25th Annual
## Virginia Tech Regional Mathematics Contest
From 8:30 a.m. to 11:00 a.m., November 1, 2003

Fill out the individual registration form

1. An investor buys stock worth $10,000 and holds it for n business days. Each day he has an equal chance of either gaining 20% or losing 10%. However in the case he gains every day (i.e. n gains of 20%), he is deemed to have lost all his money, because he must have been involved with insider trading. Find a (simple) formula, with proof, of the amount of money he will have on average at the end of the n days.

2. Find $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n(n+1)} = \frac{x}{1\cdot 2} + \frac{x^2}{2\cdot 3} + \frac{x^3}{3\cdot 4} + \cdots$ for $|x| < 1$.

3. Determine all invertible 2 by 2 matrices A with complex numbers as entries satisfying $A = A^{-1} = A'$, where A' denotes the transpose of A.

4. It is known that $2\cos^3 \dfrac{\pi}{7} - \cos^2 \dfrac{\pi}{7} - \cos \dfrac{\pi}{7}$ is a rational number. Write this rational number in the form p/q, where p and q are integers with q positive.

5. In the diagram below, X is the midpoint of BC, Y is the midpoint of AC, and Z is the midpoint of AB. Also $\angle ABC + \angle PQC = \angle ACB + \angle PRB = 90°$. Prove that $\angle PXC = 90°$.
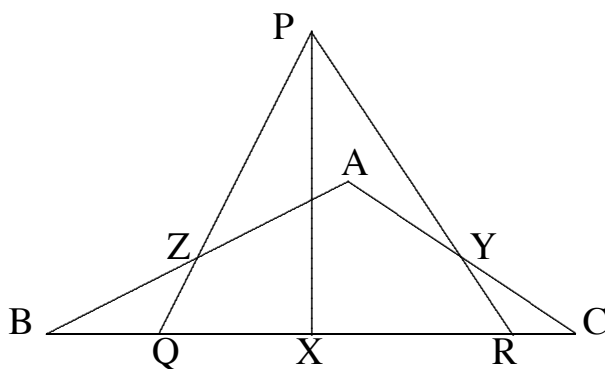
6. Let $f\colon [0,1] \to [0,1]$ be a continuous function such that $f(f(f(x))) = x$ for all $x \in [0,1]$. Prove that $f(x) = x$ for all $x \in [0,1]$. Here $[0,1]$ denotes the closed interval of all real numbers between 0 and 1, including 0 and 1.

7. Let T be a solid tetrahedron whose edges all have length 1. Determine the volume of the region consisting of points which are at distance at most 1 from some point in T (your answer should involve $\sqrt{2}, \sqrt{3}, \pi$).