6th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to noon November 3, 1984

Fill out the individual registration form

1. Find the units digit (base 10) in the sum $\sum_{k=1}^{99} k!$.

2. Consider any three consecutive positive integers. Prove that the cube of the largest cannot be the sum of the cubes of the other two.

3. A sequence $\{u_n\}$, $n = 0, 1, 2, \ldots$, is defined by $u_0 = 5$, $u_{n+1} = u_n + n^2 + 3n + 3$, for $n = 0, 1, 2, \ldots$. If u_n is expressed as a polynomial $u_n = \sum_{k=0}^{d} c_k n^k$, where d is the degree of the polynomial, find the sum $\sum_{k=0}^{d} c_k$.

4. Let the (x, y)-plane be divided into regions by n lines, any two of which may or may not intersect. Describe a procedure whereby these regions may be colored using only two colors so that regions with a common line segment as part of their boundaries have different colors.

5. Let $f(x)$ satisfy the conditions for Rolle's theorem on $[a, b]$ with $f(a) = f(b) = 0$. Prove that for each real number k the function $g(x) = f'(x) + kf(x)$ has at least one zero in (a, b).

6. A matrix is called excellent if it is square and the sum of its elements in each row and column equals the sum of its elements in every other row and column. Let V_n denote the collection of excellent $n \times n$ matrices.

   (a) Show that V_n is a vector space under addition and scalar multiplication (by real numbers).

   (b) Find the dimensions of V_2, V_3, and V_4.

   (c) If $A \in V_n$ and $B \in V_n$, show that $AB \in V_n$.

7. Find the greatest real r such that some normal line to the graph of $y = x^3 + rx$ passes through the origin, where the point of normality is not the origin.

8. Let $f = f(x)$ be an arbitrary differentiable function on $I = [x_0 - h, x_0 + h]$ with $|f'(x)| \leq M$ on I where $M \geq \sqrt{3}$. Let $f(x_0 - h) \leq f(x_0)$ and $f(x_0 + h) \leq$

$f(x_0)$. Find the smallest positive number r such that at least one local maximum of f lies inside or on the circle of radius r centered at $(x_0, f(x_0))$. Express your answer in terms of h, M and $d = \min\{f(x_0) - f(x_0 - h), f(x_0) - f(x_0 + h)\}$.