# 5th Annual
# Virginia Tech Regional Mathematics Contest

From 9:30 a.m. to 12:00 noon, November 5, 1983

## Fill out the individual registration form

1. In the expansion of $(a+b)^n$, where $n$ is a natural number, there are $n+1$ dissimilar terms. Find the number of dissimilar terms in the expansion of $(a+b+c)^n$.

2. A positive integer $N$ (in base 10) is called *special* if the operation $C$ of replacing each digit $d$ of $N$ by its nine's-complement $9-d$, followed by the operation $R$ of reversing the order of the digits, results in the original number. (For example, 3456 is a special number because $R[(C3456)] = 3456$.) Find the sum of all special positive integers less than one million which do not end in zero or nine.

3. Let a triangle have vertices at $O(0,0), A(a,0)$, and $B(b,c)$ in the $(x,y)$-plane.

    (a) Find the coordinates of a point $P(x,y)$ in the exterior of $\triangle OAB$ satisfying $\text{area}(OAP) = \text{area}(OBP) = \text{area}(ABP)$.

    (b) Find a point $Q(x,y)$ in the interior of $\triangle OAQ$ satisfying $\text{area}(OAQ) = \text{area}(OBQ) = \text{area}(ABQ)$.

4. A finite set of roads connect $n$ towns $T_1, T_2, \ldots, T_n$ where $n \geq 2$. We say that towns $T_i$ and $T_j$ $(i \neq j)$ are *directly connected* if there is a road segment connecting $T_i$ and $T_j$ which does not pass through any other town. Let $f(T_k)$ be the number of other towns directly connected to $T_k$. Prove that $f$ is not one-to-one.

5. Find the function $f(x)$ such that for all $L \geq 0$, the area under the graph of $y = f(x)$ and above the $x$-axis from $x = 0$ to $x = L$ equals the arc length of the graph from $x = 0$ to $x = L$. (Hint: recall that $\frac{d}{dx}\cosh^{-1} x = 1/\sqrt{x^2-1}$.)

6. Let $f(x) = 1/x$ and $g(x) = 1-x$ for $x \in (0,1)$. List all distinct functions that can be written in the form $f \circ g \circ f \circ g \circ \cdots \circ f \circ g \circ f$ where $\circ$ represents composition. Write each function in the form $\dfrac{ax+b}{cx+d}$, and prove that your list is exhaustive.

7. If $a$ and $b$ are real, prove that $x^4 + ax + b = 0$ cannot have only real roots.

8. A sequence $f_n$ is generated by the recurrence formula

$$f_{n+1} = \frac{f_n f_{n-1} + 1}{f_{n-2}},$$

for $n = 2, 3, 4, \ldots$, with $f_0 = f_1 = f_2 = 1$. Prove that $f_n$ is integer-valued for all integers $n \geq 0$.