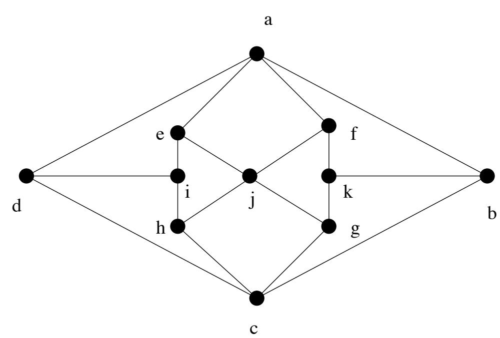
PROBLEMS ON GRAPHS

Problem 1. Cosmic highways are established among the nine planets of the solar system. Rockets travel along the following routes: Earth - Mercury, Pluto - Venus, Earth - Pluto, Pluto - Mercury, Mercury - Venus, Uranus - Neptune, Neptune - Saturn, Saturn - Jupiter, Jupiter - Mars and Mars - Uranus. Can a traveler get from Earth to Mars ?

Problem 2. Several knights are situated on a 3×3 chessboard, as in the attached figure 1. Can they move, using the usual chess knight's move, to the position shown in figure 2 ? (see attached paper for figures.)

Definition 0.1. *A diagram which consists of points (called* **vertices***) and some lines joining these points (called* **edges***) is called a* **graph***.*

An example of a graph is in the figure below.



Problem 3. In the Emirate of the Nine Cities there are 9 cities with names $1, 2, 3, \ldots, 9$. A traveler finds that two cities are connected by an airplane route if and only if the two-digit number formed by naming one of the cities, then the other, is divisible by 3. Can the traveler get from City 1 to City 9 ?

Definition 0.2. *Two graphs are* **isomorphic** *if one can redraw one of them and get the other.*

Note that isomorphic graphs have the same properties: same number of vertices, edges, etc.

Problem 4. Consider the graphs in figure 3 of the attached paper. Which pairs are isomorphic graphs ?

Definition 0.3. *The* **degree** *of a vertex is the number of edges containing that vertex.*

For example, in the graph above, $\deg a = 4; \deg f = 3$.

Problem 5. Draw 4 graphs of your choice. Calculate degrees of all the vertices, and count the number of edges. Can you see any relation between the degrees and the edge number ? Can you prove your guess for *all* graphs ?

FIGURE 3.

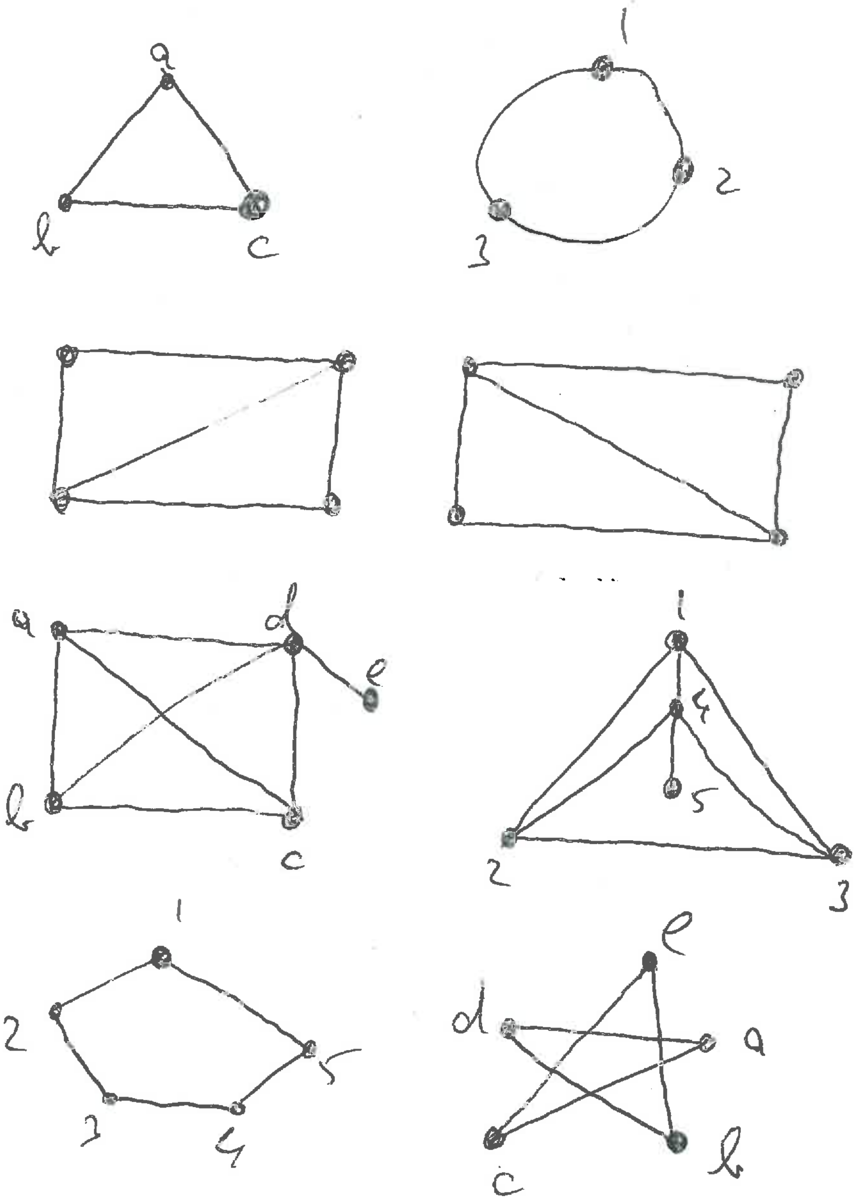


FIG. 1

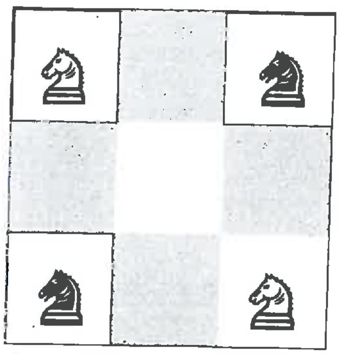
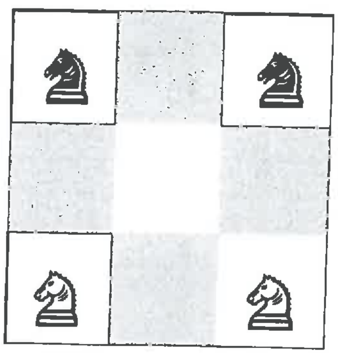


FIG. 2



Problem 6. In a Computer Science class there are 15 computers. Is it possible to connect them by cables so that each computer is connected to *exactly* 5 other computers ?

Problem 7. There are 30 students in a class. Can it happen that 9 of them have exactly 3 friends each (in the class), 11 have 4 friends each, and 10 have 5 friends each ?

Problem 8.* There used to be 26 football teams in NFL with 13 teams in each of the two conferences. An NFL guideline said that each team's 14 game schedule should include exactly 11 games against teams in its own conference, and 3 games against teams in the other conference. Decide whether this NFL rule can actually be satisfied or not.

Problem 9. In a small county from Transylvania there are 15 villages, each connected to 7 other villages. Prove that one can travel between any two villages (perhaps passing through some other villages in between).

Definition 0.4. *A graph is called* **connected** *if any two vertices can be connected by a sequence of (successive) edges. A* **path** *in graph is a sequence of edges. If the initial point and the end point of a path coincide, this is called a* **cycle**.

Can you point out some paths and some cycles in the graph above ?

Problem 10*. Prove that a graph with n vertices in which each vertex has degree at least $\frac{n-1}{2}$ is connected.

Note: Problems with $\star$ have a higher degree of difficulty.