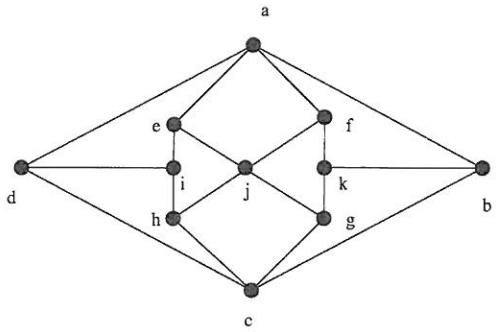
PROBLEMS ON GRAPHS, PART II

Recall:

Definition 0.1. *A diagram which consists of points (called* **vertices***) and some lines joining these points (called* **edges***) is called a* **graph**.

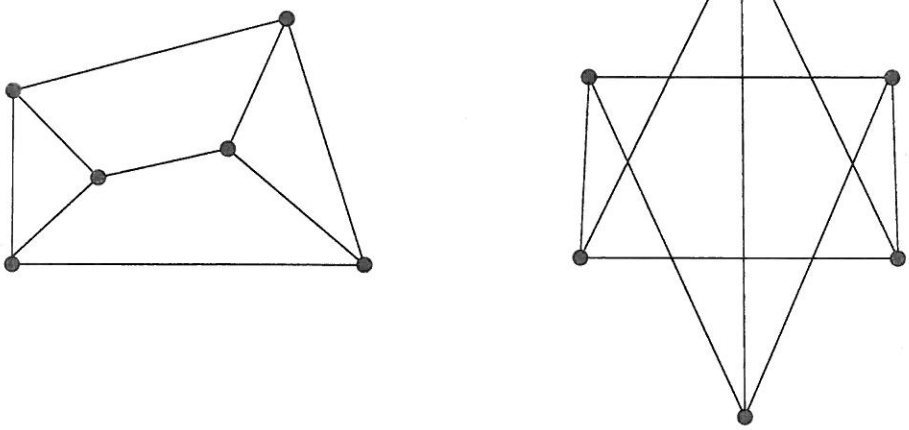
An example of a graph is in the figure below.



Definition 0.2. *Two graphs are* **isomorphic** *if one can redraw one of them and get the other.*

Note that isomorphic graphs have the same properties: same number of vertices, edges, etc.

Problem 1. Consider the graphs in figure below. Are they isomorphic ?



Definition 0.3. *The* **degree** *of a vertex is the number of edges containing that vertex.*

For example, in the graph above, $\deg a = 4; \deg f = 3$.

Theorem 1. *For a vertex v let $\deg v$ be the degree of v of vertices in a graph and let $v_1, \ldots, v_k$ be all the vertices. Denote by E be the total number of edges. Then*

$$\deg v_1 + \ldots + \deg v_k = 2E.$$

In particular, the number of vertices of odd degree is even.

Problem 2. In a certain kingdom there are 100 cities, and 4 roads lead out of each city. How many roads are in total in this kingdom ?

Problem 3. At a table there are 15 people. Each shakes hands with everyone else. How many handshakes are there in total ?

Problem 4.* There used to be 26 football teams in NFL with 13 teams in each of the two conferences. An NFL guideline said that each team's 14 game schedule should include exactly 11 games against teams in its own conference, and 3 games against teams in the other conference. Decide whether this NFL rule can actually be satisfied or not.

Definition 0.4. *A graph is called* **connected** *if any two vertices can be connected by a sequence of (successive) edges. A* **path** *in graph is a sequence of edges. If the initial point and the end point of a path coincide, this is called a* **cycle**.

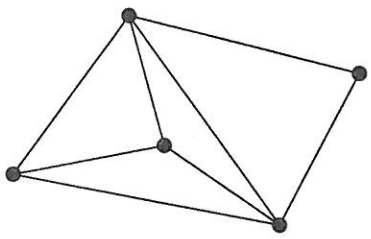
Q1: Can you point out some paths and some cycles in the graph above ?

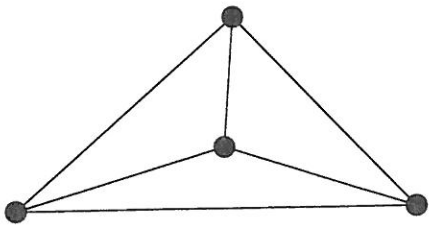
Q2: Draw two examples of connected, and two examples of non-connected graphs.

Problem 5. In a small county from Transylvania there are 15 villages, each connected to 7 other villages. Prove that one can travel between any two villages (perhaps passing through some other villages in between).

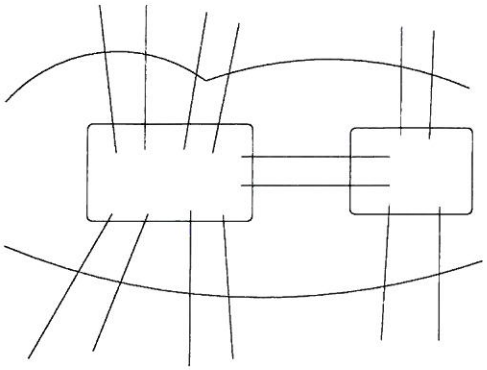
Problem 6.* In Mount Moria, the Dwarfs live in a number of cities, connected by tunnels. Their main city, called Khazad-dum is connected to exactly 7 other smaller cities. The farthest city, called Nogrod, is connected to another city by exactly one tunnel. All the remaining cities are connected to each other by exactly 4 tunnels. Show that it is possible for Gimli the Dwarf to travel from Khazad-dum to Nogrod.

Problem 7. Consider the two graphs in the figure below. Is it possible to draw these graphs with a pencil so that we do not lift the pencil from the paper and we trace each edge exactly once ?





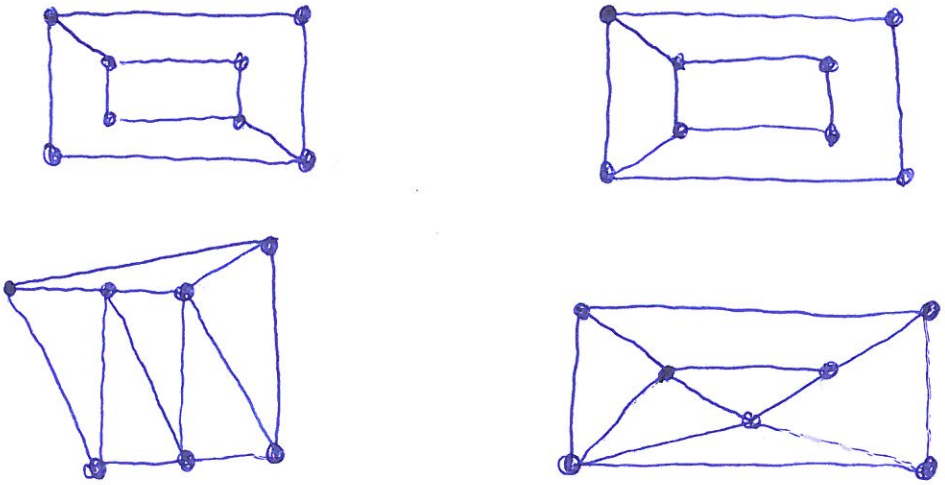
Problem 8.* A map of the city of Königsburg is given in the picture below. The city lies on both banks of a river, and there are two islands in the river. There are seven bridges, connecting the various parts of the city. Can one stroll around the town, crossing each bridge exactly once ?



Problem 9.* Is it possible for a knight to move around on a 8×8 chessboard so that it makes every possible move exactly once ? (Consider a move between two squares connected by a knight to be completed when the move is made in either direction.)

0.1. **Additional problems.** 1. Draw all non-isomorphic graphs with 3 vertices. (And explain why your graphs are non-isomorphic!)

2. Consider the two pairs of graphs below. Decide whether they are isomorphic or not.



3*. Prove that a graph with n vertices in which each vertex has degree at least $\frac{n-1}{2}$ is connected.

Note: Problems with $\star$ have a higher degree of difficulty.