32nd VTRMC, 2010, Solutions

1. It is easily checked that 101 is a prime number (divide 101 by the primes whose square is less than 101, i.e. the primes ≤ 7). Therefore for $1 \le r \le 100$, we may choose a positive integer q such that $rq \equiv 1 \mod 101$. Since $(I+A+\cdots+A^{100})(I-A) = I-A^{101}$, we see that $A^{101} = I$, in particular A is invertible with inverse A^{100}. Suppose $1 \le n \le 100$ and set $r = 101-n$. Then $1 \le r \le 100$ and $A^n+\cdots+A^{100}$ is invertible if and only if $I+\cdots+A^{r-1}$ is invertible. We can think of $I+\cdots+A^{r-1}$ as $(I-A^r)/(I-A)$, which should have inverse $(I-A)/(I-A^r)$. However $A = (A^r)^q$ and so $(I-A)/(I-A^r) = I+A^r+\cdots+(A^r)^{q-1}$. It is easily checked that

$$(I+\cdots+A^{r-1})(I+A^r+\cdots+(A^r)^{q-1}) = I.$$

It follows that $A^n+\cdots+A^{100}$ is invertible for all positive integers $n \le 100$. We conclude that $A^n+\cdots+A^{100}$ has determinant ± 1 for all positive integers $n \le 100$.

2. First we will calculate $f_n(75) \mod 16$. Note that if a,b are odd positive integers and $a \equiv b \mod 16$, then $a^a \equiv b^b \mod 16$. Also $3^3 \equiv 11 \mod 16$ and $11^{11} \equiv 3 \mod 16$. We now prove by induction on n that $f_{2n-1}(75) \equiv 11 \mod 16$ for all $n \in \mathbb{N}$. This is clear for $n = 1$ so suppose $f_{2n-1}(75) \equiv 11 \mod 16$ and set $k = f_{2n-1}(75)$ and $m = f_{2n}(75)$. Then

$$f_{2n}(75) \equiv k^k \equiv 11^{11} \equiv 3 \mod 16$$
$$f_{2n+1}(75) \equiv m^m \equiv 3^3 \equiv 11 \mod 16$$

and the induction step is complete. We now prove that $f_n(a) \equiv f_{n+2}(a) \mod 17$ for all $a,n \in \mathbb{N}$ with a prime to 17 and n even. In fact we have

$$f_{n+1}(a) \equiv a^3 \mod 17, \quad f_{n+2}(a) \equiv (a^3)^{11} \equiv a \mod 17.$$

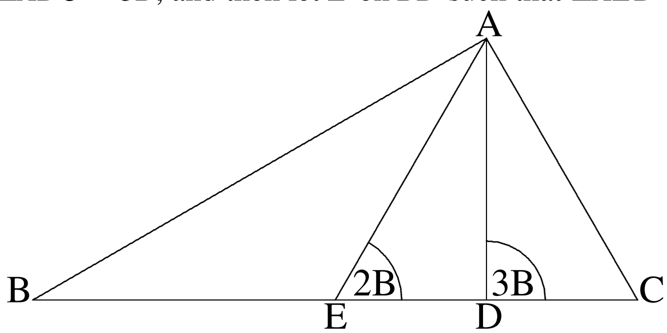
Thus $f_{100}(75) \equiv f_2(75) \mod 17$. Therefore $f_{100}(75) \equiv 7^{11} \equiv 14 \mod 17$.

3. First note the $e^{2\pi i/7}$ satisfies $1+x+\cdots+x^6 = 0$, so by taking the real part, we obtain $\sum_{n=0}^{n=6} \cos 2n\pi/7 = 0$. Since $\cos 2\pi/7 = \cos 12\pi/7$, $\cos 4\pi/7 = \cos 10\pi/7 = -\cos 3\pi/7$ and $\cos 6\pi/7 = \cos 8\pi/7 = -\cos \pi/7$, we see that $1 - 2\cos \pi/7 + 2\cos 2\pi/7 - 2\cos 3\pi/7 = 0$.

   Observe that if $1 - 2\cos\theta + 2\cos 2\theta - 2\cos 3\theta = 0$, then by using $\cos 2\theta = 2\cos^2\theta - 1$ and $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, we find that $\cos\theta$ satisfies

$8x^3-4x^2-4x+1=0$. Thus in particular $\cos\pi/7$ satisfies this equation. Next note that $1-2\cos 3\pi/7+2\cos 6\pi/7-2\cos 9\pi/7 = 1-2\cos 3\pi/7-2\cos\pi/7+2\cos 2\pi/7$, so $\cos 3\pi/7$ is also a root of $8x^3-4x^2-4x+1$. Finally since the sum of the roots of this equation is 1/2, we find that $-\cos 2\theta$ is also a root. Thus the roots of $8x^3-4x^2-4x+1$ are $\cos\pi/7$, $-\cos 2\pi/7$, $\cos 3\pi/7$.

4. The equation $4A+3C=540^\circ$ tells us that $A=3B$. Let D on BC such that $\angle ADC=3B$, and then let E on BD such that $\angle AED=2B$.



Since triangles ABD and AED are similar, we see that

$$\frac{BD}{AD}=\frac{AD}{ED}=\frac{AB}{AE}.$$

Also $BE=AE$ because $B=\angle BAE$, and $BE=BD-BE$. We deduce that $BD^2=AD^2+AB\cdot AD$. Since $BD=BC-CD$, we conclude that $(BC-CD)^2=AD^2+AB\cdot AD$.

Next triangles ABC and ADC are similar, consequently

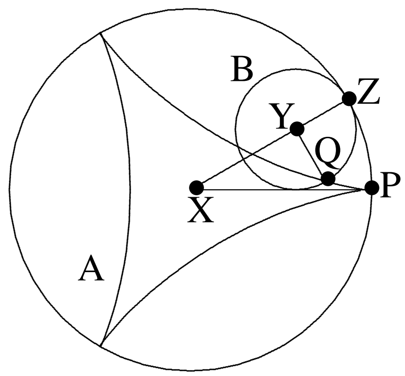
$$\frac{BC}{AC}=\frac{AB}{AD}=\frac{AC}{CD}.$$

Thus $AD=AB\cdot AC/BC$ and $CD=AC^2/BC$. We deduce that

$$(a-b^2/a)^2=c^2b^2/a^2+c^2b/a.$$

Therefore $(a^2-b^2)^2=bc^2(a+b)$ and the result follows.

5. Let X denote the center of A, let Y denote the center of B, let Z be where A and B touch (so X,Y,Z are collinear), and let $\theta=\angle PXY$. Note that YQ makes an angle 2θ downwards with respect to the horizontal, because $\angle QYZ=3\theta$.



Choose (x,y)-coordinates such that X is the origin and XP is on the line $y=0$. Let (x,y) denote the coordinates of Q. Then we have

$$x = 2\cos\theta + \cos 2\theta$$
$$y = 2\sin\theta - \sin 2\theta.$$

By symmetry the area above the x-axis equals the area below the x-axis (we don't really need this observation, but it may make things easier to follow). Also θ goes from 2π to 0 as circle B goes round circle A. Therefore the area enclosed by the locus of Q is

$$\begin{aligned}
2\int_{\pi}^{0} y\frac{dx}{d\theta}\,d\theta &= 2\int_{\pi}^{0}(2\sin\theta - \sin 2\theta)(-2\sin\theta - 2\sin 2\theta)\,d\theta \\
&= 2\int_{0}^{\pi}(4\sin^2\theta + 2\sin\theta\sin 2\theta - 2\sin^2 2\theta)\,d\theta \\
&= \int_{0}^{\pi}(4 - 4\cos 2\theta + 2\cos\theta - 2\cos 3\theta - 2 + 2\cos 4\theta)\,d\theta = 2\pi.
\end{aligned}$$

6. Note that if $0 < x, y < 1$, then $0 < 1 - y/2 < 1$ and $0 < x(1-y/2) < 1$, and it follows that (a_n) is a positive monotone decreasing sequence consisting of numbers strictly less that 1. This sequence must have a limit z where $0 \le z \le 1$. In particular $a_{n+2} - a_{n+1} = a_n a_{n+1}/2$ has limit 0, so $\lim_{n\to\infty} a_n a_{n+1} = 0$. It follows that $z = 0$.

   Set $b_n = 1/a_n$. Then $b_{n+2} = b_{n+1}/(1 - a_n/2) = b_{n+1}(1 + a_n/2 + O(a_n^2))$. Therefore $b_{n+2} - b_{n+1} = b_{n+1}(a_n/2 + O(a_n^2))$. Also

$$a_n/a_{n+1} = (1 - a_{n-1}/2)^{-1} = 1 + O(a_n)$$

and we deduce that $b_{n+2} - b_{n+1} = b_n(1+O(a_n))(a_n/2 + O(a_n^2))$. Therefore $b_{n+2} - b_{n+1} = 1/2 + O(a_n)$. Thus given $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $|b_{n+1} - b_n - 1/2| < \varepsilon$ for all $n > N$. We deduce that if k is a positive integer, then $|b_{n+k}/k - b_n/k - 1/2| < \varepsilon$. Thus for k sufficiently large, $|b_{n+k}/(n+k) - 1/2| < 2\varepsilon$. We conclude that $\lim_{n\to\infty} b_n/n = 1/2$ and hence $\lim_{n\to\infty} na_n = 2$.

7. It will be sufficient to prove that $\sum_{n=1}^{\infty} \frac{n^2}{1/a_1^2 + \cdots + 1/a_n^2}$ is convergent. Note we may assume that (a_n) is monotonic decreasing, because rearranging the terms in series $\sum a_n$ does not affect its convergence, whereas the terms of the above series become largest when (a_n) is monotonic decreasing. Next observe that if $\sum_{n=1}^{\infty} a_n = S$, then $a_n \le S/n$ for all positive integers n. Now consider $\frac{(2n)^2}{1/a_1^2 + \cdots + 1/a_{2n}^2}$. This is $\le \frac{(2n)^2 S}{1/a_1 + 2/a_2 + \cdots + 2n/a_{2n}} \le \frac{4n^2 S}{n^2/a_n} = 4Sa_n$. The result follows.