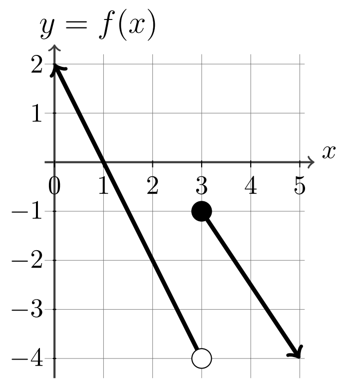
S20 Feb 12, 2020 **NAME: SOLUTIONS** **CRN:________**

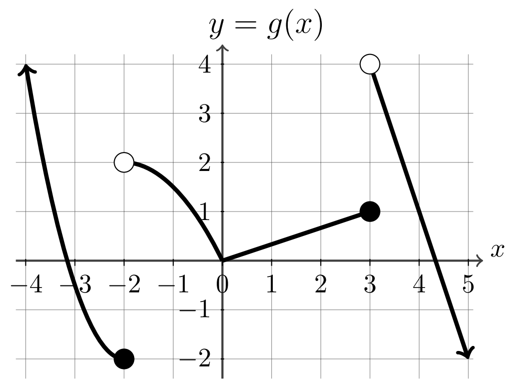
MATH 1225 Test 1 (44 pts)

Multiple Choice

(2 pts each) • **No partial credit will be given.** • **Clearly circle your answer.** • **No calculator!**

1. Consider the graphs of $y = f(x)$ and $y = g(x)$ given below.





Consider the following two statements:

Statement 1 $\lim_{x\to 3} f(x)g(x) = -4$

Statement 2 $\lim_{x\to -2} |g(x)| = 2$

Which of the following **must** be true?

(A) Only Statement 1 is correct.

(B) Only Statement 2 is correct.

(C) Both Statement 1 and Statement 2 are correct.

(D) Neither Statement 1 nor Statement 2 is correct.

C

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2. The vertical line $x = 0$ is a vertical asymptote of which of the following functions? Circle only **one** answer choice.

(A) $f(x) = e^x$

(B) $g(x) = 1 + \ln(x)$

(C) $h(x) = \dfrac{x(x-3)}{x(x+7)}$

(D) $k(x) = \dfrac{1}{x-3}$

B

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3. Let $f(x) = x^3 - 2x + 5$. Which of the following is **not** equal to the slope of the tangent line to the curve $y = f(x)$ at $x = 2$? Circle only **one** answer choice.

(A) The instantaneous rate of change of $y = f(x)$ with respect to x at $x = 2$

(B) $f'(2)$

(C) $\lim_{x\to 2} \dfrac{(x^3 - 2x + 5) - 9}{x - 2}$

(D) $\lim_{h\to 2} \dfrac{((x+h)^3 - 2(x+h) + 5) - (x^3 - 2x + 5)}{h}$

D

Free Response

- **Show reasoning that is complete and correct by the standards of this course.**
- **Whenever using theorems, you should explicitly check that all hypotheses are satisfied.**
- **Improper use of (or the absence of) proper notation will be penalized.** • **No calculator!**

4. (14 pts) Calculate the following limits. Give the limit if it exists, and where no finite limit exists be as precise as possible among $+\infty$, $-\infty$, or DNE. Tables alone will not suffice as justification. Cite theorems when appropriate.

(a) (7 pts) $\displaystyle\lim_{t\to 3^-} \frac{t^2-3t}{|3-t|}$

Note that $\displaystyle\lim_{t\to 3^-} \frac{t^2-3t}{|3-t|} = \lim_{t\to 3^-} \frac{t(t-3)}{|3-t|}$.

Observe $|3-t| = \begin{cases} -(3-t) & t \geq 3 \\ (3-t) & t < 3 \end{cases}$

For $t < 3$, $(3-t) > 0$, so $|3-t| = (3-t)$

So

$$\lim_{t\to 3^-} \frac{t(t-3)}{|3-t|} = \lim_{t\to 3^-} \frac{t(t-3)}{(3-t)} = \lim_{t\to 3^-} \frac{t(t-3)}{-(t-3)} = \lim_{t\to 3^-} -t = -3$$

(b) (7 pts) $\displaystyle\lim_{x\to -\infty} \left(5 + \frac{\sin(3x)}{x^2+10}\right)$

Notice that $-1 \leq \sin(3x) \leq 1$

Then $5 + \dfrac{-1}{x^2+1} \leq 5 + \dfrac{\sin(3x)}{x^2+1} \leq 5 + \dfrac{1}{x^2+1}$ for all x.

Note $\displaystyle\lim_{x\to -\infty} \left(5 + \frac{-1}{x^2+1}\right) = 5 + \lim_{x\to -\infty} \frac{-1}{x^2+1} = 5 - 0 = 5$

Similarly $\displaystyle\lim_{x\to -\infty} \left(5 + \frac{1}{x^2+1}\right) = 5 + \lim_{x\to -\infty} \frac{1}{x^2+1} = 5 + 0 = 5$

So $\displaystyle\lim_{x\to -\infty} \left(5 + \frac{-1}{x^2+1}\right) = \lim_{x\to -\infty} \left(5 + \frac{1}{x^2+1}\right) = 5$

So by the Squeeze Theorem:

$\displaystyle\lim_{x\to -\infty} (5 + \frac{sin(3x)}{x^2+1}) = 5$

5. (10 pts) Provide an example for each of the following.
<u>**No justification is needed,**</u> however, incorrect answers with justification may receive some partial credit.

(a) (4 pts) The **equation** of a function that has a horizontal asymptote $y = 7$, vertical asymptotes at $x = 1$ and $x = 5$, and a removable singularity at $x = -1$.

One possible function is

$$f(x) = \frac{x+1}{(x+1)(x-1)(x-5)} + 7$$

because:

- Needing vertical asymptotes at $x = 1$ and $x = 5$ is a sign that one could have $x - 1$ and $x - 5$ in the denominator.
- We can multiply by $\frac{x+1}{x+1}$ for a removable singularity.
- Since the limit as x goes to infinity of $\frac{x+1}{(x+1)(x-1)(x-5)}$ is 0, and so has a horizontal asymptote at $y = 0$, just add 7 to the whole thing.

Another correct solution is

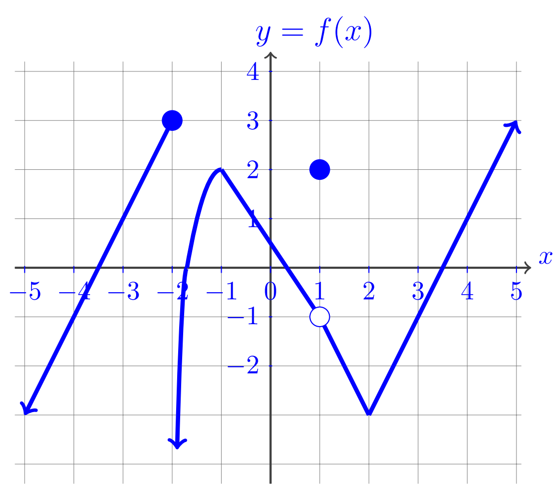
$$g(x) = \frac{7(x+1)(x+2)(x+3)}{(x+1)(x-1)(x-5)}$$

(b) (2 pts) The **equation** of a function that is continuous on $(-\infty, \infty)$.

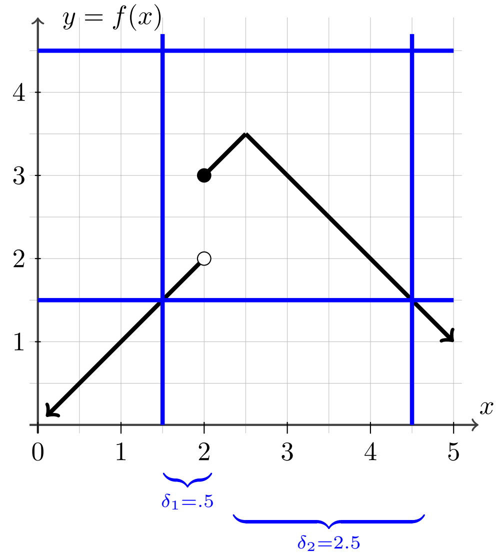
From class and/or the text, any polynomial is continuous on $(-\infty, \infty)$, so one could pick $f(x) = x$. Other functions, like $g(x) = \sin(x)$, would also be acceptable.

(c) (4 pts) The **graph** of a function, $y = f(x)$, that is not continuous on $(-\infty, \infty)$ because $\lim_{x\to 1} f(x) \neq f(1)$ and $\lim_{x\to -2^+} f(x) = -\infty$.

One possible graph is given below:



6. (6 pts) Consider the graph of $y = f(x)$ below.
Find the largest value of δ such that $|f(x) - 3| < 1.5$ whenever $0 < |x - 2| < \delta$, if it exists.
If no such value exists, explain why.
Support your answer graphically (on the graph below) or by providing a written explanation, consistent with the methods demonstrated in class.



If we require that $|f(x) - 3| < 1.5$, then this is the same as $1.5 < f(x) < 4.5$. The maximum value of $f(x)$ is 3.5, so we only focus on the lower bound of $y = 1.5$. $f(x) = 1.5$ when $x = 0.5$ and $x = 4.5$ which are distance 0.5 and 2.5 from $x = 2$ respectively.
Of those distances, we pick $\delta = 0.5$, because if we were to pick $\delta = 2.5$, we would have values which are 2.5 away from $x = 2$ where $|f(x) - 3| > 1.5$, as is the case at $x = 0$, since $|f(0) - 3| = |0 - 3| = 3 > 1.5$.

7. (8 pts) Show that the function $f(x) = \dfrac{1}{x} + 5\cos(x) + 7$ must attain the value $y = 4$ at least once on the interval $[\pi, 3\pi]$. Justify your answer using a theorem from class, and give the explicit conclusion of the theorem.

First note that $\frac{1}{x}$ is continuous on $[\pi, 3\pi]$ because it excludes the only point at which $\frac{1}{x}$ is not continuous - $x = 0$. Note that $5\cos(x) + 7$ is a trig function and continuous on its domain of $(-\infty, \infty)$. Thus, f is a continuous function on $[\pi, 3\pi]$ because it is the sum of the two continuous functions $\frac{1}{x}$, $5\cos(x) + 7$ on that interval.
We observe that

$$f(\pi) = \frac{1}{\pi} + 5\cos(\pi) + 7 = \frac{1}{\pi} - 5 + 7 = 2 + \frac{1}{\pi} < 3 < 4,$$

and

$$f(2\pi) = \frac{1}{2\pi} + 5\cos(2\pi) + 7 = \frac{1}{2\pi} + 5 + 7 = 12 + \frac{1}{2\pi} > 12 > 4.$$

Thus, $f(\pi) < 4 < f(2\pi)$. So we can apply the Intermediate Value Theorem on the interval $[\pi, 2\pi]$ to conclude that there exists a c in $(\pi, 2\pi)$ with $f(c) = 4$. We conclude f must attain the value $y = 4$ at least once on the interval $[\pi, 3\pi]$.