26th VTRMC, 2004, Solutions

1. The answer is no. One example is

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

   Then $\det M \neq 0$ (expand by the fourth row), whereas $\det N = 0$ (fourth row consists entirely of 0's). Therefore M is invertible and N is not invertible, as required.

2. For n a non-negative integer, as n increases from n to $n+6$, we add 3 twice, 1 twice and 2 twice to $f(n)$; in other words $f(n+6) = f(n)+12$. We deduce that $f(n) = 2n$ when $n = 0$ (mod 6).

3. Let s_n denote the number of strings of length n with no three consecutive $A's$. Thus $s_1 = 3$, $s_2 = 9$ and $s_3 = 26$. We claim that we have the following recurrence relation:

$$s_n = 2s_{n-1} + 2s_{n-2} + 2s_{n-3} \quad (n \geq 3).$$

   The first term on the right hand side indicates the number of such strings which begin with B or C; the second term indicates the number of such strings which begin with AB of AC, and the third term indicates the number of such strings which begin with AAB or AAC. Using this recurrence relation, we find that $s_4 = 76$, $s_5 = 222$ and $s_6 = 648$. Since the total number of strings is 3^6, we conclude that the probability of a string on 6 symbols not containing 3 successive A's is $648/3^6 = 8/9$.

4. The answer is no. Let us color the chess board in the usual way with alternating black and white squares, say the corners are colored with black squares. Then by determining the number of black squares in each row, working from top to bottom, we see that the number of black squares is

$$4+4+5+4+4+4+5+4+4 = 38.$$

   Since there are 78 squares in all, we see that the number of white squares is 40. Now each domino cover 1 white and 1 black square, so if the board could be covered by dominoes, then there would be an equal number of black and white squares, which is not the case.

5. Expanding the sine, we have

$$f(x) = \cos x \int_0^x \sin(t^2 - t)\,dt + \sin x \int_0^x \cos(t^2 - t)\,dt.$$

Therefore

$$\begin{aligned}
f'(x) &= \cos x \sin(x^2 - x) - \sin x \int_0^x \sin(t^2 - t)\,dt \\
&\quad + \sin x \cos(x^2 - x) + \cos x \int_0^x \cos(t^2 - t)\,dt, \\
f''(x) &= -\sin x \sin(x^2 - x) + (2x - 1)\cos x \cos(x^2 - x) \\
&\quad - \sin x \sin(x^2 - x) - \cos x \int_0^x \sin(t^2 - t)\,dt \\
&\quad + \cos x \cos(x^2 - x) + (1 - 2x)\sin x \sin(x^2 - x) \\
&\quad - \sin x \int_0^x \sin(t^2 - t)\,dt + \cos x \sin(x^2 - x).
\end{aligned}$$

We deduce that

$$\begin{aligned}
f(x) + f''(x) &= (2x+1)(\cos x \cos(x^2 - x) - \sin x \sin(x^2 - x)) \\
&= (2x+1)\cos x^2.
\end{aligned}$$

Set $g(x) = f''(x) + f(x)$. To find $f^{(12)}(0) + f^{(10)}(0)$, we need to compute $g^{(10)}(0)$, which we can find by considering the coefficient of x^{10} in the Maclaurin series expansion for $(2x+1)\cos x^2$. Since $\cos x^2 = 1 - x^4/2! + x^8/4! - x^{12}/6! + \cdots$, we see that this coefficient is 0. Therefore $g^{(10)}(0) = 0$ and the result follows.

6. Suppose first that there is an infinite subset S such that each person only knows a finite number of people in S. Then pick a person A_1 in S. Then there is an infinite subset S_1 of S containing A_1 such that A_1 knows nobody in S_1. Now choose a person A_2 other than A_1 in S_1. Then there is an infinite subset S_2 of S_1 containing $\{A_1, A_2\}$ such that nobody in S_2 knows A_2. Of course nobody in S_2 will know A_1 either. Now choose a person A_3 in S_2 other than A_1 and A_2. Then nobody from $\{A_1, A_2, A_3\}$ knows each other. Clearly we can continue this process indefinitely to obtain an arbitrarily large number of people who don't know each other.

Therefore we may assume in any infinite subset of people there is a person who knows an infinite number of people. So we can pick a person B_1 who knows infinitely many people T_1. Then we can pick a person B_2 in T_1 who knows an infinite number of people T_2 of T_1, because we are assuming in any infinite subset of people, there is somebody who knows infinitely many of them. Of course, B_1 and B_2 know all the people in T_2. Now choose a person B_3 in T_2 who knows an infinite number of people in T_2. Then $\{B_1, B_2, B_3\}$ know each other. Clearly we can continue this process indefinitely to obtain an arbitrarily large number of people who know each other.

Remark A simple application of the axiom of choice shows that we can find an infinite number of people in the party such that either they all know each other or they all don't know each other.

7. Set $b_n = 1 - a_{n+1}/a_n$. Let us suppose to the contrary that $\sum |b_n|$ is convergent. Then $\lim_{n\to\infty} b_n = 0$, so may assume that $|b_n| < 1/2$ for all n. Now

$$a_{n+1} = a_1(1-b_1)(1-b_2)\dots(1-b_n),$$

hence

$$\ln(a_{n+1}) = \ln a_1 + \ln(1-b_1) + \ln(1-b_2) + \cdots + \ln(1-b_n).$$

Since $\lim_{n\to\infty} a_n = 0$, we see that $\lim_{n\to\infty} \ln(a_n) = -\infty$. Now $\ln(1-b) \geq -2|b|$ for $|b| < 1/2$; one way to see this is to observe that $1/(1-x) \leq 2$ for $0 \leq x \leq 1/2$ and then to integrate between 0 and $|b|$. Therefore $\lim_{n\to\infty}(-2|b_1| - \cdots - 2|b_n|) = -\infty$. This proves that $\sum |b_n|$ is divergent and the result follows.