39th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 21, 2017

Fill out the individual registration form

1. Determine the number of real solutions to the equation $\sqrt{2-x^2} = \sqrt[3]{3-x^3}$.

2. Evaluate $\displaystyle\int_0^a \frac{dx}{1+\cos x+\sin x}$ for $-\pi/2 < a < \pi$. Use your answer to show that $\displaystyle\int_0^{\pi/2} \frac{dx}{1+\cos x+\sin x} = \ln 2$.

3. Let ABC be a triangle and let P be a point in its interior. Suppose $\angle BAP = 10^\circ$, $\angle ABP = 20^\circ$, $\angle PCA = 30^\circ$ and $\angle PAC = 40^\circ$. Find $\angle PBC$.

4. Let P be an interior point of a triangle of area T. Through the point P, draw lines parallel to the three sides, partitioning the triangle into three triangles and three parallelograms. Let a, b and c be the areas of the three triangles. Prove that $\sqrt{T} = \sqrt{a}+\sqrt{b}+\sqrt{c}$.

5. Let $f(x,y) = \dfrac{x+y}{2}$, $g(x,y) = \sqrt{xy}$, $h(x,y) = \dfrac{2xy}{x+y}$, and let
$$S = \{(a,b) \in \mathbb{N}\times\mathbb{N} \mid a \neq b \text{ and } f(a,b), g(a,b), h(a,b) \in \mathbb{N}\},$$
where $\mathbb{N}$ denotes the positive integers. Find the minimum of f over S.

6. Let $f(x) \in \mathbb{Z}[x]$ be a polynomial with integer coefficients such that $f(1) = -1$, $f(4) = 2$ and $f(8) = 34$. Suppose $n \in \mathbb{Z}$ is an integer such that $f(n) = n^2 - 4n - 18$. Determine all possible values for n.

7. Find all pairs (m,n) of nonnegative integers for which $m^2 + 2\cdot 3^n = m(2^{n+1}-1)$.