3rd Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, November 7, 1981

Fill out the individual registration form

1. The number $2^{48}-1$ is exactly divisible by what two numbers between 60 and 70?

2. For which real numbers b does the function $f(x)$, defined by the conditions $f(0)=b$ and $f'=2f-x$, satisfy $f(x)>0$ for all $x\geq 0$?

3. Let A be non-zero square matrix with the property that $A^3=0$, where 0 is the zero matrix, but with A being otherwise arbitrary.

   (a) Express $(I-A)^{-1}$ as a polynomial in A, where I is the identity matrix.

   (b) Find a 3×3 matrix satisfying $B^2\neq 0, B^3=0$.

4. Define $F(x)$ by $F(x)=\sum_{n=0}^{\infty}F_n x^n$ (wherever the series converges), where F_n is the nth Fibonacci number defined by $F_0=F_1=1, F_n=F_{n-1}+F_{n-2}$, $n>1$. Find an explicit closed form for $F(x)$.

5. Two elements A,B in a group G have the property $ABA^{-1}B=1$, where 1 denotes the identity element in G.

   (a) Show that $AB^2=B^{-2}A$.

   (b) Show that $AB^n=B^{-n}A$ for any integer n.

   (c) Find u and v so that $(B^aA^b)(B^cA^d)=B^uA^v$.

6. With k a positive integer, prove that $(1-k^{-2})^k\geq 1-1/k$.

7. Let $A=\{a_0,a_1,\ldots\}$ be a sequence of real numbers and define the sequence $A'=\{a'_0,a'_1,\ldots\}$ as follows for $n=0,1,\ldots$: $a'_{2n}=a_n$, $a'_{2n+1}=a_n+1$. If $a_0=1$ and $A'=A$, find

   (a) a_1,a_2,a_3 and a_4

   (b) a_{1981}

   (c) A simple general algorithm for evaluating a_n, for $n=0,1,\ldots$.

8. Let

(i) $0 < a < 1$,

(ii) $0 < M_{k+1} < M_k$, for $k = 0, 1, \ldots$,

(iii) $\lim_{k \to \infty} M_k = 0$.

If $b_n = \sum_{k=0}^{\infty} a^{n-k} M_k$, prove that $\lim_{n \to \infty} b_n = 0$.