Challenge of the Week

Make sure that your solution is correct, complete, and clearly written. You should not expect much credit if your proof refers to a false statement, or even if all your statements are true but you forgot to tell us"why?" It is one of the purposes of the Circle to help you improve your "essay-proof" writing style as well as your logical skills.

Please remember that the Challenge is individual. Although we strongly encourage co-operation and help among the participants of the Circle, the Weekly Challenge will be one exception to this rule: you may consult your notes, but you may not ask other people to help you.

Problem 4.

Find the last two digits of $7^{7^{7^{7}}}$. (Recall that $7^{7^7} = 7^{(7^7)} = 7^{823543}$ and $7^{7^7} \neq 823543^7$.)

Finding the last two digits of $7^{7^{7^{7}}}$ is equivalent to finding the remainder of $7^{7^{7^{7}}}$ when divided by 100, so we want to take the modulus to be $d = 100$. Notice the following pattern: $7^1 \equiv 7 \pmod{100}$, $7^2 \equiv 49 \pmod{100}$, $7^3 \equiv 43 \pmod{100}$, $7^4 \equiv 1 \pmod{100}$, and $7^5 \equiv 7 \pmod{100}$. We have the period of 4, and

$$7^n \equiv \begin{cases} 1 \pmod{100} & \text{if } n \equiv 0 \pmod{4} \\ 7 \pmod{100} & \text{if } n \equiv 1 \pmod{4} \\ 49 \pmod{100} & \text{if } n \equiv 2 \pmod{4} \\ 43 \pmod{100} & \text{if } n \equiv 3 \pmod{4} \end{cases}.$$

For example, $7^{321} \equiv 7^1 \equiv 7 \pmod{100}$ because $321 \equiv 1 \pmod{4}$.

First, we see that $7^7 \equiv 7^3 \equiv 43 \pmod{100}$. In other words, $7^7 - 43$ is divisible by 100, so we have $7^7 = 100n_1 + 43$ for some integer n_1. Now, because $7^7 \equiv 100n_1 + 43 \equiv 43 \equiv 3 \pmod{4}$, we see that $7^{7^7} \equiv 7^3 \equiv 43 \pmod{100}$, and hence $7^{7^7} = 100n_2 + 43$ for some integer n_2 (which is much bigger than n_1). This pattern repeats, and it is clear that we get $7^{7^{7^{7}}} \equiv 7^3 \equiv 43 \pmod{100}$ and $7^{7^{7^{7^{7}}}} \equiv 7^3 \equiv 43 \pmod{100}$ as well. In fact, the last two digits are always 43 no matter how many 7's we have.