10th Annual
Virginia Tech Regional Mathematics Contest
From 9:30 a.m. to 12:00 noon, October 22, 1988

Fill out the individual registration form

1. A circle C of radius r is circumscribed by a parallelogram S. Let θ denote one of the interior angles of S, with $0 < \theta \leq \pi/2$. Calculate the area of S as a function of r and θ.

2. A man goes into a bank to cash a check. The teller mistakenly reverses the amounts and gives the man cents for dollars and dollars for cents. (Example: if the check was for \$5.10, the man was given \$10.05.). After spending five cents, the man finds that he still has twice as much as the original check amount. What was the original check amount? Find all possible solutions.

3. Find the general solution of $y(x) + \int_1^x y(t)\,dt = x^2$.

4. Let a be a positive integer. Find all positive integers n such that $b = a^n$ satisfies the condition that $a^2 + b^2$ is divisible by $ab + 1$.

5. Let f be differentiable on $[0,1]$ and let $f(\alpha) = 0$ and $f(x_0) = -.0001$ for some α and $x_0 \in (0,1)$. Also let $|f'(x)| \geq 2$ on $[0,1]$. Find the smallest upper bound on $|\alpha - x_0|$ for all such functions.

6. Find positive real numbers a and b such that $f(x) = ax - bx^3$ has four extrema on $[-1,1]$, at each of which $|f(x)| = 1$.

7. For any set S of real numbers define a new set $f(S)$ by $f(S) = \{x/3 \mid x \in S\} \cup \{(x+2)/3 \mid x \in S\}$.
   (a) Sketch, carefully, the set $f(f(f(I)))$, where I is the interval $[0,1]$.
   (b) If T is a bounded set such that $f(T) = T$, determine, *with proof*, whether T can contain 1/2.

8. Let $T(n)$ be the number of incongruent triangles with integral sides and perimeter $n \geq 6$. Prove that $T(n) = T(n-3)$ if n is even, or disprove by a counterexample. (*Note*: two triangles are *congruent* if there is a one-to-one correspondence between the sides of the two triangles such that corresponding sides have the same length.)