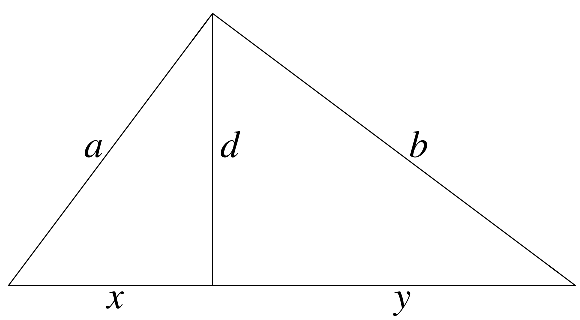
9th VTRMC, 1987, Solutions

1. The length of $\overline{P_0P_1}$ is $\sqrt{2}/2$. For $n \geq 1$, the horizontal distance from P_n to P_{n+1} is 2^{-n-1}, while the vertical distance is $3 \cdot 2^{-n-1}$. Therefore the length of $\overline{P_n, P_{n+1}}$ is $2^{-n-1}\sqrt{10}$ for $n \geq 1$, and it follows that the distance of the path is $(\sqrt{2}+\sqrt{10})/2$.

2. We want to solve in positive integers $a^2 = x^2 + d^2$, $b^2 = d^2 + y^2$, $a^2 + b^2 = (x+y)^2$. These equations yield $xy = d^2$, so we want to find positive integers x, y such that $x(x+y)$ and $y(x+y)$ are perfect squares. One way to do this is to choose positive integers x, y such that $x, y, x+y$ are perfect squares, so one possibility is $x = 9$ and $y = 16$. Thus we could have $a = 15$, $b = 20$ and $c = 25$.



3. If n is odd, then there are precisely $(n+1)/2$ odd integers in $\{1, 2, \dots\}$. Since $(n+1)/2 > n/2$, there exists an odd integer r such that a_r is odd, and then $a_r - r$ is even. It follows that $(a_1 - 1)(a_2 - 2)\cdots(a_n - n)$ is even.

4. (a) Since $|a_0| = |p(0)| \leq |0|$, we see that $a_0 = 0$.

   (b) We have $|p(x)/x| \leq 1$ for all $x \neq 0$ and $\lim_{x\to 0} p(x)/x = a_1$. Therefore $|a_1| \leq 1$.

5. (a) Set $n_1 = 2^{31}$, which has binary representation 1 followed by 31 0's. Since 31 has binary representation 11111, we see that $n_i = 31$ for all $i \geq 2$.

   (b) It is clear that $\{n_i\}$ is monotonic increasing for $i \geq 2$, so we need to prove that $\{n_i\}$ is bounded. Suppose $2^k \leq n_i < 2^{k+1}$ where $i \geq 2$. If ℓ is the number of zeros in the binary representation of n_i, then $n_i < 2^{k+1} - \ell$ and we see that $n_{i+1} < 2^{k+1}$. We deduce that $n_i < 2^{k+1}$ for all $i \geq 2$ and the result follows.

6. Of course, $\{a_n\}$ is the Fibonacci sequence (so in particular $a_n > 0$ for all n), and it is obvious that $x = -1$ is a root of $p_n(x)$ for all n (because $a_{n+2} - a_{n+1} - a_n = 0$). Since the roots of $p_n(x)$ are real and their product is $-a_n/a_{n+2}$, we see that $\lim_{n\to\infty} r_n = -1$. Finally $s_n = \lim_{n\to\infty} a_n/a_{n+2}$. If $f = \lim_{n\to\infty} a_{n+1}/a_n$, then $f^2 = f + 1$, so $f = (1+\sqrt{5})/2$ because $f > 0$. Thus $f^2 = (3+\sqrt{5})/2$ and we deduce that $\lim_{n\to\infty} s_n = (3-\sqrt{5})/2$.

7. Let $D = \{d_{ij}\}$ be the diagonal matrix with $d_{nn} = t$, $d_{ii} = 1$ for $i \neq n$, and $d_{ij} = 0$ if $i \neq j$. Then $A(t) = DA$ and $B(t) = DB$. Therefore

$$A(t)^{-1}B(t) = (DA)^{-1}DB = A^{-1}D^{-1}DB = A^{-1}B$$

as required.

8. (a) $x'(t) = u(t) - x(t)$, $y'(t) = v(t) - y(t)$, $u'(t) = -x(t) - u(t)$, $v'(t) = y(t) - v(t)$.

   (b) Set $Y = \begin{pmatrix} u(t) \\ x(t) \end{pmatrix}$ and $A = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix}$. Then we want to solve $Y' = AY$. The eigenvalues of A are $-1 \pm i$, and the corresponding eigenvectors are $\begin{pmatrix} \pm i \\ 1 \end{pmatrix}$. Therefore $u(t) = e^{-t}(-A\sin t + B\cos t)$ and $x(t) = e^{-t}(A\cos t + B\sin t)$, where A, B are constants to be determined. However when $t = 0$, we have $u(t) = x(t) = 10$, so $A = B = 10$. Therefore $u(t) = 10e^{-t}(\cos t - \sin t)$ and $x(t) = 10e^{-t}(\cos t + \sin t)$. Finally the cat will hit the mirror when $u(t) = 0$, that is when $t = \pi/4$.