# THE MATH BATTLE OF BLACKSBURG MATH CIRCLE: SATURDAY, APRIL 9, 2016

## Math battle problems

**Problem 1.** (`Easy`) Find a solution of the equation $pi + is + sp = sip$, where each letter represent a different digit between 1 and 9 in the corresponding 2- and 3-digit numbers.

*Answer:* $s = 1, i = 9, p = 8$.

**Problem 2.** (`Medium`) You work in a pharmacy, and received a package containing 10 identical bottles of pills. A day later the pharmaceutical company that sent the package informs you that in one of the bottles sent the pills are 10 mg heavier than in the rest of the bottles. What is the minimal number of weightings you can do in order to determine which bottle contains the heavier pills, using only a scale?

*Answer:* Assuming we cannot open the bottles there are 4 weightings necessary (divide in half first, then divide $2 + 3$; in the worst case scenario you need two more). If you can open the bottles, then only one weighting is necessary (you put one pill from the first bottle, two from the second etc, then weigh them).

**Problem 3.** (`Medium`) Given 12 integers, prove that 2 of these integers can be selected such that their difference is divisible by 11.

*Solution: Let $\{a_1, \dots, a_{12}\}$ be the set of remainders of these integers $\mod 11$. The original problem is equivalent to showing that at least 2 numbers in this set are equal (why?). But the remainders of division by 11 can take only 11 distinct values $\{0, \dots, 10\}$, so at least 2 of the 12 numbers $\{a_1, \dots, a_{12}\}$ are equal.* $\square$

**Problem 4.** (`More difficult`) For a positive integer $N$, let $\mathcal{S}(N)$ denote the sum of its digits.

(i) Find the largest value of $N$ such that $N + \mathcal{S}(N) = 2015$;

*Answer:* 2011.

(ii) Find the largest value of $N$ such that $N + \mathcal{S}(N) + \mathcal{S}(\mathcal{S}(N)) = 2015$.

*Solution:* $N \mod 3 = \mathcal{S}(N) \mod 3 = \mathcal{S}(\mathcal{S}(N)) \mod 3$, *hence*

$$N + \mathcal{S}(N) + \mathcal{S}(\mathcal{S}(N))$$

*is divisible by* 3*. But* $2015 \bmod 3 = 2$*, so no such* $N$ *exists.* □

**Problem 5.** (`Difficult`) In a triangle $ABC$, $BC = 2AB$. Let $D$ be the midpoint of the side $BC$ and let $K$ be the midpoint of $BD$. Prove that $AC = 2AK$.

*Solution: Let* $E$ *be the midpoint of the side* $AC$*. Then the lines* $\overline{AB}$ *and* $\overline{ED}$ *are parallel and* $2ED = AB$ *(why?). Hence* $\triangle KDE$ *is isosceles with* $ED = \frac{1}{2}AB = \frac{1}{4}BC = KD$*. The triangle* $\triangle ABD$ *is also isosceles (*$AB = \frac{1}{2}BC = BD$*), so* $\angle BAD = \angle BDA$*. Since* $\overline{AB} \parallel \overline{ED}$*, by the alternate interior angles property we conclude that* $\angle BDA = \angle BAD = \angle ADE$*. Thus if we denote by* $O$ *the intersection of segments* $AD$ *and* $KE$*, the segment* $DO$ *is the bisector of the isosceles triangle* $\triangle KDE$*. That means that it is also its altitude and the median. So* $AO$ *is the altitude and the median in* $\triangle AKE$*, which implies that* $\triangle AKE$ *is also isosceles (why?). So* $AK = AE = \frac{1}{2}AC$*.* □