14th VTRMC, 1992, Solutions

1. First make the substitution $y = x^3$. Then $dF/dx = (dF/dy)(dy/dx) = e^{y^2}3x^2 = 3x^2e^{x^6}$ by the chain rule. Therefore $d^2F/dx^2 = 6xe^{x^6} + 18x^7e^{x^6}$. To find the point of inflection, we set $d^2F/dx^2 = 0$; thus we need to solve $6x + 18x^7 = 0$. The only solution is $x = 0$, so this is the point of inflection (perhaps we should note that d^3F/dx^3 is $6 \neq 0$ at $x = 0$, so $x = 0$ is indeed a point inflection).

2. The shortest path will first be reflected off the x-axis, then be reflected off the y-axis. So we reflect (x_2, y_2) in the y-axis, and then in the x-axis, which yields the point $(-x_2, -y_2)$. Thus the length of the shortest path is the distance from (x_1, y_1) to $(-x_2, -y_2)$, which is $((x_1+x_2)^2 + (y_1+y_2)^2)^{1/2}$.

3. (i) We have $f(f(x)) = 1 + \sin(f(x) - 1) = 1 + \sin(\sin(x-1))$, so $f_2(x) = x$ if and only if $x - 1 = \sin\sin(x-1)$. If y is a real number, then $|\sin y| \leq |y|$ with equality if and only if $y = 0$. It follows that $y = \sin\sin(y)$ if and only if $y = 0$ and we deduce that there is a unique point x_0 such that $f_2(x_0) = x_0$, namely $x_0 = 1$.

   (ii) From (i), we have $x_n = 1$ for all n. Thus we need to find $\sum_{n=0}^{\infty} 1/3^n$. This is a geometric series with first term 1 and ratio between successive terms 1/3. Therefore this sum is $1/(1-1/3) = 3/2$.

4. Clearly $t_n \geq 1$ for all n. Set $T = (1+\sqrt{5})/2$ (the positive root of $x^2 - x - 1$). Note that if $1 \leq x < T$, then $x^2 < 1 + x < T^2$. This shows that $t_n < T$ for all n, and also that t_n is an increasing sequence, because $t_{n+1}^2 - t_n^2 = 1 + +t_n - t_n^2$. Therefore this sequence converges to a number between 1 and T. Since the number must satisfy $x^2 = x + 1$, we deduce that $\lim_{n\to\infty} t_n = T = (1+\sqrt{5})/2$.

5. First we find the eigenvalues and eigenvectors of A. The eigenvalues satisfy $x(x-3) + 2 = 0$, so the eigenvalues are 1,2. To find the eigenvectors corresponding to 1, we need to solve the matrix equation

$$\begin{pmatrix} -1 & -2 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

   One solution is $u = 2$, v$= -1$ and we see that $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to 1.

To find the eigenvectors corresponding to 2, we need to solve

$$\begin{pmatrix} -2 & -2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

One solution is $u = 1$, $v = -1$ and we see that $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector corresponding to 2.

We now know that if $T = \begin{pmatrix} 2 & 1 \\ -1 & -1 \end{pmatrix}$, then $T^{-1}AT = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$. We deduce that $T^{-1}A^{100}T = \begin{pmatrix} 1 & 0 \\ 0 & 2^{100} \end{pmatrix}$. Therefore

$$A^{100} = \begin{pmatrix} 2 - 2^{100} & 2 - 2^{101} \\ -1 + 2^{100} & -1 + 2^{101} \end{pmatrix}.$$

6. Since $p(r) = 0$, we may write $p(x) = q(x)(x-r)$, where $q(x)$ is of the form $x^2 + dx + e$. Then $p(x)/(x-r) - 2p(x+1)/(x+1-r) + p(x+2)/(x+2-r) = q(x) - 2q(x+1) + q(x+2) = 2$ as required.

7. Assume that log means natural log. Note that $\log x$ is a positive increasing function for $x > 1$. Therefore

$$\int_1^n x \log x \, dx \leq \sum_{t=2}^{n} t \log t \leq \int_2^{n+1} x \log x \, dx.$$

Since $\int x \log x = (x^2 \log x)/2 - x^2/4$, we see that

$$(n^2 \log n)/2 - n^2/4 \leq \sum_{t=2}^{n} t \log t \leq ((n+1)^2 \log n)/2 - (n+1)^2/4 - 2\log 2 + 1.$$

Now divide by $n^2 \log n$ and take $\lim_{n\to\infty}$. We obtain

$$1/2 \leq \lim_{n\to\infty} \frac{\sum_{t=2}^{n} t \log t}{n^2 \log n} \leq 1/2.$$

We conclude that the required limit is 1/2.

8. For $G(3)$ we have 8 possible rows of goblins, and by writing these out we see that $G(3) = 17$. Similarly for $G(4)$ we have 16 possible rows of goblins, and by writing these out we see that $G(4) = 44$.

In general, let X be a row of N goblins. Then the rows with $N+1$ columns are of the form $X2$ or $X3$ (where $X2$ indicates adding a goblin with height $2'$ to the end of X). If X ends in 3 or 32 ($2^{N-1}+2^{N-2}$ possible rows), then $X3$ has 1 more LGG than X; on the other hand if X ends in 22, then $X3$ has the same number of LGG's as X. If X ends in 2 (2^{N-1} possible rows), then $X2$ has 1 more LGG than X, while if X ends in 3, then $X2$ has the same number of LGG's as X. We conclude that for $N \geq 2$,

$$G(N+1) = 2G(N) + 2^{N-1} + 2^{N-1} + 2^{N-2} = 2G(N) + 5 \cdot 2^{N-2}.$$

We now solve this recurrence relation in a similar way to solving a linear differential equation. The general solution will be $G(N) = C \cdot 2^N + aN2^{N-2}$, where a is to be determined. Then $G(N+1) = 2G(N) + 5 \cdot 2^{N-2}$ yields $a = 5/2$, so $G(N) = C \cdot 2^N + 5N2^{N-3}$. The initial condition $G(2) = 6$ shows that $C = 1/4$ and we conclude that $G(N) = 2^{N-2} + 5N2^{N-3}$ for all $N \geq 2$. We also have $G(1) = 2$.