28th VTRMC, 2006, Solutions

1. If we write such an integer n in base 3, then it must end in $200\ldots0$, because n contains no 1's. But then n^2 will end in $100\ldots0$ and we conclude that there are *no* positive integers n for which neither n nor n^2 contain a 1 when written out in base 3.

2. The format of such a sequence must either consist entirely of A's and B's, or must be a block of A's, followed by a single B, followed by a block of C's, followed by a string of A's and B's. In the former case, there are 2^n such sequences. In the latter case, the number of such sequences which have k A's and m C's (where $m \geq 1$) is $2^{n-m-k-1}$. Therefore the number of such sequences with k A's is

$$\sum_{m=1}^{n-k-1} 2^{n-m-k-1} = 2^{n-k-1} - 1.$$

We deduce that the total number of such sequences is

$$\sum_{k=0}^{n-2} (2^{n-k-1} - 1) + 2^n = 2^n - 2 - (n-1) + 2^n = 2^{n+1} - (n+1).$$

We conclude that $S(10) = 2^{11} - 11 = 2037$.

3. From the recurrence relation $F(n) = F(n-1) + F(n-2)$, we obtain

$$\begin{aligned} F(n+5) &= F(n+4) + F(n+3) = 2F(n+3) + F(n+2) \\ &= 3F(n+2) + 2F(n+1) = 5F(n+1) + 3F(n). \end{aligned}$$

Thus $F(n+20) = 3^4 F(n) = F(n) \mod 5$ and we deduce that $F(2006) = F(6) \mod 20$. Since $F(6) = 5F(2) + 3F(1) = 8$. it follows that $F(2006)$ has remainder 3 after being divided by 5. Also $F(n+5) = 2F(n+3) + F(n+2)$ tells us that $F(n+5) = F(n+2) \mod 2$ and hence $F(2006) = F(2) = 1 \mod 2$. We conclude that $F(2006)$ is an odd number which has remainder 3 after being divided by 5, consequently the last digit of $F(2006)$ is 3.

4. Set $c_n = (-b_{3n-2})^n - (-b_{3n-1})^n + (-b_{3n})^n$. Then the series $\sum_{n=1}^{\infty} c_n$ can be written as the sum of the three series

$$\sum_{n=1}^{\infty} (-1)^n b_{3n-2}, \quad \sum_{n=1}^{\infty} -(-1)^n b_{3n-1}, \quad \sum_{n=1}^{\infty} (-1)^n b_{3n}.$$

Since each of these three series is alternating in sign with the absolute value of the terms monotonically decreasing with limit 0, the alternating series test tells us that each of the series is convergent. Therefore the sequence $s_k := \sum_{n=1}^{3k}(-1)^n b_n$ is convergent, with limit S say. Since $\lim_{n\to\infty} b_n = 0$, it follows that $\sum_{n=1}^{\infty}(-1)^n b_n$ is also convergent with sum S.

5. We will model the solution on the method reduction of order; let us try a solution of the form $y = u\sin t$ where u is a function of t. Then $y' = u'\sin t + u\cos t$ and $y'' = u''\sin t + 2u'\cos t - u\sin t$. Plugging into $y'' + py' + qy = 0$, we obtain $u''\sin t + u'(2\cos t + p\sin t) + u(p\cos t + q\sin t - \sin t) = 0$. We set

$$u''\sin t + u'(2\cos t + p\sin t) = 0 \quad \text{and} \quad p\cos t + q\sin t - \sin t = 0.$$

There are many possibilities. We want $u = t^2$ to satisfy $u'' + u'(2\cos t + p\sin t)/\sin t = 0$. Since $u = t^2$ satisfies $u'' - u'/t = 0$, we set $2\cot t + p = -1/t$, and then

$$\begin{aligned} p &= -1/t - 2\cot t, \\ q &= 1 - p\cot t = 1 + \frac{\cot t}{t} + 2\cot^2 t, \\ f &= t^2 \sin t. \end{aligned}$$

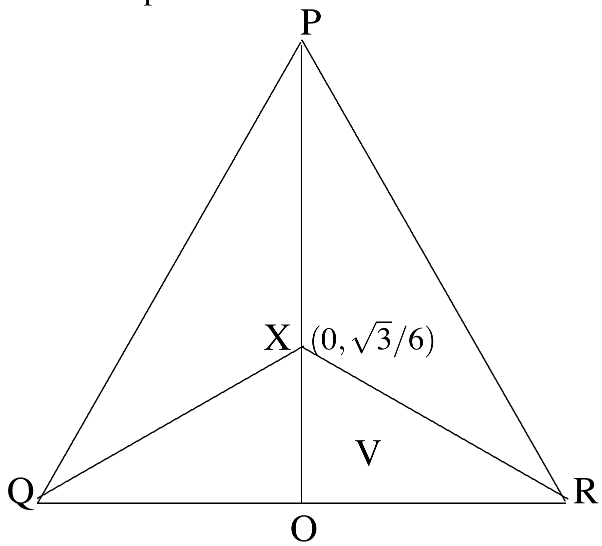
Then p and q are continuous on $(0,\pi)$ (because $1/t$ and $\cot t$ are continuous on $(0,\pi)$), and $y = \sin t$ and $y = f(t)$ satisfy $u'' + pu' + qu = 0$. Also f is infinitely differentiable on the whole real line $(-\infty,\infty)$ and $f(0) = f'(0) = f''(0) = 0$.

6. Let $\beta = \angle QBP$ and $\gamma = \angle QCP$. Then the sine rule for the triangle ABC followed by the double angle formula for sines, and then the addition rules for sines and cosines yields

$$\begin{aligned} \frac{AB+AC}{BC} &= \frac{\sin 2\beta + \sin 2\gamma}{\sin(2\beta + 2\gamma)} = \frac{2\sin(\beta+\gamma)\cos(\beta-\gamma)}{2\sin(\beta+\gamma)\cos(\beta+\gamma)} \\ &= \frac{\cos\beta\cos\gamma + \sin\beta\sin\gamma}{\cos\beta\cos\gamma - \sin\beta\sin\gamma} = \frac{1+\tan\beta\tan\gamma}{1-\tan\beta\tan\gamma}. \end{aligned}$$

Since $\tan\beta\tan\gamma = \dfrac{PQ}{BQ}\dfrac{PQ}{QC} = \dfrac{1}{2}$, we see that $\dfrac{AB+AC}{BC} = 3$ and the result is proven.

7. We will call the three spheres A, B, D and let their centers be P, Q, R respectively. Then PQR is an equilateral triangle with sides of length 1. So we will let $O = (0,0,0)$, $P = (0, \sqrt{3}/2, 0)$, $Q = (-1/2, 0, 0)$, $R = (1/2, 0, 0)$, and $X = (0, 1/(2\sqrt{3}), 0)$. Then M can be described as the cylinder C with cross-section PQR which is bounded above and below by the spheres A, B, D. Let V denote the space above ORX. We now have the following diagram.



By symmetry, the mass of M is $12 \iiint_V z\,dV$. Also above QRX, the mass M is bounded above by the A, which has equation $z = \sqrt{1 - x^2 - (y - \sqrt{3}/2)^2}$, and the equation of the line XR in the xy-plane is $x + \sqrt{3}y = 1/2$. Therefore the mass of M is

$$
\begin{aligned}
& 12 \int_0^{1/(2\sqrt{3})} \int_0^{1/2-\sqrt{3}y} \int_0^{\sqrt{1-x^2-(y-\sqrt{3}/2)^2}} z\,dz\,dx\,dy \\
&= 6 \int_0^{1/(2\sqrt{3})} \int_0^{1/2-\sqrt{3}y} (1 - x^2 - (y - \sqrt{3}/2)^2)\,dx\,dy \\
&= 2 \int_0^{1/(2\sqrt{3})} [3x - x^3 - 3x(y - \sqrt{3}/2)^2]_0^{1/2-\sqrt{3}y} dy \\
&= \int_0^{1/(2\sqrt{3})} (1 - 2\sqrt{3}y)(1/2 + 4\sqrt{3}y - 6y^2)\,dy \\
&= \int_0^{1/(2\sqrt{3})} (12\sqrt{3}y^3 - 30y^2 + 3\sqrt{3}y + 1/2)\,dy \\
&= [3\sqrt{3}y^4 - 10y^3 + 3\sqrt{3}y^2/2 + y/2]_0^{\sqrt{3}/6}
\end{aligned}
$$

$$= \sqrt{3}(1/48 - 5/36 + 1/8 + 1/12) = \frac{13}{48\sqrt{3}}.$$