27th VTRMC, 2005, Solutions

1. We note that if $p = 2,3,5,11,19$, then $6(p^3+1) \equiv 4,3,1,2,0 \mod 5$ respectively. Therefore if $n \geq 20$, we may choose p such that $n+6(p^3+1)$ is divisible by 5. Obviously if $n+6(p^3+1)$ is prime, then n is not divisible by 2 or 3. If $12 \leq n \leq 19$, then p may be chosen so that $6(p^3+1) \equiv 1,2,3,4$ mod 5, so 15 is the only possibility for $n+6(p^3+1)$ to be prime; however the above remark tells us this cannot be prime. So we need only check the numbers $1,5,7,11$. For $n = 7,11$, we see that $n+6(p^3+1)$ is divisible by 5 by choosing $p = 3$ and 2 respectively. Finally $5+6(p^3+1)$ is 59 when $p = 2$ and 173 when $p = 3$. Since 59 and 173 are both prime, we conclude that 5 is the largest integer required.

2. First, we must have $p(20) = 12$, $p(19) = 13$, $p(18) = 14$, $p(17) = 15$, $p(16) = 16$, $p(8) = 8$, $p(9) = 7$, $p(10) = 6$, $p(11) = 5$. Then $p(4)$ cannot be 12, hence $p(4) = 4$, $p(3) = 1$, $p(2) = 2$, $p(1) = 3$. Now if $p(n) = 20$, then n must be 12, and we have $p(13) = 19$, $p(14) = 18$ and $p(15) = 17$. Finally we must have $p(5) = 11$, $p(6) = 10$ and $p(7) = 9$. Thus the permutation required is
3,2,1,4,11,10,9,8,7,6,5,20,19,18,17,16,15,14,13,12.

3. If the end of the strip consists of one or two squares, then the number of ways of tiling the strip is $t(n-1)$, which makes a total of $2t(n-1)$. If the end of the strip consists of one or two dominos, then the number of ways of tiling the strip is $t(n-2)$, making for a total of $2t(n-2)$ ways. Finally if the strip ends in one domino and one square, then there may or may not be a square in the penultimate position, and here we get a total of $2t(n-2)$ ways. We conclude that $t(n) = 2t(n-1)+4t(n-2)$. We now have $t(3) = 24$, $t(4) = 80$, $t(5) = 256$ and $t(6) = 832$.

4. The x-coordinate of the beam of light will return to 0 after traveling distance 14. During this period, the y and z coordinates will each have traveled distance 28, and thus will have also returned to their original positions. It follows that the total distance traveled by the beam of light will be $\sqrt{14^2+28^2+28^2} = 14\sqrt{1^2+2^2+2^2} = 42$.

5. Set $z = y\ln(x^2)$. Then $f(x,y) = \dfrac{xz}{(x^2+z^2)\ln(x^2)}$. Since $|xz| \leq x^2+z^2$, we see that $|f(x,y)| \leq 1/\ln(x^2)$. Since $1/\ln(x^2) \to 0$ as $(x,y) \to (0,0)$, it follows that $\lim_{(x,y)\to(0,0)}$ exists and is equal to 0.

6. Divide the rectangle in the xy-plane $0 \leq x \leq 1, 0 \leq y \leq e$ into two regions A, the area above $y = e^{x^2}$, and B, the area below $y = e^{x^2}$. We have area(A) + area(B) = e and area(B) = $\int_0^1 e^{x^2}\,dx$. Also, by interchanging the rôles of x and y, so $y = e^{x^2}$ becomes $x = \sqrt{\ln y}$, we see that area(A) = $\int_1^e \sqrt{\ln y}\,dy$. Now make the substitution $x = (y-1)/(e-1)$, so $y = 1 + ex - x$, we see that this last integral is $\int_0^1 (e-1)\ln(1+ex-x)\,dx$. We conclude that $\int_0^1 ((e-1)\sqrt{\ln(1+ex-x)} + e^{x^2})\,dx = e$.

7. Suppose $AA'\mathbf{x} = \mathbf{0}$, where $\mathbf{x}$ is a column vector with 5 components. Then $\mathbf{x}'AA'\mathbf{x} = \mathbf{0}$, hence $(A'\mathbf{x})'(A'\mathbf{x}) = \mathbf{0}$. Since all the entries of $A'\mathbf{x}$ are real numbers, we deduce that $A'\mathbf{x} = \mathbf{0}$. Thus A' and AA' have the same null space. By hypothesis $\operatorname{rank}(A) = 5$, hence $\operatorname{rank}(A') = 5$ and we deduce that the null space of A' is 0. Therefore AA' has zero null space and we deduce that $\operatorname{rank}(AA') = 5$. This means that every 5×1 matrix can be written in the form $AA'\mathbf{v}$.