30th Annual Virginia Tech Regional Mathematics Contest

From 9:00 a.m. to 11:30 a.m., November 1, 2008

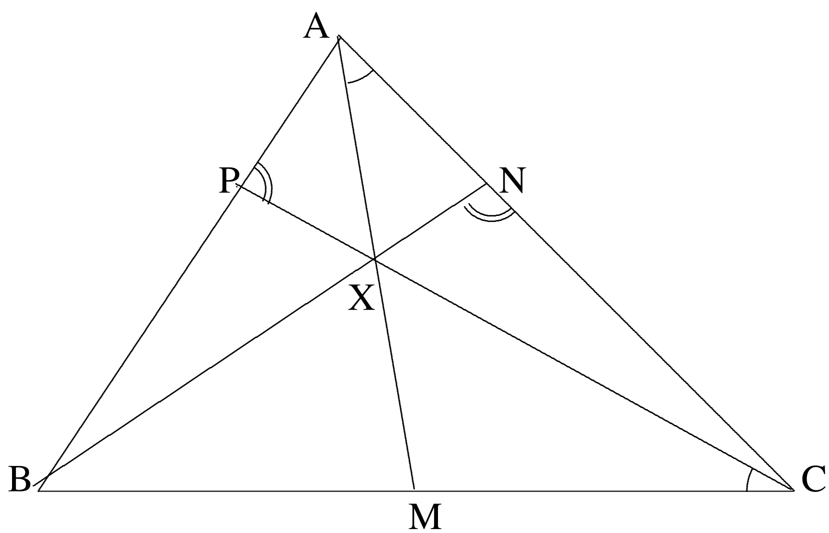
Fill out the individual registration form

1. Find the maximum value of $xy^3 + yz^3 + zx^3 - x^3y - y^3z - z^3x$ where $0 \leq x \leq 1, 0 \leq y \leq 1, 0 \leq z \leq 1$.

2. How many sequences of 1's and 3's sum to 16? (Examples of such sequences are $\{1,3,3,3,3,3\}$ and $\{1,3,1,3,1,3,1,3\}$.)

3. Find the area of the region of points (x,y) in the xy-plane such that $x^4 + y^4 \leq x^2 - x^2y^2 + y^2$.

4. Let ABC be a triangle, let M be the midpoint of BC, and let X be a point on AM. Let BX meet AC at N, and let CX meet AB at P. If $\angle MAC = \angle BCP$, prove that $\angle BNC = \angle CPA$.



5. Let $a_1, a_2, \ldots$ be a sequence of nonnegative real numbers and let π, ρ be permutations of the positive integers $\mathbb{N}$ (thus $\pi, \rho \colon \mathbb{N} \to \mathbb{N}$ are one-to-one and onto maps). Suppose that $\sum_{n=1}^{\infty} a_n = 1$ and ε is a real number such that $\sum_{n=1}^{\infty} |a_n - a_{\pi n}| + \sum_{n=1}^{\infty} |a_n - a_{\rho n}| < \varepsilon$. Prove that there exists a finite subset X of $\mathbb{N}$ such that $|X \cap \pi X|, |X \cap \rho X| > (1-\varepsilon)|X|$ (here $|X|$ indicates the number of elements in X; also the inequalities $<, >$ are strict).

(Please turn over)

6. Find all pairs of positive (nonzero) integers a, b such that $ab - 1$ divides $a^4 - 3a^2 + 1$.

7. Let $f_1(x) = x$ and $f_{n+1}(x) = x^{f_n(x)}$ for n a positive integer. Thus $f_2(x) = x^x$ and $f_3(x) = x^{(x^x)}$. Now define $g(x) = \lim_{n\to\infty} 1/f_n(x)$ for $x > 1$. Is g continuous on the open interval $(1, \infty)$? Justify your answer.