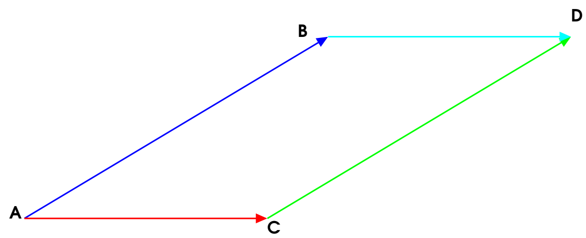
BLACKSBURG MATH CIRCLE

Lesson I: Vector calculus and its applications

To represent a vector, we draw an arrow connecting two points in space (as in figure below), and write it as $\overrightarrow{AB}$. A is the **tail** and B is the **head** of this vector. Two vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are called **equal** if they are equal in both magnitude and direction, meaning that one can be obtained from another by translation. Note that $\overrightarrow{AB} = \overrightarrow{CD}$ exactly when the quadrilateral $ABDC$ is a parallelogram. We will denote by $|\overrightarrow{AB}| = AB$ the **magnitude** of vector $\overrightarrow{AB}$.



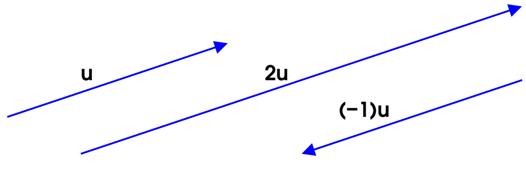
Addition of vectors. Given two vectors, $\vec{u}$ and $\vec{v}$, their sum $\vec{u}+\vec{v}$ is defined as follows. Pick any $\overrightarrow{AB} = \vec{u}$ and choose $\overrightarrow{BC} = \vec{v}$. Then $\vec{u}+\vec{v} = \overrightarrow{AC}$ (tail to head rule).

$$\textit{Commutativity}: \ \vec{u}+\vec{v} = \vec{v}+\vec{u} \text{ for all vectors } \vec{u},\ \vec{v}.$$

$$\textit{Associativity}: \ (\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w}) \text{ for all vectors } \vec{u},\ \vec{v},\vec{w}.$$

Problem 0.1. *Prove the relations above.*

Multiplication of vectors by scalars. Given a scalar (i.e. a real number) α and a vector $\vec{v}$, one can form a new vector denoted $\alpha\vec{v}$ and called the **product** of the scalar and the vector. Let $\vec{v} = \overrightarrow{AB}$, then we fix the head A, and send the head B to another point C such that the segment AC is on the same line as AB, but is scaled by a factor α. If $\alpha > 0$ than points B, C are chosen to lie on the same side from the point A (the vectors $\alpha\vec{v}$ and $\vec{v}$ point in the same direction). If $\alpha < 0$ than A lies between B and C (i.e. the vector $\alpha\vec{v}$ reverse direction if $\alpha < 0$). When $\alpha = 0$ we get the **zero vector** $\vec{0}$ (meaning that points A and C coincide).



Distributivity and **compatibility**:

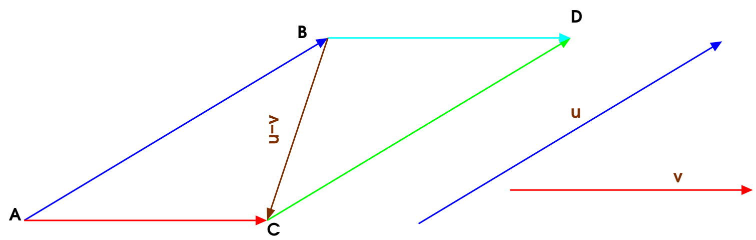
$$\alpha(\vec{u}+\vec{v}) = \alpha\vec{v}+\alpha\vec{u};\ (\alpha+\beta)\vec{u} = \alpha\vec{u}+\beta\vec{u};\ (\alpha\beta)\vec{u} = \alpha(\beta\vec{u})$$

for all vectors $\vec{u},\ \vec{v}$ and all scalars α, β.

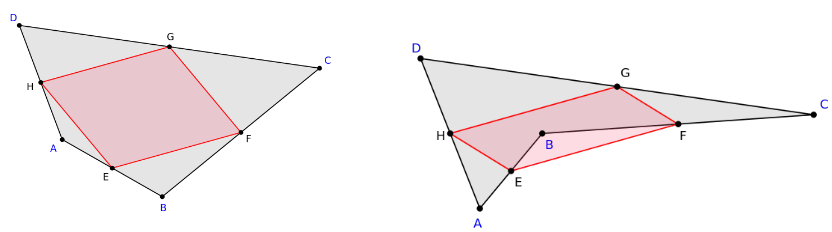
Problem 0.2. *Check the relations above.*

The vector $(-1)\vec{u}$ is denoted simply by $-\vec{u}$. If $\vec{u} = \overrightarrow{AB}$, then by distributive property $-\vec{u} = \overrightarrow{BA}$ (the opposite vector).

If $\vec{u} = \overrightarrow{AB}$ and $\vec{v} = \overrightarrow{AC}$ (common tail), then the **difference** of vectors $\vec{v}$ and $\vec{u}$ is $\vec{v} - \vec{u} = \overrightarrow{BC}$ (because $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$).

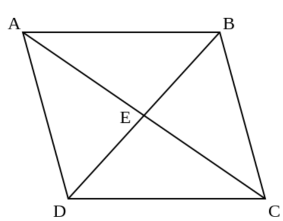


Problem 0.3. *Prove the Varignon's theorem (1731): The midpoints of the sides of an arbitrary quadrangle form a parallelogram.*

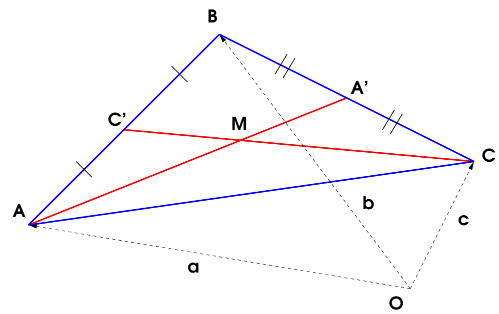


Problem 0.4. *Let A, B, C, D be points on a plane such that AC and BD are non-collinear segments. Suppose that numbers α, β are such that the vectors $\vec{u} = \alpha\overrightarrow{AC}$ and $\vec{v} = \beta\overrightarrow{BD}$ satisfy $\vec{u} + \vec{v} = 0$. Prove that $\alpha = \beta = 0$.*

Problem 0.5. *Prove that the diagonals of a parallelogram bisect one another.*



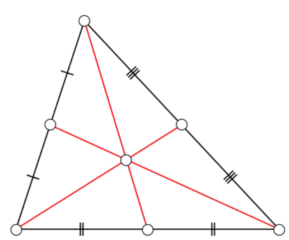
Problem 0.6. *Prove that two medians of a triangle ABC meet at the point M dividing each of them in the proportion $2:1$ counting from the vertex.*



In applications, we will often represent all vectors by segments with a common tail, called the **origin**. Once the origin O is chosen, each point A in space becomes represented by a unique vector $\overrightarrow{OA}$, called the **radius-vector** or **position vector** with respect to the origin O.

Problem 0.7. *Compute the radius-vector of the point M in the previous problem, given the radius-vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ of its vertices.*

Corollary 0.1. *The medians of a triangle meet at a single point (are concurrent) which divides each median in the $2:1$ ratio, counting from the vertex. This point is called the* ***barycenter*** *(or* ***centroid****) of the triangle.*



Appendix: Mathematical Propositions

Definitions. Definitions are propositions which explain what meaning one attributes to a name or expression.

Axioms. Axioms are the fact which are accepted without proof.

Theorems. Theorems are those propositions whose truth is only found through a certain reasoning process (proof). In any theorem one can distinguish two parts: the hypothesis and the conclusion. The **hypothesis** expresses what is considered given, the **conclusion** what is required to prove.

Corollaries. Corollaries are those propositions which follow directly from an axiom or a theorem.