32nd Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., October 30, 2010

Fill out the individual registration form

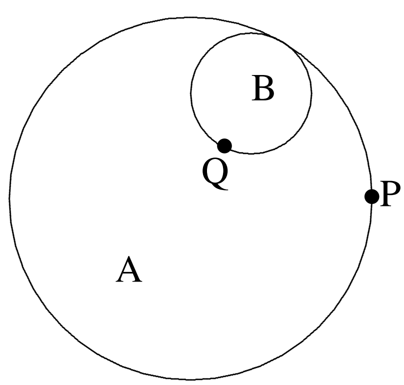
1. Let d be a positive integer and let A be a $d \times d$ matrix with integer entries. Suppose $I + A + A^2 + \cdots + A^{100} = 0$ (where I denotes the identity $d \times d$ matrix, so I has 1's on the main diagonal, and 0 denotes the zero matrix, which has all entries 0). Determine the positive integers $n \le 100$ for which $A^n + A^{n+1} + \cdots + A^{100}$ has determinant ± 1.

2. For n a positive integer, define $f_1(n) = n$ and then for i a positive integer, define $f_{i+1}(n) = f_i(n)^{f_i(n)}$. Determine $f_{100}(75) \mod 17$ (i.e. determine the remainder after dividing $f_{100}(75)$ by 17, an integer between 0 and 16). Justify your answer.

3. Prove that $\cos(\pi/7)$ is a root of the equation $8x^3 - 4x^2 - 4x + 1 = 0$, and find the other two roots.

4. Let $\triangle ABC$ be a triangle with sides a, b, c and corresponding angles A, B, C (so $a = BC$ and $A = \angle BAC$ etc.). Suppose that $4A + 3C = 540°$. Prove that $(a-b)^2(a+b) = bc^2$.

5. Let A, B be two circles in the plane with B inside A. Assume that A has radius 3, B has radius 1, P is a point on A, Q is a point on B, and A and B touch so that P and Q are the same point. Suppose that A is kept fixed and B is rolled once round the inside of A so that Q traces out a curve starting and finishing at P. What is the area enclosed by this curve?



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6. Define a sequence by $a_1 = 1$, $a_2 = 1/2$, and $a_{n+2} = a_{n+1} - \dfrac{a_n a_{n+1}}{2}$ for n a positive integer. Find $\lim_{n\to\infty} n a_n$.

7. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms (so $a_i > 0$ for all i) and set $b_n = \dfrac{1}{n a_n^2}$ for $n \geq 1$. Prove that $\displaystyle\sum_{n=1}^{\infty} \frac{n}{b_1 + b_2 + \cdots + b_n}$ is convergent.