ANALYTIC VERSIONS OF THE ZERO DIVISOR CONJECTURE

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1. Introduction

This is an expanded version of the three lectures I gave at the Durham conference. The material is mainly expository, though there are a few new results, and for those I have given complete proofs. While the subject matter involves analysis, it is written from an algebraic point of view. Thus hopefully algebraists will find the subject matter comprehensible, though analysts may find the analytic part rather elementary.

The topic considered here can be considered as an analytic version of the zero divisor conjecture over $\mathbb{C}$: recall that this states that if G is a torsion free group and $0 \neq \alpha, \beta \in \mathbb{C}G$, then $\alpha\beta \neq 0$. Here we will study the conjecture that if $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^p(G)$, then $\alpha\beta \neq 0$ (precise definitions of some of the terminology used in this paragraph can be found in later sections). We shall also discuss applications to L^p-cohomology.

Since these notes were written, the work of Rosenblatt and Edgar [19, 54] has come to my attention. This is closely related to the work of Section 6.

2. Notation and Terminology

All rings will have a 1, and to say that R is a field will imply that R is commutative (because we use the terminology division ring for not necessarily commutative "fields"). A nonzero divisor in a ring R will be an $a \in R$ such that $ab \neq 0 \neq ba$ for all $b \in R \backslash 0$. To say that the ring R is a domain will mean that if $a, b \in R \backslash 0$, then $ab \neq 0$; equivalently $R \backslash 0$ is the set of nonzero divisors of R. We shall use the notation $\mathbb{C}$, $\mathbb{R}$, $\mathbb{Z}$, $\mathbb{N}$ and $\mathbb{P}$ for the complex numbers, real numbers, integers, nonnegative integers and positive integers respectively. Ring homomorphisms will preserve the 1, and unless otherwise stated, mappings will be on the left and modules will be right modules. If $n \in \mathbb{N}$, then M^n will indicate the direct sum of n copies of the R-module M. As usual, $\ker\theta$ and $\operatorname{im}\theta$ will denote the kernel and image of the map θ. The closure of a subset X in a Banach space will be denoted by $\overline{X}$; in particular if θ is a continuous map between Banach spaces, then $\overline{\operatorname{im}\theta}$ denotes the closure of the image of θ. If $\mathcal{H}$ is a Hilbert space and $\mathcal{K}$ is a subspace of $\mathcal{H}$, we shall let $\mathcal{L}(\mathcal{H})$ denote the set of bounded linear operators on $\mathcal{H}$, and $\mathcal{K}^\perp$ denote the orthogonal complement of $\mathcal{K}$ in $\mathcal{H}$. We shall let $\mathrm{M}_n(R)$ indicate the set of $n \times n$ matrices over a ring R, $\mathrm{GL}_n(R)$ the set of invertible elements of $\mathrm{M}_n(R)$, 1_n the identity matrix of $\mathrm{M}_n(R)$, and 0_n the zero matrix of $\mathrm{M}_n(R)$. If $t \in \mathbb{P}$ and $A_i \in \mathrm{M}_{n_i}(R)$ $(1 \leq i \leq t)$, then $\operatorname{diag}(A_1, \dots, A_t)$ denotes the matrix in $\mathrm{M}_{n_1+\dots+n_t}(R)$

$$\begin{pmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_t \end{pmatrix}.$$

For any ring R, we let $K_0(R)$ denote the Grothendieck group associated with the category of all finitely generated projective R-modules: thus $K_0(R)$ has generators $[P]$ where P runs through the class of finitely generated projective R-modules, and relations $[P] = [Q] \oplus [U]$ whenever P, Q and U are finitely generated projective R-modules and $P \cong Q \oplus U$.

When R is a right Noetherian ring, the Grothendieck group associated with the category of all finitely generated R-modules will be denoted by $G_0(R)$: thus $G_0(R)$ has generators $[M]$ where M runs through the class of finitely generated R-modules, and relations $[L] = [M] \oplus [N]$ whenever L, M and N are finitely generated R-modules and there is a short exact sequence $0 \to M \to L \to N \to 0$. There is then a natural map $K_0(R) \to G_0(R)$ given by $[P] \to [P]$, and in the case R is semisimple Artinian, this map is an isomorphism.

We shall use the notation $G *_A H$ for the free product of the groups G and H amalgamating the subgroup A, $[G : A]$ for the index of A in G, G' for the commutator subgroup of G, and $\mathcal{F}(G)$ for the set of finite subgroups of G. If the orders of the subgroups in $\mathcal{F}(G)$ are bounded, we shall let $\operatorname{lcm}(G)$ stand for the lcm (lowest common multiple) of the orders of the subgroups in $\mathcal{F}(G)$. The characteristic subgroup of G generated by its finite normal subgroups will be indicated by $\Delta^+(G)$. If S is a subset or an element of G, then $\langle S \rangle$ will denote the subgroup generated by S. For $g \in G$, we shall let $C_G(g)$ indicate the centralizer of $g \in G$. If $\mathcal{X}$ and $\mathcal{Y}$ are classes of groups, then $G \in \mathcal{X}\mathcal{Y}$ will mean that G has a normal subgroup $X \in \mathcal{X}$ such that $G/X \in \mathcal{Y}$.

3. Definitions and $L^p(G)$

Here we will define the Banach spaces $L^p(G)$ and discuss some elementary results from functional analysis. Throughout this section G will be a group.

As usual, we define the complex group ring

$$\mathbb{C}G = \{\sum_{g \in G} \alpha_g g \mid \alpha_g \in \mathbb{C} \text{ and } \alpha_g = 0 \text{ for all but finitely many } g\}.$$

For $\alpha = \sum_{g \in G} \alpha_g g$, $\beta = \sum_{g \in G} \beta_g g \in \mathbb{C}G$, the multiplication is defined by

$$\alpha\beta = \sum_{g,h \in G} \alpha_g \beta_h gh = \sum_{g \in G} \big(\sum_{x \in G} \alpha_{gx^{-1}} \beta_x\big) g.$$

Then for $1 \le p \in \mathbb{R}$, we define

$$L^p(G) = \{\alpha = \sum_{g \in G} \alpha_g g \mid \alpha_g \in \mathbb{C} \text{ and } \sum_{g \in G} |\alpha_g|^p < \infty\},$$
$$\|\alpha\|_p = \big(\sum_{g \in G} |\alpha_g|^p\big)^{1/p}.$$

Thus $L^p(G)$ is a Banach space under the norm $\|.\|_p$ (of course $L^p(G)$ can also be defined for $p < 1$, but then it would no longer satisfy the triangle inequality $\|\alpha+\beta\|_p \le \|\alpha\|_p + \|\beta\|_p$ and so would not be a Banach space). Also we define

$$\begin{aligned}
L^\infty(G) &= \{\alpha = \sum_{g\in G} \alpha_g g \mid \alpha_g \in \mathbb{C} \text{ and } \sup_{g\in G} |\alpha_g| < \infty\},\\
C_0(G) &= \{\alpha = \sum_{g\in G} \alpha_g g \mid \alpha_g \in \mathbb{C} \text{ and given } \epsilon > 0,\\
&\qquad \text{there exist only finitely many } g \text{ such that } |\alpha_g| > \epsilon\},\\
\|\alpha\|_\infty &= \sup_{g\in G} |\alpha_g|.
\end{aligned}$$

Then $L^\infty(G)$ and $C_0(G)$ are Banach spaces under the norm $\|.\|_\infty$. If $\alpha \in L^\infty(G)$, then $\alpha_g \in \mathbb{C}$ is determined by the formula $\alpha = \sum_{g\in G} \alpha_g g$. For $p < q$,

$$\mathbb{C}G \subseteq L^p(G) \subseteq L^q(G) \subseteq C_0(G) \subseteq L^\infty(G),$$

and there is equality everywhere if and only if $|G| < \infty$ and strict inequality everywhere if and only if $|G| = \infty$. The multiplication in $\mathbb{C}G$ extends to a multiplication

$$L^1(G) \times L^\infty(G) \to L^\infty(G)$$

according to the formula

$$\sum_{g\in G} \alpha_g g \sum_{g\in G} \beta_g g = \sum_{g,h\in G} \alpha_g \beta_h gh = \sum_{g\in G} \big(\sum_{x\in G} \alpha_{gx^{-1}} \beta_x\big) g, \tag{3.1}$$

and this also induces a multiplication $L^1(G) \times L^p(G) \to L^p(G)$ for all $p \ge 1$; in the case $p = 1$, this makes $L^1(G)$ into a ring. Another multiplication is $L^2(G) \times \mathbb{C}G \to L^2(G)$; this is useful because it means that $L^2(G)$ can be viewed as a right $\mathbb{C}G$-module, as we do in Section 11.

The central topic of these notes is the following:

Problem 3.1. *Let G be a torsion free group and let $1 \le p \le \infty$. Does $0 \ne \alpha \in \mathbb{C}G$ and $0 \ne \beta \in L^p(G)$ imply $\alpha\beta \ne 0$?*

We shall also consider generalizations of this to groups with torsion and to matrix rings. Since this can be considered as an extension of the classical zero divisor conjecture, let us consider the current status of that problem.

4. The classical zero divisor conjecture

We shall briefly review the status of the classical zero divisor conjecture. Recall that the group G is right ordered means that there exists a total order $\le$ on G such that $x \le y$ implies that $xz \le yz$ for all $x, y, z \in G$. The class of right ordered groups includes all torsion free abelian groups, all free groups, and is closed under taking subgroups, directed unions, free products, and group extension (i.e. H and G/H are right ordered implies that G is right ordered). It also includes the class of locally indicable groups, where G is locally indicable means that if $H \ne 1$ is a finitely generated subgroup of G, then there exists $H_0 \lhd H$ such that $H/H_0 \cong \mathbb{Z}$. Furthermore if G has a family of normal subgroups $\{H_i \mid i \in \mathcal{I}\}$ for some indexing set $\mathcal{I}$ such that G/H_i is right orderable for all $i \in \mathcal{I}$ and $\bigcap_{i\in\mathcal{I}} H_i = 1$, then G is right orderable. These results can be found in [44, §7.3]. Then the usual argument which shows that a polynomial ring is a domain can be extended to show

Theorem 4.1. *Let k be a field and let G be a right ordered group. Then kG is a domain.*

Variants of this result have been around in the literature for a long time. For instance back in 1940, Higman [29] proved the above result in the case G is locally indicable.

Little further progress was made until the 1970's, though in 1959 Cohn proved that the free product of two domains amalgamating a common division ring is also a domain [13, theorem 2.5]. The significance of this result was not realized for group rings until Lewin applied it to show that under fairly mild restrictions, the group ring of a free product with amalgamation is a domain. To describe his results, we need to recall the definition of the Ore condition.

Let R be a ring, let S be the set of nonzero divisors in R, and let S_0 be a subset of R which is closed under multiplication and contains 1. Then R satisfies the right Ore condition with respect to S_0 means that for each $r \in R$ and $s \in S_0$, there exists $r_1 \in R$ and $s_1 \in S_0$ such that $rs_1 = sr_1$, and then we can form the ring RS_0^{-1} which consists of elements $\{rs^{-1} \mid r \in R,\, s \in S_0\}$. Normally S_0 will be contained in S, but this is not essential. We say that R satisfies the right Ore condition if it satisfies the right Ore condition with respect S. Also a classical right quotient ring for R is a ring Q which contains R such that every element of S is invertible in Q, and every element of Q can be written in the form rs^{-1} with $r \in R$ and $s \in S$. If such a ring Q exists, then R satisfies the right Ore condition and $RS^{-1} \cong Q$. In the case that R is also domain, this is equivalent to saying that R can be embedded as a right order in a division ring D; in other words, each element of D can be written in the form rs^{-1} where $r, s \in R$ and $s \neq 0$. It is well known that a semiprime right Noetherian ring satisfies the right Ore condition.

A right Ore domain will mean a domain which satisfies the right Ore condition; thus by the above, a right Noetherian domain is a right Ore domain. Of course one can replace "right" with "left" in all of the above, and then an Ore domain will mean a domain which satisfies the Ore condition; i.e. both the right and left Ore condition. If G is a solvable group and k is a field such that kG is a domain, then the proposition of [36] shows that kG satisfies the Ore condition. Then one of the consequences of Lewin's results for example, is (see [36, theorem 1])

Theorem 4.2. *Let k be a field and let $G = G_1 *_H G_2$ be groups such that $H \lhd G$. Suppose kG_1 and kG_2 are domains, and kH satisfies the right Ore condition. Then kG is a domain.*

This result was applied by Formanek [25] to prove that if k is a field and G is a torsion free supersolvable group, then kG is a domain.

The next step was made by Brown, Farkas and Snider [6, 24] who realized that a combination of ring and K-theoretic techniques could be applied to the problem, especially solvable groups. Their techniques established that if k is a field of characteristic zero and G is a torsion free polycyclic-by-finite group, then kG is a domain. Building on these ideas, Cliff [8] established the zero divisor conjecture for group rings of polycyclic-by-finite groups over fields of arbitrary characteristic.

At this time it was already folklore that a suitable generalization of some well known K-theoretic theorems on polynomial rings, in particular on the Grothendieck group G_0, would yield stronger results for the zero divisor conjecture, especially for solvable groups. Let G be a group, let R be a ring, and let $R * G$ be a crossed product (see [47]). Thus $R*G$ is an associative ring with a 1, and it may be viewed as a free R-module with basis $\{\bar{g} \mid g \in G\}$, where each $\bar{g}$ is a unit in $R*G$. Another way of describing $R*G$ is that it is a G-graded ring with a unit in each degree (see [47, chapter 1, §2]). Of course $R*G$ is not uniquely determined by R and G in general, but this never seems to cause any confusion. Also it is clear that if $H \leqslant G$, then $R*H$ (the free R-submodule of $R*G$ with R-basis the elements of H) is also a crossed product and is a subring of $R*G$. Many theorems for group rings go over immediately to the crossed product situation. Thus for example, Theorem 4.1 becomes

Theorem 4.3. *Let k be a domain, let G be a right ordered group, and let $k*G$ be a crossed product. Then $k*G$ is a domain.*

To make induction arguments work, we would prefer to work with $R*G$ rather than the group ring RG. Indeed if $H \lhd G$, then a crossed product $R*G$ can be expressed as the crossed product $RH*[G/H]$, whereas the corresponding result for group rings, that if k is a field then kG can be expressed as the group ring $kH[G/H]$, is decidedly false.

The importance of G_0 for the zero divisor conjecture is as follows. If G is a torsion free group and $k*G$ is a crossed product, then one can often prove that $k*G$ can be embedded in a matrix ring $\mathrm{M}_n(D)$ over a division ring D for some $n \in \mathbb{P}$ in a "nice way". Clearly what we need is that $n = 1$. If I is a minimal right ideal of $\mathrm{M}_n(D)$, then $G_0(\mathrm{M}_n(D)) = \langle [I] \rangle$, so we would like to prove that $G_0(\mathrm{M}_n(D)) = \langle [\mathrm{M}_n(D)] \rangle$. With the right setup, the inclusion of $k*G$ in $\mathrm{M}_n(D)$ induces an epimorphism of $G_0([k*G])$ onto $G_0(\mathrm{M}_n(D))$, so it will be sufficient to prove that $G_0(k*G) = \langle [k*G] \rangle$.

If G is a finitely generated free abelian group, k a right Noetherian ring, and $k*G$ a crossed product, then by exploiting the fact that G can be ordered it has been known for a long time that the natural map $G_0(k) \to G_0(k*G)$ is an epimorphism; in particular if k is a field, then $G_0(k*G) = \langle [k*G] \rangle$. However for a long time better K-theoretic results (at least for applications to the zero divisor conjecture) seemed hard to come by. Then in 1986, John Moody came up with the following remarkable theorem (proved in [43, theorem 1]).

Theorem 4.4. *Let G be a finitely generated abelian-by-finite group, let R be a right Noetherian ring, and let $R*G$ be a crossed product. Then the induced map*

$$\bigoplus_{H \in \mathcal{F}(G)} G_0(R*H) \to G_0(R*G)$$

is surjective.

For an exposition of this result, see [9, 23] and [47, chapter 8]. Thus in the special case R is a division ring and G is torsion free finitely generated abelian-by-finite, we have that $G_0(R*G) = \langle [R*G] \rangle$, and using earlier remarks of this section, it is not difficult to prove that $R*G$ is a domain. Also an easy induction argument shows that Theorem 4.4 remains valid if G is replaced by an arbitrary polycyclic-by-finite group (this is in fact how Theorem 4.4 is stated in [43, theorem 1]). Another consequence of Theorem 4.4 is the following result, well known from when Theorem 4.4 was established.

Corollary 4.5. *Let G be an abelian-by-finite group, let k be a division ring, and let $k * G$ be a crossed product. If $k * H$ is a domain whenever H is a finite subgroup of G, then $k * G$ is an Ore domain.*

Proof (sketch). We may assume that G is finitely generated and $\Delta^+(G) = 1$. Let $A \lhd G$ with A free abelian and $[G : A] < \infty$. If $S = k * A \backslash 0$, then we can form the ring $k * GS^{-1}$, which will be an $n \times n$ matrix ring over a division ring for some $n \in \mathbb{P}$. Note that $k * H$ is a division ring whenever H is a finite subgroup of G. By Theorem 4.4 $G_0(k * G) = \langle [k * G] \rangle$, and by [34, lemma 2.2] the inclusion $k * G \to k * GS^{-1}$ induces an epimorphism $G_0(k * G) \to G_0(k * GS^{-1})$. Therefore $G_0(k * GS^{-1}) = \langle [k * GS^{-1}] \rangle$ and we deduce that $n = 1$, i.e. $k * GS^{-1}$ is a division ring. The result follows. □

Another induction argument now gives the zero divisor conjecture for crossed products of torsion free solvable groups over right Noetherian domains; in fact it shows that if G is a torsion free solvable group, R is a right Ore domain and $R * G$ is a crossed product, then $R * G$ is also a right Ore domain. Roughly the argument goes as follows. To prove that $R * G$ is a right Ore domain, we may assume that G is finitely generated. Then there exists $H \lhd G$ such that G/H is finitely generated abelian-by-finite and H is "smaller" than G, so by induction we may assume that $R * F$ is a right Ore domain whenever F/H is a finite subgroup of G/H; let us say that $R * H$ is a right order in the division ring D. We now form the crossed product $D * [G/H]$, and since $D * [F/H]$ is a domain for all finite subgroups F/H of G/H, we deduce from Corollary 4.5 that $D * [G/H]$ is an Ore domain. It now follows easily that $R * G$ is a right Ore domain.

These arguments also apply to the case when G is an elementary amenable group. Recall that the class of elementary amenable groups, which we shall denote by $\mathcal{C}$, is the smallest class of groups which

(i) Contains all cyclic and all finite groups,
(ii) Is closed under taking group extension,
(iii) Is closed under directed unions.

Then $\mathcal{C}$ contains all solvable groups, and every elementary amenable group is amenable (see [48, 49] for much information on amenable groups). Then the arguments of above establish the following result.

Theorem 4.6. *Let $G \in \mathcal{C}$ and let R be a right Noetherian domain. If G is torsion free, then $R * G$ is a domain. In fact, $R * G$ is a right order in a division ring.*

More results along these lines can be found in [34].

Theorem 4.4 is very useful for Problem 3.1 and related problems. Whenever you can prove a conjecture related to zero divisors for a class of groups $\mathcal{D}$, then with the aid of Theorem 4.4, it is usually easy to prove it also for the class of groups $\mathcal{DC}$; an exception to this is Theorem 4.1.

Finally results of Lazard [35] imply that if p is an odd prime and G is the kernel of the natural epimorphism $\mathrm{GL}_n(\mathbb{Z}) \to \mathrm{GL}_n(\mathbb{Z}/p\mathbb{Z})$ (i.e. G is a congruence subgroup), then $\mathbb{Z}_p G$ is a domain (where $\mathbb{Z}_p$ denotes the p-adic integers; a similar result holds for $p = 2$). This is described in [23]; see also [46].

When proving the zero divisor conjecture and related problems, it seems in nearly all cases that one needs to not only show that the group ring is domain, but that it embeds in a division ring in some nice way. This is the case, for example, in Theorem 4.6.

We shall see that for the case $p = 2$, many of the above techniques are still relevant for Problem 3.1, but in the case $p > 2$, at least at the moment, they do not seem to be helpful and methods from Fourier analysis appear to be more useful.

5. Elementary Results and L^p-cohomology

If G is a group with torsion, say $g \neq 1 = g^n$ for some $g \in G$ and $n \in \mathbb{P}$, then $(1 + g + \cdots + g^{n-1})(1 - g) = 0$, so there are zero divisors. Thus the simplest nontrivial case to consider is when G is infinite cyclic, say $G = \langle x \rangle$ where x has infinite order. If $L = L^p(G)$, $C_0(G)$ or $\mathbb{C}G$, and $\alpha \in \mathbb{C}G$, let us say that α is a zero divisor in L if there exists $\beta \in L \backslash 0$ such that $\alpha\beta = 0$, and that α is a nonzero divisor in L if no such β exists.

Theorem 5.1. *Let $G = \langle x \rangle$ where x has infinite order, and let $\xi \in \mathbb{C}$ where $|\xi| = 1$. Then*

(i) *$x - \xi$ is a zero divisor in $L^\infty(G)$.*
(ii) *If $0 \neq \alpha \in \mathbb{C}G$, then α is a nonzero divisor in $C_0(G)$.*

Proof. (i) $(x - \xi) \sum_{n=-\infty}^{\infty} \xi^{-n} x^n = 0$.
(ii) Write $\alpha = cx^m(x - a_1) \dots (x - a_n)$ where $c, a_i \in \mathbb{C}$, $m \in \mathbb{Z}$, and $c \neq 0$. Then by induction on n, we may assume that $n = 1$, $m = 0$ and $c = 1$; in other words we may assume that $\alpha = x - a$ where $a \in \mathbb{C}$. Suppose $\alpha\beta = 0$ where $\beta \in C_0(G)$. Write $\beta = \sum_{n=-\infty}^{\infty} b_n x^n$ where $b_n \in \mathbb{C}$ for all n. Equating coefficients of x^{n+1}, we obtain $b_n = ab_{n+1}$ for all $n \in \mathbb{Z}$. Without loss of generality, we may assume that $|a| \leq 1$ and $b_1 \neq 0$. But then our equation on the coefficients yields $|b_n| \geq |b_1|$ for all $n \in \mathbb{P}$, which contradicts the hypothesis that $\beta \in C_0(G)$. □

Thus though we cannot expect Problem 3.1 to have an affirmative answer in the case $p = \infty$, it seems plausible that it may have an affirmative answer in all other cases (and also in the case when $L^p(G)$ is replaced by $C_0(G)$).

Let us give some motivation for the problem from L^p-cohomology. For more detailed information we refer the reader to [7, 10, 11] and [26, §8]. Let X be a simplicial complex on which G acts freely, let X_r denote the set of r-simplices of X, let $C_r(X)$ denote the free abelian group with basis X_r, and let $\partial_r \colon C_r(X) \to C_{r-1}(X)$ denote the boundary map. For simplicity, we shall assume that X_r has only finitely many orbits for each $r \in \mathbb{N}$. Now define

$$L^p(X_r) = \{f \colon X_r \to \mathbb{C} \mid \sum_{\sigma \in X_r} |f(\sigma)|^p < \infty\}.$$

Then $L^p(X_r)$ is a Banach space under the norm $\|f\| = \left(\sum_{\sigma \in X_r} |f(\sigma)|^p\right)^{1/p}$; in fact it is isomorphic to $L^p(G)^{d_r}$ where d_r is the number of orbits of X_r. The coboundary map $\delta_r \colon L^p(X_r) \to L^p(X_{r+1})$ which obeys the rule $(\delta_r f)\sigma = f(\partial_{r+1}\sigma)$ for all $\sigma \in X_{r+1}$, is clearly a well defined bounded linear operator on $L^p(X_r)$. Thus $\ker \delta_r$ is a closed subspace of $L^p(X_r)$, but $\operatorname{im} \delta_r$ need not be closed. We now define the L^p-cohomology groups by

$$l_p \, \bar{\mathrm{H}}^r(X) = \frac{\ker \delta_r}{\overline{\operatorname{im} \delta_{r-1}}}.$$

Since ∂_r commutes with the action of G, it follows that ∂_{r+1} is described by a $d_r \times d_{r+1}$ matrix all of whose entries are in $\mathbb{Z}G$, and δ_r is described by the transpose of this matrix. Therefore δ_r is described by a matrix all of whose entries are in $\mathbb{Z}G$.

To determine $l_p \bar{\mathrm{H}}^r(X)$, we need to know about $\ker \delta_r$ and in particular when it is nonzero. The simplest case is when δ_r is 1×1 matrix. Thus we have come up against the problem stated in Problem 3.1.

In the case of L^2-cohomology, we can exploit the fact that $L^2(G)$ is a Hilbert space (see Section 8). Let M_r denote the orthogonal complement of $\operatorname{im} \delta_{r-1}$ in $\ker \delta_r$. Then M_r is a closed subspace and also a $\mathbb{C}G$-submodule of $L^2(X_r)$. It follows that M has a well defined von Neumann dimension $\dim_G(M)$ (which will be described precisely in Section 11). Then for $r \in \mathbb{N}$, the L_2-Betti numbers are defined by $b^r_{(2)}(X : G) = \dim_G M_r$. In the case G is a group whose finite subgroups have bounded order, results from studying Problem 3.1 show for example, that if G has a normal subgroup F such that F is a direct product of free groups and G/F is elementary amenable, then $\operatorname{lcm}(G)\, b^r_{(2)}(X : G) \in \mathbb{N}$ for all $r \in \mathbb{N}$ and for all X.

6. The case $p > 2$ and G abelian

In view of Theorem 5.1, it seems surprising that the answer to Problem 3.1 is negative if G is a noncyclic abelian group and $p > 2$. The work of this section describes work of my research student Mike Puls.

Throughout this section $d \in \mathbb{P}$, and G is a finitely generated free abelian group of rank d. Let $\mathbb{T}$ denote the torus, which we will think of as $[-\pi, \pi]/\{-\pi \sim \pi\}$, and let $\mathbb{T}^d = \mathbb{T} \times \cdots \times \mathbb{T}$, the d-torus. We can view $\mathbb{T}$ as the abelian group $\mathbb{R}/2\pi\mathbb{Z}$, and then $\mathbb{T}^d$ is also a group. This means that we can talk about cosets in $\mathbb{T}^d$; a coset of $\mathbb{T}^d$ will mean a coset of the form Ht where $H \leqslant \mathbb{T}^d$ and $t \in \mathbb{T}^d$, and the coset will be proper if $H \neq \mathbb{T}^d$. Let $\{x_1, \dots, x_d\}$ be a $\mathbb{Z}$-basis for G. If $g = x_1^{n_1} \dots x_d^{n_d} \in G$ (where $n_i \in \mathbb{Z}$), then we can define the Fourier transform $\hat{g} \colon \mathbb{T}^d \to \mathbb{C}$ by

$$\hat{g}(t_1, \dots, t_d) = e^{i(n_1 t_1 + \cdots + n_d t_d)}$$

(where $t_i \in \mathbb{T}$). If $\alpha = \sum_{g \in G} \alpha_g g \in L^1(G)$, then we set

$$\hat{\alpha} = \sum_{g \in G} \alpha_g \hat{g} \colon \mathbb{T}^d \longrightarrow \mathbb{C},$$

and this extends the Fourier transform to $L^1(G)$. Set $Z(\alpha) = \{t \in \mathbb{T}^d \mid \hat{\alpha}(t) = 0\}$. Then Puls [52] proved the following result.

Theorem 6.1. *Suppose $\alpha \in L^1(G)$ and $Z(\alpha)$ is contained in a finite union of proper closed cosets. Then α is a nonzero divisor in $C_0(G)$.*

Let us indicate how this theorem is proved. If E is a closed subset of $\mathbb{T}^d$, then we define $I(E) = \{\beta \in L^1(G) \mid E \subseteq Z(\beta)\}$, $j(E)$ to be the set of all $\beta \in L^1(G)$ such that there exists an open subset O in $\mathbb{T}^d$ such that $E \subseteq O \subseteq Z(\beta)$, and $J(E)$ to be the closure of $j(E)$ in $L^1(G)$. Then $j(E) \subseteq J(E) \subseteq I(E)$, $J(E)$ and $I(E)$ are closed ideals in $L^1(G)$, and $\alpha \in I(Z(\alpha))$. We say that E is an *S-set* (or set of *spectral synthesis*) if $J(E) = I(E)$. We require the next result on the existence of S-sets, which follows from [55, Theorem 7.5.2], the remark just preceding that theorem, namely that C-sets are S-sets, and the remark immediately following that theorem, namely that C-sets are invariant under translation.

Proposition 6.2. *A finite union of closed cosets is an S-set.*

Define

$$\Phi(E) = \{h \in L^\infty(G) \mid \beta h = 0 \text{ for all } \beta \in I(E)\},$$
$$\Psi(E) = \{h \in L^\infty(G) \mid \beta h = 0 \text{ for all } \beta \in J(E)\}.$$

Then $\Phi(E) \subseteq \Psi(E)$ because $J(E) \subseteq I(E)$, and if F is a closed subset of E, then $\Psi(F) \subseteq \Psi(E)$. Now for $\alpha \in L^1(G)$, it follows from [55, Corollary 7.2.5a] that $J(Z(\alpha)) \subseteq \overline{\alpha L^1(G)}$, where $\overline{}$ denotes the closure in $L^1(G)$. Therefore if $h \in L^\infty(G)$, then

$$\alpha h = 0 \quad \text{implies} \quad h \in \Psi(Z(\alpha)). \tag{6.1}$$

We say that E is a set of *uniqueness* if $\Psi(E) \cap C_0(G) = 0$; clearly if E is a set of uniqueness and F is a closed subset of E, then F is also a set of uniqueness. It follows from (6.1) that if $Z(\alpha)$ is contained in a set of uniqueness, then α is a nonzero divisor in $C_0(G)$. Conversely if α is a nonzero divisor in $C_0(G)$ and $Z(\alpha)$ is an S-set, then $\Phi(Z(\alpha)) = \Psi(Z(\alpha))$ and we deduce that $Z(\alpha)$ is a set of uniqueness. Thus we have

Lemma 6.3. *Let $\alpha \in L^1(G)$.*

(i) *If $Z(\alpha)$ is contained in a set of uniqueness, then α is a nonzero divisor in $C_0(G)$.*
(ii) *If α is a nonzero divisor in $C_0(G)$ and $Z(\alpha)$ is an S-set, then $Z(\alpha)$ is a set of uniqueness.*

Proof of Theorem 6.1. For this proof, let us say that a hypercoset in $\mathbb{T}^d$ is a set of the form $Z(g - \xi)$ where $g \in G\backslash 1$, $\xi \in \mathbb{C}$ and $|\xi| = 1$. From [55, section 2.1], it is not difficult to see that every proper closed coset of $\mathbb{T}^d$ is contained in a hypercoset. Since $Z(\beta\gamma) = Z(\beta) \cup Z(\gamma)$ for $\beta, \gamma \in L^1(G)$, we see that any finite union of hypercosets in $\mathbb{T}^d$ is of the form $Z\bigl(\prod_i (g_i - \xi_i)\bigr)$ where $g_i \in G\backslash 1$, $\xi_i \in \mathbb{C}$ and $|\xi_i| = 1$.

If $1 \neq g \in G$ and $\xi \in \mathbb{C}$, then the same argument as in Theorem 5.1(ii) shows that $g - \xi$ is a nonzero divisor in $C_0(G)$. It follows that $\prod_i (g_i - \xi_i)$ is a nonzero divisor in $C_0(G)$ whenever $g_i \in G\backslash 1$ and $\xi_i \in \mathbb{C}$; the relevant case here is when $|\xi_i| = 1$ for all i. Using Proposition 6.2 and Lemma 6.3(ii), we see that any finite union of hypercosets is a set of uniqueness. Therefore α is a nonzero divisor in $C_0(G)$ by Lemma 6.3(i). □

Let us now describe Puls's proof that if $G \cong \mathbb{Z}^2$, then there exists $\alpha \in \mathbb{C}G\backslash 0$ which is a zero divisor in $L^q(G)$ for some $q < \infty$ (we shall consider the case $G \cong \mathbb{Z}^d$ where $d > 2$ later, where it will be seen that better values of q can be obtained). Let $\{x, y\}$ be a basis for G. For $i, j \in \mathbb{Z}$ and $\beta \in L^\infty(G)$, we shall write β_{ij} or $\beta_{i,j}$ for $\beta_{x^i y^j}$. Given a bounded measure μ on $\mathbb{T}^2$, we can define its Fourier transform $\tilde{\mu} \in L^\infty(G)$ by

$$\tilde{\mu} = \sum_{m,n \in \mathbb{Z}} \tilde{\mu}_{mn} x^m y^n \quad \text{where} \quad \tilde{\mu}_{mn} = \int_{\mathbb{T}^2} e^{-i(ms+nt)}\, d\mu(s,t).$$

Then we can state

Proposition 6.4. *Let $\alpha \in L^1(G)$ and let μ be a bounded measure on $\mathbb{T}^2$. If μ is concentrated on $Z(\alpha)$, then $\alpha\tilde{\mu} = 0$.*

Proof. We need to prove that $(\alpha\tilde{\mu})_{ij} = 0$ for all $i, j \in \mathbb{Z}$. Replacing α with $\alpha x^{-i}y^{-j}$, we see that it is sufficient to prove that $(\alpha\tilde{\mu})_1 = 0$. Now

$$\begin{aligned}(\alpha\mu)_1 &= \sum_{m,n} \alpha_{mn}\tilde{\mu}_{-m,-n} = \sum_{m,n} \alpha_{mn} \int_{\mathbb{T}^2} e^{i(ms+nt)} d\mu(s,t) \\ &= \int_{\mathbb{T}^2} \sum_{m,n} \alpha_{mn} e^{ims} e^{int} d\mu(s,t) = \int_{Z(\alpha)} \hat{\alpha}(s,t) d\mu(s,t) = 0,\end{aligned}$$

as required. □

Thus it is easy to construct zero divisors α in $L^p(G)$ by choosing a nonzero μ; all that we need to verify is that $\tilde{\mu} \in L^p(G)$. To make this verification, we require theorems from Fourier analysis. Let $a, b \in \mathbb{R}$ such that $-\pi \le a < b \le \pi$, and let $\alpha \in L^1(G)$. Suppose $Z(\alpha)$ contains $\{(t, \theta(t)) \mid a \le t \le b\}$ where $\theta\colon [a,b] \to [-\pi, \pi]$ is smooth (i.e. infinitely differentiable). Define a measure μ on $\mathbb{T}^2$ by $\int_{\mathbb{T}^2} f d\mu = \int_a^b f(t, \theta(t))\, dt$ for all measurable f. Then

$$\tilde{\mu}_{mn} = \int_a^b e^{-i(mt+n\theta(t))}\, dt$$

and $\tilde{\mu} \neq 0$ because $\tilde{\mu}_{00} = b - a$. What we need is that $\sum_{m,n\in\mathbb{Z}} \left|\tilde{\mu}_{mn}\right|^p < \infty$ for p large enough. This certainly will not be true in general, for example take $\theta = 0$. In fact if $\frac{d^2\theta}{dt^2}(t) = 0$ for all $t \in (a, b)$, then it is not difficult to see that $\tilde{\mu} \notin C_0(G)$. This is not surprising in view of Theorem 6.1, which in this case says that if $Z(\alpha)$ is contained in a finite union of lines with rational slope, then α is a nonzero divisor in $C_0(G)$. Let us assume that there exists $k \in \mathbb{P}$ such that for each $t \in [a, b]$, there exists $l \in \mathbb{P}$ such that $l \le k$ and $\frac{d^l\theta}{dt^l}(t) \neq 0$ (where l depends on t). We need the following result from Fourier analysis, for which we refer to [57, §8.3].

Proposition 6.5. *In the above situation, there exists $A \in \mathbb{R}$ such that $\left|\tilde{\mu}_{mn}\right| \le A(m^2 + n^2)^{-1/(2k)}$ for all $m, n \in \mathbb{Z}$.*

It now follows easily that if $p > 2k$, then $\sum_{m,n\in\mathbb{Z}} \left|\tilde{\mu}_{mn}\right|^p < \infty$ and hence $\tilde{\mu} \in L^p(G)$ for all $p > 2k$.

Example 6.6. *Let $\alpha = 2xy - x + y - 2 \in \mathbb{C}G$. Then α is a zero divisor in $L^p(G)$ for all $p > 4$.*

Proof. For $(s,t) \in \mathbb{T}^2$ (where $-\pi \le s, t \le \pi$), we have $\hat{\alpha}(s,t) = 2e^{is}e^{it} - e^{is} + e^{it} - 2$, thus $\hat{\alpha}(s,t) = 0$ when $e^{it} = \frac{e^{is}+2}{2e^{is}+1}$. Therefore $Z(\alpha) = \{(t, \theta(t)) \mid -\pi \le t \le \pi\}$ where $e^{i\theta(t)} = \frac{e^{it}+2}{2e^{it}+1}$ and we may write $\theta(t) = -i \log\left(\frac{e^{it}+2}{2e^{it}+1}\right)$, where we have taken the branch of log which satisfies $\log 1 = 0$. It is easily checked that θ is smooth and $\frac{d^2\theta}{dt^2}(t) \neq 0$ for all $t \in (-\pi, \pi)\backslash\{0\}$, in particular for all $t \in [\pi/4, 3\pi/4]$. As above, define a measure μ on $\mathbb{T}^2$ by $\int_{\mathbb{T}^2} f d\mu = \int_{\pi/4}^{3\pi/4} f(t, \theta(t))\, dt$ for all measurable f. We can now apply Proposition 6.5 with $a = \pi/4$ and $b = 3\pi/4$ to deduce that $\tilde{\mu} \in L^p(G)$ for all $p > 4$, and Proposition 6.4 to deduce that $\alpha\tilde{\mu} = 0$. Also $\tilde{\mu} \neq 0$, so we have shown that α is a zero divisor in $L^p(G)$ for all $p > 4$. □

It is interesting to actually compute $\tilde{\mu}$ explicitly, though in the above example this seems somewhat messy. We could define a measure ν on $\mathbb{T}^2$ by $\int_{\mathbb{T}^2} f d\nu = \int_{-\pi}^{\pi} f(t, \theta(t))\, dt$ for all measurable f, and then as above, $\tilde{\nu} \neq 0$ and $\alpha\tilde{\nu} = 0$. Since

$\frac{d^2\theta}{dt^2}(t) = 0$ when $t = 0$ or $\pm\pi$, we cannot assert from Proposition 6.5 that $\tilde{\nu} \in L^p(G)$ for $p > 4$, but we do have $\frac{d^3\theta}{dt^3}(t) \neq 0$ for $t = 0$ or $\pm\pi$, so we can assert that $\tilde{\nu} \in L^p(G)$ for all $p > 6$. We now determine $\tilde{\nu}_{mn}$, which is

$$\begin{aligned}\int_{-\pi}^{\pi} e^{-i(mt+n\theta(t))}\,dt &= \int_{-\pi}^{\pi} e^{-imt}\left(e^{-i\theta(t)}\right)^n\,dt\\ &= \int_0^{2\pi} e^{-imt}\left(\frac{2e^{it}+1}{e^{it}+2}\right)^n\,dt.\end{aligned}$$

For $m < 0$ and $n \geq 0$ contour integration shows that $\tilde{\nu}_{mn} = 0$, and then using the substitution $t \to -t$, we see that $\tilde{\nu}_{mn} = 0$ for $m > 0$ and $n \leq 0$. Also, $\nu_{00} = 2\pi$. Now the equality $\alpha\tilde{\nu} = 0$ yields $2\tilde{\nu}_{rs} - \tilde{\nu}_{r,s+1} + \tilde{\nu}_{r+1,s} - 2\tilde{\nu}_{r+1,s+1} = 0$, so we have a recurrence relation from which to calculate the other $\tilde{\nu}_{rs}$. This determines $\tilde{\nu}$ because $\tilde{\nu} = \sum_{r,s}\tilde{\nu}_{rs}x^r y^s$.

Of course, this argument can be generalized to the case $G = \mathbb{Z}^d$ where $d > 2$. To state Puls's results in this case, we need the concept of Gaussian curvature. We shall describe this here: for more details, see [57, §8.3]. Let S be a smooth $(d-1)$-dimensional submanifold of $\mathbb{R}^d$ and let $x_0 \in S$. Then after a change of coordinates (specifically a rotation), we may assume that in a sufficiently small open neighborhood of x_0, the surface is of the form $\{(x, \theta(x)) \mid x \in U\}$, where U is a bounded open subset of $\mathbb{R}^{d-1}$ and $\theta \colon U \to \mathbb{R}$ is a smooth function. Then we say that S has nonzero Gaussian curvature at x_0 if the $(d-1) \times (d-1)$ matrix

$$\left(\frac{\partial^2\theta}{\partial x_i \partial x_j}(x_0)\right)$$

is nonsingular. Then in [52], Puls proved the following.

Theorem 6.7. *Let $\alpha \in \mathbb{C}\mathbb{Z}^d$ where $2 \leq d \in \mathbb{P}$, and suppose there exists $x_0 \in Z(\alpha)$ such that there is a neighborhood S of x_0 in $Z(\alpha)$ which is a smooth $(d-1)$-dimensional manifold. If S has nonzero Gaussian curvature at x_0, then α is a zero divisor in $L^p(G)$ for all $p > \frac{2d}{d-1}$.*

He uses the above theorem to give the following set of examples of zero divisors in $L^p(G)$. Let G be the free abelian group of rank d and as before let $\{x_1, \dots, x_d\}$ be a $\mathbb{Z}$-basis for G. Let

$$\alpha = \frac{2d-1}{2} - \frac{1}{2}\sum_{i=1}^{d}(x_i + x_i^{-1}).$$

Then $\alpha \in \mathbb{C}G$ and $\hat{\alpha}(t_1, \dots, t_d) = \frac{2d-1}{2} - \sum_{i=1}^d \cos t_i$. In a neighborhood of $(0, \dots, 0, \pi/3)$, we have that $Z(\alpha)$ is of the form $\{(t, \theta(t)) \mid t \in U\}$, where U is a bounded open neighborhood of the origin in $\mathbb{R}^{d-1}$, $t = (t_1, \dots, t_{d-1})$, and $\theta(t) = \cos^{-1}(\frac{2d-1}{2} - \sum_{i=1}^{d-1}\cos t_i)$. A computation shows that the matrix $\left(\frac{\partial^2\theta(t)}{\partial t_i \partial t_j}\right)$ is nonsingular at $t = 0$, hence $Z(\alpha)$ has nonzero Gaussian curvature. Therefore α is a zero divisor in $L^p(G)$ for all $p > \frac{2d}{d-1}$.

Puls has also covered many other cases in [52], in which he requires the concept of the "type" of a manifold (see [57, §8.3.2]). Let us say that M is a hyperplane in $\mathbb{T}^d$ if there exists a hyperplane N in $\mathbb{R}^d$ such that $M = N \cap [-\pi, \pi]^d$. (We have been a little sloppy here: what we really mean is that we consider $\mathbb{T}^d$ as $[-\pi, \pi]^d$ with opposite faces identified, and let M' be the inverse image of M in $[-\pi, \pi]^d$. Then we say that M is a hyperplane to mean that M' is the intersection of a hyperplane in $\mathbb{R}^d$ with $[-\pi,\pi]^d$. Perhaps this is not a very good definition because for example, it allows points to be hyperplanes.) Then the results of [52] make it seem very likely that the following conjecture is true.

Conjecture 6.8. *Let G be a free abelian group of finite rank, and let $\alpha \in \mathbb{C}G$. Then α is a nonzero divisor in $L^p(G)$ for some $p \in \mathbb{P}$ (where $p > 2$) if and only if $Z(\alpha)$ is not contained in a finite union of hyperplanes. Furthermore if α is a zero divisor in $C_0(G)$, then α is a zero divisor in $L^p(G)$ for some $p < \infty$.*

7. The case $p > 2$ and G free

This section also describes work of Mike Puls. It will show that when $p > 2$ and G is a nonabelian free group, then the answer to Problem 3.1 is even more in the negative than in the case of G a noncyclic free abelian group of the last section.

Let G denote the free group of rank two on the generators $\{x, y\}$, let E_n denote the words of length n on $\{x, y\}$ in G, and let $e_n = |E_n|$. Thus $E_0 = \{1\}$, $E_1 = \{x, y, x^{-1}, y^{-1}\}$, $E_2 = \{x^2, y^2, x^{-2}, y^{-2}, xy, yx, x^{-1}y^{-1}, y^{-1}x^{-1}, xy^{-1}, y^{-1}x, x^{-1}y, yx^{-1}\}$ etc. It is well known that $e_n = 4 \cdot 3^{n-1}$ for all $n \in \mathbb{P}$. We shall let χ_n denote the characteristic function of E_n, i.e.

$$\chi_n = \sum_{g \in E_n} g \in \mathbb{C}G.$$

These elements of $\mathbb{C}G$ are often called radial functions and were studied in [12], which is where some of the ideas for what follows were obtained.

Let

$$\Theta = 1 - \frac{1}{3}\chi_2 + \frac{1}{3^2}\chi_4 + \cdots + \frac{1}{(-3)^n}\chi_{2n} + \cdots .$$

Then for $p > 2$,

$$\begin{aligned}\|\Theta\|_p^p &= 1 + \frac{e_1}{3^p} + \frac{e_2}{3^{2p}} + \cdots + \frac{e_n}{3^{np}} + \cdots \\ &= 1 + \frac{4}{3} \cdot 3^{-(p-1)} + \frac{4}{3} \cdot 3^{-2(p-1)} + \cdots + \frac{4}{3} \cdot 3^{-n(p-1)} + \cdots .\end{aligned}$$

This is a geometric series with ratio between successive terms $3^{-(p-1)}$, so it is convergent when $p - 1 > 1$. It follows that $\Theta \in L^p(G)$ for all $p > 2$.

We now set $\theta = \chi_1\Theta$ and show that $\theta = 0$. If $m \in \mathbb{P}$, $g \in E_m$, and $g = g_1g_2$ with $g_1 \in E_1$, then $g_2 \in E_{m-1} \cup E_{m+1}$. Furthermore there is exactly one choice for (g_1, g_2) if $g_2 \in E_{m-1}$, and exactly three if $g_2 \in E_{m+1}$. It follows for $n \in \mathbb{N}$ that $\theta_g = 0$ for $g \in E_{2n}$, and $\theta_g = \dfrac{1}{(-3)^n} + 3 \cdot \dfrac{1}{(-3)^{n+1}} = 0$ for $g \in E_{2n+1}$. Thus we have shown that χ_1 is a zero divisor in $L^p(G)$ for all $p > 2$.

Of course there are similar results for radial functions of free groups on more than two generators, and these are established in [53].

8. Group von Neumann Algebras

We saw in Section 6 and Theorem 6.7 that for $p > 2$, one can construct many elements in $\mathbb{C}G$ which are zero divisors in $L^p(G)$. The situation for $L^2(G)$ is different, and there is evidence that the following conjecture is true.

Conjecture 8.1. *Let G be a torsion free group. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^2(G)$, then $\alpha\beta \neq 0$.*

The reason for this is that $L^2(G)$ is a Hilbert space, whereas the spaces $L^p(G)$ are not (unless G is finite). Indeed $L^2(G)$ becomes a Hilbert space with inner product

$$\langle \sum_{g\in G} \alpha_g g, \sum_{h \in G} \beta_h h \rangle = \sum_{g\in G} a_g \bar{b}_g,$$

where $\bar{\ }$ denotes complex conjugation. This inner product satisfies $\langle ug, v\rangle = \langle u, vg^{-1}\rangle$ for all $g \in G$, so if U is a right $\mathbb{C}G$-submodule of $L^2(G)$, then so is $U^{\perp}$. In the case of right ordered groups, the argument of Theorem 4.1 can be extended to show (see [40, thèorem II])

Theorem 8.2. *Let $H \lhd G$ be groups such that G/H is right orderable. Suppose that nonzero elements of $\mathbb{C}H$ are nonzero divisors in $L^2(H)$. Then nonzero elements of $\mathbb{C}G$ are nonzero divisors in $L^2(G)$.*

Thus taking $H = 1$ in the above theorem, we immediately see that Problem 3.1 has an affirmative answer in the case G is right orderable.

As mentioned at the end of Section 4, a key ingredient in proving the classical zero divisor conjecture is to embed the group ring in a division ring in some nice way, and the same is true here. To accomplish this, we need the concept of the group von Neumann algebra of G.

The formula of (3.1) also yields a multiplication $L^2(G)\times L^2(G) \to L^\infty(G)$ defined by

$$\sum_{g\in G} \alpha_g g \sum_{g\in G} \beta_g g = \sum_{g\in G} (\sum_{x\in G} \alpha_{gx^{-1}}\beta_x) g.$$

Now $\mathbb{C}G$ acts faithfully and continuously by left multiplication on $L^2(G)$, so we may view $\mathbb{C}G \subseteq \mathcal{L}(L^2(G))$. Let $W(G)$ denote the group von Neumann algebra of G: thus by definition, $W(G)$ is the weak closure of $\mathbb{C}G$ in $\mathcal{L}(L^2(G))$. For $\theta \in \mathcal{L}(L^2(G))$, the following are standard facts.

(i) $\theta \in W(G)$ if and only if there exist $\theta_n \in \mathbb{C}G$ such that $\lim_{n\to\infty}\langle \theta_n u, v\rangle \to \langle \theta u, v\rangle$ for all $u, v \in L^2(G)$.
(ii) $\theta \in W(G)$ if and only if $(\theta u)g = \theta(ug)$ for all $g \in G$.

Another way of expressing (ii) above is that $\theta \in W(G)$ if and only if θ is a right $\mathbb{C}G$-map. Using (ii), we see that if $\theta \in W(G)$ and $\theta 1 = 0$, then $\theta g = 0$ for all $g \in G$ and hence $\theta\alpha = 0$ for all $\alpha \in \mathbb{C}G$. It follows that $\theta = 0$ and so the map $W(G) \to L^2(G)$ defined by $\theta \mapsto \theta 1$ is injective. Therefore the map $\theta \mapsto \theta 1$ allows us to identify $W(G)$ with a subspace of $L^2(G)$. Thus algebraically we have

$$\mathbb{C}G \subseteq W(G) \subseteq L^2(G).$$

It is not difficult to show that if $\theta \in L^2(G)$, then $\theta \in W(G)$ if and only if $\theta\alpha \in L^2(G)$ for all $\alpha \in L^2(G)$. For $\alpha = \sum_{g\in G} \alpha_g g \in L^2(G)$, define $\alpha^* = \sum_{g\in G} \bar{\alpha}_g g^{-1} \in L^2(G)$. Then for $\theta \in W(G)$, we have $\langle \theta u, v\rangle = \langle u, \theta^* v\rangle$ for all $u, v \in L^2(G)$; thus θ^* is the adjoint of the operator θ.

If $\theta = \sum_{g\in G} \theta_g g \in W(G)$, then we define the trace map $\operatorname{tr}_G \colon W(G) \to \mathbb{C}$ by $\operatorname{tr}_G \theta = \theta_1$. Then for $\theta, \phi \in W(G)$, we have $\operatorname{tr}_G(\theta+\phi) = \operatorname{tr}_G \theta + \operatorname{tr}_G \phi$, $\operatorname{tr}_G \theta^* = \bar{\operatorname{tr}}_G \theta$ (where the bar denotes complex conjugation), $\operatorname{tr}_G(\theta\phi) = \operatorname{tr}_G(\phi\theta)$, and $\operatorname{tr}_G \theta = \langle \theta 1, 1\rangle$. For $n \in \mathbb{P}$, this trace map extends to $\mathrm{M}_n(W(G))$ by setting $\operatorname{tr}_G \theta = \sum_{i=1}^n \theta_{ii}$ when $\theta \in \mathrm{M}_n(W(G))$ is a matrix with entries θ_{ij} in $W(G)$, and then $\operatorname{tr}_G \theta\phi =$

$\mathrm{tr}_G\, \phi\theta$ for $\phi \in \mathrm{M}_n(W(G))$. This will be more fully described in Section 11. An important property of the trace map is given by Kaplansky's theorem (see [42] and [38, proposition 9]) which states that if $e \in \mathrm{M}_n(W(G))$ is an idempotent and $e \neq 0$ or 1, then $\mathrm{tr}_G\, e \in \mathbb{R}$ and $0 < \mathrm{tr}_G\, e < n$.

At first glance, it seems surprising that $W(G)$ is useful for proving Conjecture 8.1 because if G contains an element of infinite order, then $W(G)$ contains uncountably many idempotents, so it is very far from being a domain. However it has a classical right quotient ring $U(G)$ which we shall now describe.

Let $\mathcal{U}$ denote the set of all closed densely defined linear operators [33, §2.7] considered as acting on the left of $L^2(G)$. These are maps $\theta\colon L \to L^2(G)$ where L is a dense linear subspace of $L^2(G)$ and the graph $\{(u, \theta u) \mid u \in L\}$ is closed in $L^2(G)^2$. The adjoint map * extends to $\mathcal{U}$ and for $\theta \in \mathcal{U}$, it satisfies $\langle \theta u, v\rangle = \langle u, \theta^* v\rangle$ whenever θu and $\theta^* v$ are defined. We now let $U(G)$ denote the operators in $\mathcal{U}$ "affiliated" to $W(G)$ [5, p. 150]; thus for $\theta \in \mathcal{U}$, we have $\theta \in U(G)$ if and only if $\theta(ug) = (\theta u)g$ for all $g \in G$ whenever θu is defined. Then $U(G) = U(G)^*$, $U(G)$ is a $*$-regular ring [4, definition 1, p. 229] containing $W(G)$, and every element of $U(G)$ can be written in the form $\gamma\delta^{-1}$ where $\gamma \in W(G)$ and δ is a nonzero divisor in $W(G)$ (see [5], especially theorem 1 and the proof of theorem 10). On the other hand, the trace map tr_G does not extend to $U(G)$. Now a $*$-regular ring R has the property that if $\alpha \in R$, then there exists a unique projection $e \in R$ (so e is an element satisfying $e = e^2 = e^*$) such that $\alpha R = eR$, in particular every element of R is either invertible or a zero divisor. Therefore we have embedded $W(G)$ into a ring in which every element is either a zero divisor or is invertible (so $U(G)$ is a classical right quotient ring for $W(G)$), and if $0 \neq \beta \in U(G)$, then $(\beta^*\beta)^n \neq 0$ for all $n \in \mathbb{N}$. Furthermore it is obvious that if γ is an automorphism of G, then γ extends in a unique way to automorphisms of $W(G)$ and $U(G)$. Given $\alpha \in L^2(G)$, we can define an element $\hat\alpha \in U(G)$ by setting $\hat\alpha u = \alpha u$ for all $u \in \mathbb{C}G$. Then $\hat\alpha$ is an unbounded operator on $L^2(G)$, densely defined because $\mathbb{C}G$ is a dense subspace of $L^2(G)$ (of course $\hat\alpha$ does not define an element of $\mathcal{L}(L^2(G))$ in general, because the product of two elements of $L^2(G)$ does not always lie in $L^2(G)$, only in $L^\infty(G)$). It is not difficult to show that $\hat\alpha$ extends to a closed operator on $L^2(G)$ (see the proof of Lemma 11.3), which we shall also call $\hat\alpha$. Thus $\hat\alpha$ is an element of $\mathcal{U}$. Since $\hat\alpha(ug) = (\hat\alpha u)g$ for all $u \in \mathbb{C}G$ and $g \in G$, and $\mathbb{C}G$ is dense in $L^2(G)$, it follows (cf. [33, remark 5.6.3]) that $\hat\alpha \in U(G)$. Thus we have a map $L^2(G) \to U(G)$ defined by $\alpha \mapsto \hat\alpha$ which is obviously an injection. Algebraically, we now have

$$\mathbb{C}G \subseteq W(G) \subseteq L^2(G) \subseteq U(G). \tag{8.1}$$

Similar properties to those of the above paragraph hold for matrix rings over $U(G)$. Let $n \in \mathbb{P}$. Then $\mathrm{M}_n(\mathbb{C}G)$ acts continuously by left multiplication on $L^2(G)^n$, and $\mathrm{M}_n(W(G))$ is the weak closure of $\mathrm{M}_n(\mathbb{C}G)$ in $\mathcal{L}(L^2(G)^n)$. Also $\mathrm{M}_n(U(G))$ is the set of closed densely defined linear operators acting on the left of $L^2(G)^n$ which are affiliated to $\mathrm{M}_n(W(G))$. For $\theta \in \mathrm{M}_n(U(G))$, the adjoint θ^* of θ satisfies $\langle \theta u, v\rangle = \langle u, \theta^* v\rangle$ for $u, v \in L^2(G)^n$ whenever θu and $\theta^* v$ are defined. If θ is represented by the matrix (θ_{ij}) where $\theta_{ij} \in U(G)$, then θ^* is represented by the matrix (θ^*_{ji}). Then $\mathrm{M}_n(U(G))$ is a $*$-regular ring containing $\mathrm{M}_n(W(G))$, and every element of $\mathrm{M}_n(U(G))$ can be written in the form $\alpha\beta^{-1}$ where $\alpha \in \mathrm{M}_n(W(G))$ and β is a nonzero divisor in $\mathrm{M}_n(W(G))$. Furthermore every projection of $\mathrm{M}_n(U(G))$ lies in $\mathrm{M}_n(W(G))$ (use [5, theorem 1]). This means that if $\alpha \in \mathrm{M}_n(U(G))$, then $\alpha\, \mathrm{M}_n(U(G)) = e\, \mathrm{M}_n(U(G))$ for a unique projection $e \in \mathrm{M}_n(W(G))$. Thus we can define $\operatorname{rank}_G \alpha = \operatorname{tr}_G e$; the following lemma (see [41, Lemma 2.3]) gives some easily derived properties of rank_G; part (ii) requires Kaplansky's theorem on the trace of idempotents mentioned earlier in this section.

Lemma 8.3. *Let G be a group and let $\theta \in \mathrm{M}_n(U(G))$. Then*

(i) $\operatorname{rank}_G \theta\alpha = \operatorname{rank}_G \theta = \operatorname{rank}_G \alpha\theta$ *for all* $\alpha \in \mathrm{GL}_n(U(G))$.
(ii) *If* $0 \neq \theta \notin \mathrm{GL}_n(U(G))$, *then* $0 < \operatorname{rank}_G \theta < n$.

Two other useful results are

Lemma 8.4. *(See* [39, lemma 13]*.) Let G be a group, let $n \in \mathbb{P}$, and let e, f be projections in $\mathrm{M}_n(U(G))$. If $f = ueu^{-1}$ for some unit $u \in \mathrm{M}_n(U(G))$, then $f = vev^{-1}$ for some unit $v \in \mathrm{M}_n(W(G))$.*

Lemma 8.5. *Let G be a group, let $n \in \mathbb{P}$, and let e, f be projections in $\mathrm{M}_n(U(G))$. Suppose that $e\,\mathrm{M}_n(U(G)) \cap f\,\mathrm{M}_n(U(G)) = 0$ and $e\,\mathrm{M}_n(U(G)) + f\,\mathrm{M}_n(U(G)) = h\,\mathrm{M}_n(U(G))$ where h is a projection in $\mathrm{M}_n(U(G))$. Then $\operatorname{tr}_G e + \operatorname{tr}_G f = \operatorname{tr}_G h$.*

Proof. This follows from the parallelogram law [4, §13]. Alternatively one could note that $e\,\mathrm{M}_n(W(G)) \cap f\,\mathrm{M}_n(W(G)) = 0$ and then apply [39, lemmas 11(i) and 12]. □

Suppose $d, n \in \mathbb{P}$, $H \leqslant G$ are groups such that $[G : H] = n$, and $\{x_1, \dots, x_n\}$ is a left transversal for H in G. Then as Hilbert spaces $L^2(G)^d = \bigoplus_{i=1}^n x_i L^2(H)^d$, hence we may view elements of $\mathcal{L}(L^2(G)^d)$ as acting on $\bigoplus_{i=1}^n x_i L^2(H)^d$ and we deduce that we have a monomorphism $\hat{}\colon \mathcal{L}(L^2(G)^d) \to \mathcal{L}(L^2(H)^{dn})$. It is not difficult to see that $\hat{}$ takes $\mathrm{M}_d(W(G))$ into $\mathrm{M}_{dn}(W(H))$, which yields the following result (cf. [2, (16) on p. 23])

Lemma 8.6. *Let $H \leqslant G$ be groups such that $[G : H] = n < \infty$, and let $d \in \mathbb{P}$. If $\theta \in \mathrm{M}_d(W(G))$, then $\operatorname{tr}_H \hat{\theta} = n \operatorname{tr}_G \theta$.*

We can now explain the usefulness of $U(G)$. Suppose we have proved Conjecture 8.1 for the torsion free group G. Then we have in particular that if $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \theta \in W(G)$, then $\alpha\theta \neq 0$. Since $U(G)$ is a classical right quotient ring for $W(G)$, it follows that α is invertible in $U(G)$. Thus in the special case that $\mathbb{C}G$ is a right order in a division ring (this will be the case when G is elementary amenable: see Theorem 4.6), we can deduce that there is a division ring D such that $\mathbb{C}G \subseteq D \subseteq U(G)$. This was exploited in [39] to obtain the following result.

Theorem 8.7. *Let G be a torsion free elementary amenable group. Then there exists a division ring D such that $\mathbb{C}G \subseteq D \subseteq U(G)$.*

Of course in the above theorem, D can be chosen so that $\mathbb{C}G$ is a right order in D, see Theorem 4.6. In view of this theorem, it seems plausible that the following conjecture is true.

Conjecture 8.8. *If G is a torsion free group, then there exists a division ring D such that $\mathbb{C}G \subseteq D \subseteq U(G)$.*

Note that the above conjecture implies Conjecture 8.1. Indeed if $0 \neq \alpha \in \mathbb{C}G$, then the above conjecture shows that α is invertible in $U(G)$, in particular $\alpha\beta \neq 0$ for all $\beta \in U(G)\backslash 0$. Then (8.1) shows that $\alpha\beta \neq 0$ for all $\beta \in L^2(G)\backslash 0$. Thus combining Theorems 8.2 and 8.7, we obtain the following.

Theorem 8.9. *Let $H \lhd G$ be groups where H is torsion free elementary amenable and G/H is right ordered. If $0 \neq \alpha \in \mathbb{C}G$ and $0 \neq \beta \in L^2(G)$, then $\alpha\beta \neq 0$.*

We conclude this section with an amusing example. Recall that the group G is of *exponential growth* (see eg. [48, p. 219]) if there is a finite subset C of G such that $\lim_{n\to\infty} |C^n|^{1/n} > 1$ (where C^n denotes the subset of G consisting of all products of at most n elements of C). We say G is exponentially bounded if it does not have exponential growth.

Example 8.10. *Let p be a prime, let $d \in \mathbb{P}$, and let H be an exponentially bounded residually finite p-group which can be generated by d elements. Write $H \cong F/K$ where F is the free group of rank d, and write $G = F/K'$. Then there exists a division ring D such that $\mathbb{C}G \subseteq D \subseteq U(G)$ and $\mathbb{C}G$ is a right order in D.*

Of course, any finite p-group will satisfy the hypothesis for H in the above example (provided that H can be generated by d-elements), but then G will be torsion free elementary amenable and we are back in the case of Theorem 8.7. However there exist infinite periodic groups satisfying the above hypothesis for H [21, 27]; also Grigorchuk has constructed examples of such groups. Now a finitely generated elementary amenable periodic group must be finite [48, §3.11], hence H and also G cannot be elementary amenable when H is infinite. On the other hand, if H is chosen to be a periodic group, then it is easy to see that G does not contain a subgroup isomorphic to a nonabelian free group.

Proof of Example 8.10. First we show that G is right orderable. Let $\{F_i/K \mid i \in \mathcal{I}\}$ be the family of normal subgroups in H of index a power of p, and set $L = \bigcap_{i\in\mathcal{I}} F_i'$. Then F/F_i' has a finite normal series whose factors are all isomorphic to $\mathbb{Z}$ [22, §4, lemma 5], thus by the remarks just before Theorem 4.1 we see that F/F_i' is right ordered and hence so is F/L. Now $K' \leqslant L \leqslant K$, so L/K' is right ordered and we deduce that G is right ordered (again, use the remarks on right ordered groups just before Theorem 4.1). It follows from Theorem 4.1 that $\mathbb{C}G$ is a domain.

Since G is exponentially bounded, G is amenable by [48, proposition 6.8]. Now [58] tells us that if k is a field and M is an amenable group such that kM is a domain, then kM is an Ore domain. Thus $\mathbb{C}G$ is a right order in a division ring D. Since nonzero elements in $\mathbb{C}G$ are nonzero divisors in $L^2(G)$ by Theorem 8.2, it follows that the inclusion of $\mathbb{C}G$ in $L^2(G)$ extends to a ring monomorphism of D into $U(G)$, and the result follows. □

9. Universal Localization

The next step is to extend Theorem 8.7 to other groups. Since "most" (but not all) nonelementary amenable groups contain a nonabelian free subgroup, it is plausible to consider nonabelian free groups next. Here we come up with the problem that although $\mathbb{C}G$ is a domain, it does not satisfy the Ore condition. Indeed if G is the free group of rank two on $\{x, y\}$, then the fact that $(x-1)\mathbb{C}G \cap (y-1)\mathbb{C}G = 0$ shows that $\mathbb{C}G$ does not satisfy the Ore condition. If R is a subring of the ring S, the division closure [15, exercise 7.1.4, p. 387] of R in S, which we shall denote by $D(R, S)$, is the smallest subring of S containing R which is closed under taking inverses (i.e. $s \in D(R, S)$ and $s^{-1} \in S$ implies $s^{-1} \in D(R, S)$); perhaps a better concept is the closely related one of "rational closure" [15, p. 382], but division closure will suffice for our purposes. The division closure of $\mathbb{C}G$ in $U(G)$ will be indicated by $D(G)$. Obviously if R is an Artinian ring, then $D(R,S) = R$. In the case S is a division ring, the division closure of R is simply the smallest division subring of S containing R; thus Conjecture 8.8 could be restated as $D(G)$ is a division ring whenever G is torsion free. The following four elementary lemmas are very useful.

Lemma 9.1. *Let $R \subseteq S$ be rings, let D denote the division closure of R in S, and let $n \in \mathbb{P}$. If D is an Artinian ring, then $\mathrm{M}_n(D)$ is the division closure of $\mathrm{M}_n(R)$ in $\mathrm{M}_n(S)$.*

Proof. Exercise, or see [41, lemma 4.1]. $\square$

Lemma 9.2. *Let G be a group and let α be an automorphism of G. Then $\alpha D(G) = D(G)^* = D(G)$.*

Proof. Of course, here we have regarded α as an automorphism of $U(G)$, and * as an antiautomorphism of $U(G)$; see Section 8. The result follows because $\alpha\mathbb{C}G = \mathbb{C}G^* = \mathbb{C}G$. $\square$

Lemma 9.3. *(cf.* [41, lemma 2.1]*.) Let $H \lhd G$ be groups, and let $D(H)G$ denote the subring of $D(G)$ generated by $D(H)$ and G. Then $D(H)G \cong D(H) * G/H$ for a suitable crossed product.*

Proof. Let T be a transversal for H in G. Since $h \hookrightarrow tht^{-1}$ is an automorphism of H, we see that $tD(H)t^{-1} = D(H)$ for all $t \in T$ by Lemma 9.2, and so $D(H)G = \sum_{t\in T} D(H)t$. This sum is direct because the sum $\sum_{t\in T} U(H)t$ is direct, and the result is established. $\square$

Lemma 9.4. *Let $H \lhd G$ be groups such that G/H is finite, and suppose $D(H)$ is Artinian. Then $D(G)$ is semisimple Artinian and is a crossed product $D(H)*G/H$.*

Proof. Let $D(H)G$ denote the subring generated by $D(H)$ and G. Then Lemma 9.3 shows that $D(H)G \cong D(H) * G/H$, hence $D(H)G$ is Artinian and we deduce that $D(H)G = D(G)$. Now $D(G) = D(G)^*$ by Lemma 9.2 and if $0 \neq \alpha \in D(G)$, then $(\alpha^*\alpha)^n \neq 0$ for all $n \in \mathbb{N}$. Therefore $D(G)$ has no nonzero nilpotent ideals, and the result follows. $\square$

More generally for $n \in \mathbb{P}$, we denote the division closure of $\mathrm{M}_n(\mathbb{C}G)$ in $\mathrm{M}_n(U(G))$ by $D_n(G)$, and let $W_n(G) = D_n(G) \cap \mathrm{M}_n(W(G))$. Then we have (see [41, proposition 5.1])

Proposition 9.5. *Let G be a group and let $n \in \mathbb{P}$. Then*

(i) *If e is an idempotent in $D_n(G)$, then there exists $\alpha \in \mathrm{GL}_1(D_n(G))$ such that $eD_n(G) = \alpha e D_n(G)$ and $\alpha e \alpha^{-1}$ is a projection; in particular $eD_n(G) = fD_n(G)$ for some projection $f \in D_n(G)$.*
(ii) *If $\alpha \in D_n(G)$, then there exists a nonzero divisor $\beta \in W_n(G)$ such that $\beta\alpha \in W_n(G)$.*

The following result shows that if $D(G)$ is Artinian, then there is a bound on the length of a descending chain of right ideals in $D(G)$ in terms of the real numbers $\mathrm{tr}_G\, e$ for e a projection in $D(G)$.

Lemma 9.6. *Let G be a group and let $l \in \mathbb{P}$. Suppose that $D(G)$ is Artinian and that $l\,\mathrm{tr}_G\, e \in \mathbb{Z}$ for all projections $e \in D(G)$. If $I_0 > I_1 > \cdots > I_r$ is a strictly descending sequence of right ideals in $D(G)$, then $r \leq l$.*

Proof. Since $D(G)$ is semisimple Artinian by Lemma 9.4, the descending sequence of right ideals yields nonzero right ideals $J_1, \dots, J_r$ of $D(G)$ such that $D(G) = J_1 \oplus \cdots \oplus J_r$. Write $1 = e_1 + \cdots + e_r$ where $e_i \in J_i$. Then $e_i^2 = e_i$ and $e_i e_j = 0$ for $i \neq j$ $(1 \leq i, j \leq r)$, hence

$$U(G) = e_1 U(G) \oplus \cdots \oplus e_r U(G).$$

In view of Proposition 9.5(i), there exist nonzero projections $f_i \in D(G)$ such that $e_i D(G) = f_i D(G)$ $(1 \leq i \leq r)$. Then $e_i U(G) = f_i U(G)$ and it now follows from Lemma 8.5 that $1 = \operatorname{tr}_G f_1 + \cdots + \operatorname{tr}_G f_r$, upon which an application of Kaplansky's theorem (see Section 8) completes the proof. □

When constructing the classical right quotient ring of a ring D which satisfies the right Ore condition, one only inverts the nonzero divisors of D, but for more general rings it is necessary to consider inverting matrices. For any ring homomorphism f, we shall let f also denote the homomorphism induced by f on all matrix rings. Let Σ be any set of square matrices over a ring R. Then in [15, §7.2], Cohn constructs a ring R_Σ and a ring homomorphism $\lambda \colon R \to R_\Sigma$ such that the image of any matrix in Σ under λ is invertible. Furthermore R_Σ and λ have the following universal property: given any ring homomorphism $f \colon R \to S$ such that the image of any matrix in Σ under f is invertible, then there exists a unique ring homomorphism $\bar{f} \colon R_\Sigma \to S$ such that $\bar{f}\lambda = f$. The ring R_Σ always exists and is unique up to isomorphism, though in general λ is neither injective nor surjective. It obviously has the following useful property: if θ is an automorphism of R such that $\theta(\Sigma) = \Sigma$, then θ extends in a unique way to an automorphism of R_Σ.

Note that if R is a subring of the ring S, $D = D(R, S)$, and Σ is the set of matrices with entries in R which become invertible over D, then the inclusion $R \hookrightarrow D$ extends to a ring homomorphism $R_\Sigma \to D$. However even in the case D is a division ring, this homomorphism need not be an isomorphism.

A notable feature of the above construction of R_Σ, which is developed by Cohn in [15, §7] and Schofield in [56], is that it extends much of the classical theory of localization of Noetherian (noncommutative) rings to arbitrary rings. Indeed if S is a subset of R which contains 1, is closed under multiplication, and satisfies the Ore condition, then $RS^{-1} \cong R_S$. On the other hand, in general it is not possible to write every element of R_Σ in the form rs^{-1} with $r, s \in R$.

There are "Goldie rank" versions of Conjecture 8.8. If k is a field, G is polycyclic-by-finite, and $\Delta^+(G) = 1$, then kG is a right order in a $d \times d$ matrix ring for some $d \in \mathbb{P}$. The Goldie rank conjecture states that $d = \operatorname{lcm}(G)$. This is now known to be true, and extensions of this were considered in [34]; in particular it was proved that if k is a field, G is an elementary amenable group with $\Delta^+(G) = 1$, and the orders of the finite subgroups of G are bounded, then kG is a right order in an $l \times l$ matrix ring over a division ring where $l = \operatorname{lcm}(G)$ [34, theorem 1.3]. The proof of this depends heavily on Moody's Theorem, as described in Theorem 4.4. We describe two versions of the Goldie rank conjecture.

Conjecture 9.7. *Let G be a group such that $\Delta^+(G) = 1$, and let Σ denote the matrices with entries in $\mathbb{C}G$ which become invertible over $D(G)$. Suppose the orders of the finite subgroups of G are bounded, and $l = \operatorname{lcm}(G)$. Then there is a division ring D such that $D(G) \cong \mathrm{M}_l(D) \cong \mathbb{C}G_\Sigma$.*

Conjecture 9.8. *Let G be a group such that the orders of the finite subgroups of G are bounded, and let $l = \operatorname{lcm}(G)$. If $n \in \mathbb{P}$ and $\alpha \in \mathrm{M}_n(\mathbb{C}G)$, then $l \operatorname{rank}_G \alpha \in \mathbb{N}$.*

10. C*-algebra Techniques

There is a close connection between problems related to zero divisors in $L^2(G)$ and projections in $W(G)$. Indeed Lemma 12.3 states that if $\operatorname{rank}_G \theta \in \mathbb{Z}$ for all $\theta \in \mathrm{M}_n(\mathbb{C}G)$ and for all $n \in \mathbb{P}$, then Conjecture 8.8 is true, and of course $\operatorname{rank}_G \theta$ is defined in terms of the trace of a projection in $\mathrm{M}_n(W(G))$ (Section 8). Recall that the *reduced group C*-algebra* $C^*_r(G)$ of G is the strong closure (as opposed to the weak closure for $W(G)$) of $\mathbb{C}G$ in $\mathcal{L}(L^2(G))$: thus $\mathbb{C}G \subseteq C^*_r(G) \subseteq W(G)$. There is a conjecture going back to Kaplansky and Kadison that if G is a torsion free group, then $C^*_r(G)$ has no idempotents except 0 and 1 (this is equivalent to $C^*_r(G)$ having no projections except 0 and 1). The special case G is a nonabelian free group is of particular interest, because at one time there was an open problem to as whether a simple C*-algebra was generated by its projections. Powers [51, theorem 2] proved that $C^*_r(G)$ is simple for G a nonabelian free group, so it was then sufficient to show that $C^*_r(G)$ had no nontrivial projections, but this property turned out to be more difficult to prove. However Pimsner and Voiculescu [50] established this property, thus obtaining a simple C*-algebra ($\neq \mathbb{C}$) with no nontrivial projections. Connes [16, §1] (see [20] for an exposition) gave a very elegant proof of the Pimsner-Voiculescu result, and his method was used in [41] to establish Conjecture 8.8 in the case G is a free group. For further information on this topic, see the survey article [59].

As has already been remarked, in view of Lemma 12.3 we want to prove that $\operatorname{tr}_G e \in \mathbb{Z}$ for certain projections e. Now in his proof that $C^*_r(G)$ has no nontrivial projections, this is exactly what Connes does. Once it is established that $\operatorname{tr}_G e \in \mathbb{Z}$, then the result follows from Kaplansky's theorem (§8). Of course Connes is dealing with projections in $C^*_r(G)$, while we are interested in projections which are only given to lie in $\mathrm{M}_n(W(G))$ for some $n \in \mathbb{P}$, but the Connes argument is still applicable. Connes uses a Fredholm module technique in which he constructs a "perturbation" π of $C^*_r(G)$ where G is the free group of rank two such that if $C^*_r(G)$ has a nontrivial projection, then there is a nontrivial projection $e \in C^*_r(G)$ such that the operator $e - \pi e$ on $L^2(G)$ is of trace class (though $\pi e \notin C^*_r(G)$), and it follows that the trace of $e - \pi e$ is an integer [20, lemma 4.1]. He then shows that this trace is in fact $\operatorname{tr}_G e$ [20, §5], thus proving that $\operatorname{tr}_G e \in \mathbb{Z}$ as required.

To apply Lemma 12.3 when G is the free group of rank two, we use the same perturbation π. This has the property that if $\theta \in \mathrm{M}_n(\mathbb{C}G)$ for some $n \in \mathbb{P}$, then the resulting operators $\theta, \pi(\theta)$ on $L^2(G)^n$ agree on a subspace of finite codimension. It follows that if e, e' are the projections of $L^2(G)^n$ onto $\overline{\operatorname{im}\theta}, \overline{\operatorname{im}\theta'}$ respectively, then $\operatorname{im}(e - e')$ has finite dimension and therefore has a well defined trace which is an integer. Then as in the previous paragraph, this integer turns out to be $\operatorname{tr}_G e$ and we deduce that $\operatorname{rank}_G \theta \in \mathbb{Z}$ as required.

The same arguments are applied in Lemma 12.2 for the case when G is a finite direct product of free groups of rank two. For the purposes of trying to extend this to other classes of groups, it seems necessary to have that $\theta, \pi(\theta)$ agree on a subspace of finite codimension: it is not enough for $\theta - \pi(\theta)$ to have trace class, because this does not imply that $e - e'$ has trace class.

To construct the perturbation π, Connes uses the following result for free groups (see [20, section 4], [32, corollary 1.5], [18, §3]). We say that a function $\phi\colon X \to Y$ between the left G-sets X and Y is an almost G-map if for all $g \in G$, the set $\{x \in X \mid g(\phi x) \neq \phi(gx)\}$ is finite.

Theorem 10.1. *Let $\kappa \in \mathbb{N}$, let G be a free group of rank κ, let κG denote the free left G-set with κ orbits, and let $\{*\}$ denote the G-set consisting of one fixed point. Then there exists a bijective almost G-map $\phi\colon G \to \kappa G \cup \{*\}$.*

In fact the above is the only property of free groups that Connes uses, and it is also the only property of free groups used in [41] in establishing Conjecture 8.8 for G a free group. Thus it was of considerable interest to determine which other groups satisfied the conclusion of the above theorem. However Dicks and Kropholler [18] showed that free groups were the only such groups.

After proving Conjecture 8.8 for free groups, the following was established in [41] (see [41, theorem 1.5] for a generalization).

Theorem 10.2. *Let $n \in \mathbb{P}$, let $F \lhd G$ be groups such that F is free, G/F is elementary amenable, and $\Delta^+(G) = 1$, and let $D_n(G)$ denote the division closure of $\mathrm{M}_n(\mathbb{C}G)$ in $\mathrm{M}_n(U(G))$. Assume that the finite subgroups of G have bounded order, and that $l = \mathrm{lcm}(G)$. Then there exists a division ring D such that $D_n(G) \cong \mathrm{M}_{ln}(D)$.*

Of course the special case $l = n = 1$ in the above theorem yields Conjecture 8.8 for groups G which have a free subgroup F such that G/F is elementary amenable. The subsequent sections will be devoted to a proof of the following result.

Theorem 10.3. *Let $F \lhd G$ be groups, and let Σ denote the set of matrices with entries in $\mathbb{C}G$ which become invertible over $D(G)$. Suppose F is a direct product of free groups, G/F is elementary amenable, and the orders of the finite subgroups of G are bounded. Then $D(G)$ is a semisimple Artinian ring and the identity map on $\mathbb{C}G$ extends to an isomorphism $\mathbb{C}G_\Sigma \to D(G)$. Furthermore if $e \in D(G)$ is a projection, then $\mathrm{lcm}(G)\,\mathrm{tr}_G\, e \in \mathbb{Z}$ for all projections $e \in D(G)$.*

It seems very plausible that it is easy to extend the above theorem to the case when F is a subgroup of a direct product of free groups, and it certainly would be nice to establish this stronger result. However subgroups of direct products can cause more difficulty than one might intuitively expect, see for example [3]. In fact if $H \lhd G$ are groups such that G is torsion free, G/H is finite, and H is a subgroup of a direct product of free groups, then it is even unknown whether $\mathbb{C}G$ is a domain.

One can easily read off a number of related results from Theorem 10.3, for example

Corollary 10.4. *Let $F \lhd G$ be groups such that F is a direct product of free groups and G/F is elementary amenable, let $n \in \mathbb{P}$, and let $D_n(G)$ denote the division closure of $\mathrm{M}_n(\mathbb{C}G)$ in $\mathrm{M}_n(U(G))$. Suppose $\Delta^+(G) = 1$ and the orders of the finite subgroups of G are bounded, and set $l = \mathrm{lcm}(G)$. Then $D_n(G) \cong \mathrm{M}_{ln}(D)$ for some division ring D.*

For further recent information on these analytic techniques, especially in the case G is a free group, we refer the reader to the survey article [28].

11. $L^2(G)$-modules

We define $\mathbb{E} = \mathbb{N} \cup \{\infty\}$, where ∞ denotes the first infinite cardinal. Let G be a group, and let $L^2(G)^\infty$ denote the Hilbert direct sum of ∞ copies of $L^2(G)$, so $L^2(G)^\infty$ is a Hilbert space. Following [10, section 1], an $L^2(G)$-module $\mathcal{H}$ is a closed right $\mathbb{C}G$-submodule of $L^2(G)^n$ for some $n \in \mathbb{E}$, an $L^2(G)$-submodule of $\mathcal{H}$ is a closed right $\mathbb{C}G$-submodule of $\mathcal{H}$, an $L^2(G)$-ideal is an $L^2(G)$-submodule of $L^2(G)$, and an $L^2(G)$-homomorphism or $L^2(G)$-map $\theta\colon \mathcal{H} \to \mathcal{K}$ between $L^2(G)$-modules is a continuous right $\mathbb{C}G$-map. If X is an $L^2(G)$-ideal, then $X^\perp$ is also an $L^2(G)$-ideal, so $L^2(G) = X \oplus X^\perp$ as $L^2(G)$-modules. The following lemma shows that there can be no ambiguity in the meaning of two $L^2(G)$-modules being isomorphic.

Lemma 11.1. *Let $\mathcal{H}$ and $\mathcal{K}$ be $L^2(G)$-modules, and let $\theta\colon \mathcal{H} \to \mathcal{K}$ be an $L^2(G)$-map. If $\ker\theta = 0$ and $\overline{\operatorname{im}\theta} = \mathcal{K}$, then there exists an isometric $L^2(G)$-isomorphism $\phi\colon \mathcal{H} \to \mathcal{K}$.*

Proof. See [10, p. 134] and [45, §21.1]. □

Lemma 11.2. *If U is an $L^2(G)$-ideal, then $U = uL^2(G)$ for some $u \in U$.*

Proof. Let e be the projection of $L^2(G)$ onto U. Then $e \in W(G)$ because U is a right $\mathbb{C}G$-module, and $eL^2(G) = U$. Thus $e1 \in U$ and we may set $u = e1$. □

Lemma 11.3. *Let $n \in \mathbb{E}$, let $u \in L^2(G)^n$, and let $U = \overline{u\mathbb{C}G}$. Then U is $L^2(G)$-isomorphic to an $L^2(G)$-ideal.*

Proof. Define an unbounded operator $\theta\colon L^2(G) \to U$ by $\theta\alpha = u\alpha$ for all $\alpha \in \mathbb{C}G$. Suppose $\alpha_n \in \mathbb{C}G$, $\alpha_n \to 0$ and $\theta\alpha_n \to v$ where $v \in U\backslash 0$. Choose a standard basis element $w = (0, 0, \dots, 0, g, 0, \dots) \in L^2(G)^n$ where $g \in G$ such that $\langle v, w\rangle \neq 0$. Then

$$\langle v, w\rangle = \lim_{n\to\infty} \langle u\alpha_n, w\rangle = \lim_{n\to\infty} \langle u, w\alpha_n^*\rangle = 0,$$

a contradiction. Therefore θ extends to a closed operator, which we shall also call θ (see [33, p. 155]). Note that $\operatorname{im}\theta$ is dense in U. Using [45, §21.1, II], we may write θ uniquely in the form $\phi\psi$ where ψ is a self adjoint unbounded operator on $L^2(G)$ and $\phi\colon L^2(G) \to U$ is a partial isometry. Since θ is a right $\mathbb{C}G$-map, we see from the uniqueness of the factorization of θ that ϕ (and ψ) is also a right $\mathbb{C}G$-map. Thus ϕ induces an $L^2(G)$-isomorphism from an $L^2(G)$-ideal onto U, as required. □

We shall say that an $L^2(G)$-module $\mathcal{H}$ is finitely generated if there exist $n \in \mathbb{P}$ and $u_1, \dots, u_n \in \mathcal{H}$ such that $u_1\mathbb{C}G + \dots + u_n\mathbb{C}G$ is dense in $\mathcal{H}$. Obviously if $\mathcal{H}$ and $\mathcal{K}$ are finitely generated, then so is $\mathcal{H} \oplus \mathcal{K}$. The next lemma gives an alternative description of this definition.

Lemma 11.4. *Let $\mathcal{H}$ be an $L^2(G)$-module. Then $\mathcal{H}$ is finitely generated if and only if $\mathcal{H}$ is isomorphic to an $L^2(G)$-submodule of $L^2(G)^n$ for some $n \in \mathbb{P}$, and in this case there exist $L^2(G)$-ideals $I_1, \dots, I_n$ such that $\mathcal{H} \cong I_1 \oplus \dots \oplus I_n$.*

Proof. First suppose that $\mathcal{H}$ is isomorphic to an $L^2(G)$-submodule of $L^2(G)^n$ where $n \in \mathbb{P}$. Write $L^2(G)^n = U \oplus V$ where $U \cong L^2(G)$, $V \cong L^2(G)^{n-1}$, and $U \perp V$. Let W be the orthogonal complement to $U \cap \mathcal{H}$ in $\mathcal{H}$, and let π be the projection of $L^2(G)^n$ onto V. Then the restriction of π to W is an $L^2(G)$-monomorphism, so by Lemma 11.1 W is isomorphic to an $L^2(G)$-submodule of V. Using induction, we may assume that W is finitely generated and isomorphic to a finite direct sum of $L^2(G)$-ideals. But $\mathcal{H} = U \cap \mathcal{H} \oplus W$ and $U \cap \mathcal{H}$ is finitely generated by Lemma 11.2, so $\mathcal{H}$ is finitely generated and isomorphic to a finite direct sum of $L^2(G)$-ideals.

Now suppose $\mathcal{H}$ is finitely generated, say $u_1\mathbb{C}G + \dots + u_n\mathbb{C}G$ is dense in $\mathcal{H}$. Let $U = \overline{u_1\mathbb{C}G}$, let $V = U^\perp$, and for $i = 2, \dots, n$, write $u_i = u_i' + v_i$ where $u_i' \in U$ and $v_i \in V$. Then $\mathcal{H} = U \oplus V$, $v_2\mathbb{C}G + \dots + v_n\mathbb{C}G$ is dense in V, and U is isomorphic to an $L^2(G)$-ideal by Lemma 11.3. Using induction on n, we may assume that V

is isomorphic to an $L^2(G)$-submodule of $L^2(G)^{n-1}$ for some $n \in \mathbb{P}$, and the result follows. $\square$

Lemma 11.5. *Let U, V and W be $L^2(G)$-modules. If $U \oplus W$ is finitely generated and $U \oplus W \cong V \oplus W$, then $U \cong V$.*

Proof. Since $U \oplus W$ is finitely generated, Lemma 11.4 shows we may assume that $U \oplus W$ is an $L^2(G)$-submodule of $L^2(G)^n$ where $n \in \mathbb{P}$. Using $U \oplus W \cong V \oplus W$, we may assume that $U \oplus W = V \oplus W_1$ where $W \cong W_1$. If X is the orthogonal complement of $U \oplus W$ in $L^2(G)^n$, then

$$U \oplus (W \oplus X) = L^2(G)^n = V \oplus (W_1 \oplus X)$$

and we need only consider the case $X = 0$.

Thus we have $U \oplus W = L^2(G)^n = V \oplus W_1$ where $W \cong W_1$. Let e and f denote the projections of $L^2(G)^n$ onto W and W_1 respectively, and let $\theta \colon W \to W_1$ be an isometric $L^2(G)$-isomorphism. Then $e, f \in \mathrm{M}_n(W(G))$ because W and W_1 are $L^2(G)$-submodules. Since

$$U \oplus L^2(G)^n = U \oplus V \oplus W_1 \cong V \oplus U \oplus W = V \oplus L^2(G)^n,$$

there is an isometric $L^2(G)$-isomorphism $\phi \colon U \oplus L^2(G)^n \to V \oplus L^2(G)^n$. If $\psi = \theta \oplus \phi$, then

$$\psi \colon L^2(G)^{2n} \to L^2(G)^{2n}$$

is a unitary operator which is also a right $\mathbb{C}G$-map, so ψ can be considered as an element of $\mathrm{M}_{2n}(W(G))$. Set

$$E = \operatorname{diag}(e, 0_n) \quad \text{and} \quad F = \operatorname{diag}(f, 0_n).$$

Then E and F are projections in $\mathrm{M}_{2n}(W(G))$ and $\psi E \psi^{-1} = F$, so E and F are equivalent [4, definition 5, §1]. By [4, proposition 8, §1],

$$\operatorname{diag}(e, 1_n) \quad \text{and} \quad \operatorname{diag}(f, 1_n)$$

are also equivalent projections. Now $\mathrm{M}_n(W(G))$ is a finite von Neumann algebra [4, definition 1, §15], [38, proposition 9], and satisfies "GC" [4, corollary 1, §14], so we may apply [4, proposition 4, §17] twice to deduce that

$$1 - \operatorname{diag}(e, 1_n) \quad \text{and} \quad 1 - \operatorname{diag}(f, 1_n)$$

are equivalent projections, and hence unitarily equivalent projections. Therefore

$$(1 - e)L^2(G)^n \cong (1 - f)L^2(G)^n$$

and the result follows. $\square$

Lemma 11.6. *Let $\mathcal{H} = L^2(G)^\infty$, let U be a finitely generated $L^2(G)$-submodule of $\mathcal{H}$, and let $V = U^\perp$. Then $V \cong \mathcal{H}$.*

Proof. Using Lemma 11.4 and induction, we may assume that U is isomorphic to an $L^2(G)$-ideal. Write $\mathcal{H} = L_1 \oplus L_2 \oplus \cdots$ where $L_i \cong L^2(G)$ for all $i \in \mathbb{P}$, let $M_n = \bigoplus_{i=1}^n L_i$, let X_n denote the orthogonal complement of $V \cap M_n$ in M_n, let T_n denote the orthogonal complement of $V \cap M_n$ in $V \cap M_{n+1}$ ($n \in \mathbb{P}$), and let π denote the projection of $\mathcal{H}$ onto U. Since $X_n \cap (V \cap M_n) = 0$ and $X_n \subseteq M_n$, we see that $X_n \cap V = 0$, hence the restriction of π to X_n is an $L^2(G)$-monomorphism and we deduce from Lemma 11.1 that X_n is isomorphic to an $L^2(G)$-submodule of

U. Therefore we may write $X_n \oplus Y_n \cong L^2(G)$ for some $L^2(G)$-ideal Y_n ($n \in \mathbb{P}$). We now have

$$\begin{aligned} V \cap M_n \oplus T_n \oplus X_{n+1} = M_{n+1} &= V \cap M_n \oplus X_n \oplus L_{n+1} \\ &\cong V \cap M_n \oplus X_n \oplus X_{n+1} \oplus Y_{n+1}, \end{aligned}$$

thus by Lemma 11.5 we obtain $T_n \cong X_n \oplus Y_{n+1}$, so we may write $T_n = X'_n \oplus Y'_{n+1}$ where $X_n \cong X'_n$ and $Y_n \cong Y'_n$ ($n \in \mathbb{P}$). For $n \in \mathbb{P}$, set $F_n = V \cap M_n \oplus X'_n$. Then $F_n \subseteq F_{n+1}$, so we may define E_{n+1} to be the orthogonal complement of F_n in F_{n+1} ($n \in \mathbb{P}$); we shall set $E_1 = F_1$. Since $F_n \cong V \cap M_n \oplus X_n = M_n$, application of Lemma 11.5 yields $E_n \cong L^2(G)$ for all $n \in \mathbb{P}$. Now

$$V \cap M_n \subseteq E_1 \oplus \cdots \oplus E_n \subseteq V \cap M_{n+1}$$

for all $n \in \mathbb{P}$, hence $\bigoplus_{i=1}^{\infty} E_i = V$ and the result follows. □

An $L^2(G)$-basis $\{e_1, e_2, \dots\}$ of the $L^2(G)$-module $\mathcal{H}$ means that there exists an isometric $L^2(G)$-isomorphism $\theta\colon \mathcal{H} \to L^2(G)^n$ for some $n \in \mathbb{E}$ such that $\theta(e_i) = (0, \dots, 0, 1, 0, \dots)$, where the 1 is in the ith position. If $\{f_1, f_2, \dots\}$ is another $L^2(G)$-basis of $\mathcal{H}$ and α is the $L^2(G)$-automorphism of $\mathcal{H}$ defined by $\alpha e_i = f_i$, then $\alpha\alpha^* = \alpha^*\alpha = 1$. Also we say that an $L^2(G)$-map θ has finite rank if $\overline{\operatorname{im}\theta}$ is finitely generated.

Suppose now $\mathcal{H} = L^2(G)^\infty$ and that $\theta\colon \mathcal{H} \to \mathcal{H}$ is a finite rank $L^2(G)$-map. Let $\mathcal{K} = \ker\theta$. Then the restriction of θ to $\mathcal{K}^\perp$ is an $L^2(G)$-monomorphism, so $\mathcal{K}^\perp$ is finitely generated by Lemma 11.4. Using Lemmas 11.4 and 11.6, there exists $n \in \mathbb{P}$ and an $L^2(G)$-basis $\{e_1, e_2, \dots\}$ of $\mathcal{H}$ such that $\operatorname{im}\theta + \mathcal{K}^\perp \subseteq \overline{U}$ where $U = e_1\mathbb{C}G + \cdots + e_n\mathbb{C}G$. We may represent θ by a matrix (θ_{ij}) where $i, j \in \mathbb{P}$ and $\theta_{ij} \in W(G)$ for all i, j (so $\theta e_i = \sum_{j=1}^{\infty} e_j\theta_{ji}$). Then we define $\operatorname{tr}_G \theta = \sum_{i=1}^{\infty} \operatorname{tr}_G \theta_{ii}$, which is well defined because $\theta_{ii} = 0$ for all $i > n$. Clearly if θ_U is the restriction of θ to $\overline{U}$, then $\operatorname{tr}_G \theta = \operatorname{tr}_G \theta_U$ (where $\operatorname{tr}_G \theta_U$ is defined as in Section 8).

Let $\{f_1, f_2, \dots\}$ be another $L^2(G)$-basis for $\mathcal{H}$. We want to show that if (ϕ_{ij}) is the matrix of θ with respect to this basis, then $\sum_{i=1}^{\infty} \operatorname{tr}_G \phi_{ii}$ is an absolutely convergent series with sum $\operatorname{tr}_G \theta$. Write $f_i = \sum_{j=1}^{\infty} e_j\alpha_{ji}$ where $\alpha_{ij} \in W(G)$, and $\sum_{k=1}^{\infty} \alpha_{ik}\alpha^*_{kj}$ is an absolutely convergent series with sum δ_{ij} for all $i, j \in \mathbb{P}$. Then

$$\begin{aligned} \operatorname{tr}_G \phi_{ii} = \langle \theta f_i, f_i \rangle &= \sum_{j,k,l=1}^{n} \langle e_j\theta_{jk}\alpha_{ki}, e_l\alpha_{li} \rangle \\ &= \sum_{j,k,l=1}^{n} \langle e_j\theta_{jk}\alpha_{ki}\alpha^*_{il}, e_l \rangle \\ &= \sum_{j,k=1}^{n} \operatorname{tr}_G(\theta_{jk}\alpha_{ki}\alpha^*_{ij}). \end{aligned}$$

Now $\sum_{i=1}^{\infty} \alpha_{ki}\alpha^*_{ij}$ is absolutely convergent with sum δ_{kj}, hence $\sum_{i=1}^{\infty} \theta_{jk}\alpha_{ki}\alpha^*_{ij}$ is absolutely convergent with sum $\theta_{jk}\delta_{kj}$, consequently $\sum_{i=1}^{\infty} \operatorname{tr}_G(\theta_{jk}\alpha_{ki}\alpha^*_{ij})$ is absolutely convergent with sum $\operatorname{tr}_G(\theta_{jk}\delta_{kj})$. Therefore $\sum_{i=1}^{\infty} \operatorname{tr}_G \phi_{ii}$ is absolutely convergent with sum $\sum_{j,k=1}^{n} \operatorname{tr}_G(\theta_{jk}\delta_{kj}) = \operatorname{tr}_G \theta$, as required.

Suppose now that $\theta, \phi\colon \mathcal{H} \to \mathcal{H}$ are finite rank $L^2(G)$-maps. Let

$$M = (\ker\theta)^\perp + (\ker\phi)^\perp + \operatorname{im}\theta + \operatorname{im}\phi.$$

Then $\overline{M}$ is finitely generated, so there exists $n \in \mathbb{P}$ and an $L^2(G)$-submodule $L \cong L^2(G)^n$ of $\mathcal{H}$ containing $\overline{M}$. Let π denote the projection of $\mathcal{H}$ onto L, and if $\alpha\colon \mathcal{H} \to \mathcal{H}$ is an $L^2(G)$-map, then α_L will denote the restriction of α to L. Then $\theta + \phi$, $\theta\phi$ and $\phi\theta$ have finite $L^2(G)$-rank, and

$$\operatorname{tr}_G(\theta+\phi) = \operatorname{tr}_G(\theta+\phi)_L = \operatorname{tr}_G \theta_L + \operatorname{tr}_G \phi_L = \operatorname{tr}_G \theta + \operatorname{tr}_G \phi,$$
$$\operatorname{tr}_G \theta\phi = \operatorname{tr}_G(\theta\phi)_L = \operatorname{tr}_G \theta_L\phi_L = \operatorname{tr}_G \phi_L\theta_L = \operatorname{tr}_G(\phi\theta)_L = \operatorname{tr}_G \phi\theta.$$

Also if α is an $L^2(G)$-automorphism of $\mathcal{H}$, then $\alpha\pi$ and $\theta\alpha^{-1}$ are finite rank $L^2(G)$-maps and $\pi\theta = \theta = \theta\pi$, hence by the above

$$\operatorname{tr}_G \alpha\theta\alpha^{-1} = \operatorname{tr}_G(\alpha\pi)(\theta\alpha^{-1}) = \operatorname{tr}_G(\theta\alpha^{-1})(\alpha\pi) = \operatorname{tr}_G \theta.$$

Suppose M is a finitely generated $L^2(G)$-submodule of $L^2(G)^m$ where $m \in \mathbb{E}$. Then $\dim_G M$ is defined to be $\operatorname{tr}_G e$ where e is the projection of $L^2(G)^m$ onto M ($\dim_G$ is precisely d_G of [10, p. 134]). In view of Kaplansky's theorem (see Section 8), $\dim_G M \geq 0$ and $\dim_G M = 0$ if and only if $M = 0$. Let N be an $L^2(G)$-submodule of $L^2(G)^n$ where $n \in \mathbb{E}$ and $N \cong M$. Then

$$M^\perp \oplus L^2(G)^n \cong N^\perp \oplus L^2(G)^m,$$

hence there is a unitary $L^2(G)$-map α of $L^2(G)^m \oplus L^2(G)^n$ which takes M to N. Therefore if f is the projection of $L^2(G)^n$ onto N, then $0 \oplus f = \alpha(e \oplus 0)\alpha^{-1}$ and it follows that $\operatorname{tr}_G f = \operatorname{tr}_G e$. Thus $\dim_G N = \dim_G M$, in other words $\dim_G M$ depends only on the isomorphism type of M. If $n \in \mathbb{P}$ and $\phi \in \mathrm{M}_n(W(G))$, we may view ϕ as an $L^2(G)$-map $L^2(G)^n \to L^2(G)^n$, and then $\dim_G \overline{\operatorname{im}\phi} = \operatorname{rank}_G \phi$. We need the following technical result.

Lemma 11.7. *Let $\mathcal{H} = L^2(G)^\infty$, let $\theta\colon \mathcal{H} \to \mathcal{H}$ be an $L^2(G)$-homomorphism, and let $\{e_1, e_2, \dots\}$ be an $L^2(G)$-basis for $\mathcal{H}$. For $r, s \in \mathbb{E}$, $r \leq s$, let $\mathcal{H}_{r,s} = \overline{e_r\mathbb{C}G + \dots + e_s\mathbb{C}G}$ ($s \neq \infty$), let $\mathcal{H}_{r,\infty} = \overline{e_r\mathbb{C}G + e_{r+1}\mathbb{C}G + \cdots}$, and let $U_{r,s} = \overline{\theta\mathcal{H}_{r,s}}$. Suppose for all $i \in \mathbb{P}$ we can write θe_i as a* finite *sum of elements $\sum_{j=1}^r e_j\alpha_j$ where $\alpha_j \in \mathbb{C}G$ for all j (where r depends on i). If $\operatorname{rank}_G \phi \in \mathbb{Z}$ for all $\phi \in \mathrm{M}_r(\mathbb{C}G)$ and for all $r \in \mathbb{P}$, then $\dim_G U_{1,m} \cap U_{n,\infty} \in \mathbb{Z}$ for all $m, n \in \mathbb{P}$.*

Proof. Suppose $a, b, c, d \in \mathbb{P}$ with $a \leq b$ and $b, c \leq d$. Using the hypothesis that θe_i can be written as a finite sum of elements of the form $e_j\alpha_j$ where $\alpha_j \in \mathbb{C}G$, there exists $r \in \mathbb{P}$, $r \geq d$ such that $U_{1,d} \subseteq \mathcal{H}_{1,r}$. Define an $L^2(G)$-map $\phi\colon \mathcal{H}_{1,r} \to \mathcal{H}_{1,r}$ by $\phi e_i = \theta e_i$ if $a \leq i \leq b$ or $c \leq i \leq d$, and $\phi e_i = 0$ otherwise. Then with respect to the $L^2(G)$-basis $\{e_1, \dots, e_r\}$ of $\mathcal{H}_{1,r}$ the matrix of ϕ is in $\mathrm{M}_r(\mathbb{C}G)$, so $\operatorname{rank}_G \phi \in \mathbb{Z}$. But $\operatorname{im}\phi = \theta(\mathcal{H}_{a,b} + \mathcal{H}_{c,d})$ and it follows that $\dim_G \overline{U_{a,b} + U_{c,d}} \in \mathbb{Z}$.

Let $s \in \mathbb{P}$ with $s \geq m, n$. Using Lemma 11.1 we can obtain standard isomorphism theorems, in particular

$$\overline{(U_{1,m} + U_{n,s})} \oplus (U_{1,m} \cap U_{n,s}) \cong U_{1,m} \oplus U_{n,s}.$$

Therefore $\dim_G \overline{U_{1,m} + U_{n,s}} + \dim_G U_{1,m} \cap U_{n,s} = \dim_G U_{1,m} + \dim_G U_{n,s}$ and we deduce from the previous paragraph that $\dim_G U_{1,m} \cap U_{n,s} \in \mathbb{Z}$. Thus as s increases, $\dim_G U_{1,m} \cap U_{n,s}$ forms an increasing sequence of integers bounded above by $\dim_G U_{1,m}$, hence there exists $t \in \mathbb{P}$ such that $\dim_G U_{1,m} \cap U_{n,s} = \dim_G U_{1,m} \cap U_{n,t}$ for all $s \geq t$. Therefore $U_{1,m} \cap U_{n,s} = U_{1,m} \cap U_{n,t}$ for all $s \geq t$, and it follows that $U_{1,m} \cap U_{n,\infty} = U_{1,m} \cap U_{n,t}$. We conclude that $\dim_G U_{1,m} \cap U_{n,\infty} = \dim_G U_{1,m} \cap U_{n,t} \in \mathbb{Z}$ as required. $\square$

12. The Special Case of a Direct Product of Free Groups

Here we generalize the theory of [41, section 3]. If $\mathcal{H}$ is a Hilbert space and G is a group of operators acting on the right of $\mathcal{H}$, then we define

$$\mathcal{L}_G(\mathcal{H}) = \{\theta \in \mathcal{L}(\mathcal{H}) \mid \theta(ug) = (\theta u)g \text{ for all } u \in \mathcal{H} \text{ and } g \in G\}.$$

Note that von Neumann's double commutant theorem [1, theorem 1.2.1] (or see (ii) after Theorem 8.2) tells us that

$$\mathcal{L}_G(L^2(G)) = \{\theta \in \mathcal{L}(L^2(G)) \mid \theta(ug) = (\theta u)g \text{ for all } g \in G\} = W(G).$$

Suppose now that H and A are groups, $G = H \times A$, $n \in \mathbb{P}$, $\theta \in \mathcal{L}_A(L^2(G))$, $\phi \in \mathrm{M}_n\bigl(\mathcal{L}_A(L^2(G))\bigr)$, and ϕ is represented by the matrix (ϕ_{ij}) where $\phi_{ij} \in \mathcal{L}_A(L^2(G))$ for all i, j. We make θ act on $L^2(G) \oplus L^2(G) \oplus L^2(A)$ by $\theta(u, v, x) = (\theta u, \theta v, 0)$, and ϕ act on $L^2(G)^n \oplus L^2(G)^n \oplus L^2(A)^n$ by $\phi(u, v, x) = (\phi u, \phi v, 0)$ $(u, v \in L^2(G)$ or $L^2(G)^n$, $x \in L^2(A)$ or $L^2(A)^n)$. Note that the actions of θ and ϕ on $L^2(G) \oplus L^2(G) \oplus L^2(A)$ and $L^2(G)^n \oplus L^2(G)^n \oplus L^2(A)^n$ are right $\mathbb{C}A$-maps.

Now let H be the free group on two generators, and let A act on the right of A by right multiplication as usual; i.e. $ab = ab$ for all $a \in A$ and $b \in A$. We also make H act trivially on A: thus $ha = a$ for all $a \in A$ and $h \in H$ (though $h1_H = h$). Theorem 10.1 shows that there is a bijection $\pi \colon H \to H \cup H \cup \{1_A\}$ (where 1_A is the identity of A) such that

$$\pi 1_H = 1_A, \tag{12.1}$$

$$\{k \in H \mid h(\pi k) \neq \pi(hk)\} \text{ is finite for all } h \in H. \tag{12.2}$$

We extend π to a right A-map

$$\pi \colon G \to G \cup G \cup A \tag{12.3}$$

by setting $\pi(ha) = (\pi h)a$ for all $h \in H$ and $a \in A$. This in turn defines a unitary operator $\alpha \colon L^2(G) \to L^2(G) \oplus L^2(G) \oplus L^2(A)$, and hence also a unitary operator (equal to the direct sum of n copies of α)

$$\beta \colon L^2(G)^n \to L^2(G)^n \oplus L^2(G)^n \oplus L^2(A)^n. \tag{12.4}$$

We note that α and β are right $\mathbb{C}A$-maps. Suppose $\phi \in \mathrm{M}_n\bigl(W(G)\bigr)$ and $\phi - \beta^{-1}\phi\beta$ has finite $L^2(A)$-rank. Then we have

Lemma 12.1. $\operatorname{tr}_G \phi = \operatorname{tr}_A(\phi - \beta^{-1}\phi\beta)$.

Proof. (cf. [20, section 5].) Let (ϕ_{ij}) denote the matrix of ϕ. Since $\phi_{ij} - \alpha^{-1}\phi_{ij}\alpha$ has finite $L^2(A)$-rank for all i, j, $\operatorname{tr}_G \phi = \sum_{i=1}^n \operatorname{tr}_G \phi_{ii}$ and $\operatorname{tr}_A(\phi - \beta^{-1}\phi\beta) = \sum_{i=1}^n \operatorname{tr}_A(\phi_{ii} - \alpha^{-1}\phi_{ii}\alpha)$, it will be sufficient to show that $\operatorname{tr}_G \theta = \operatorname{tr}_A(\theta - \alpha^{-1}\theta\alpha)$ for all $\theta \in W(G)$ such that $\theta - \alpha^{-1}\theta\alpha$ has finite $L^2(A)$-rank. If $\theta = \sum_{g \in G} \theta_g g$ where $\theta_g \in \mathbb{C}$ for all $g \in G$, then $\operatorname{tr}_G \theta = \theta_1$ and $\langle \theta g, g\rangle = \theta_1$ for all $g \in G$. Using (12.1), we see that $\langle \theta\pi h, \pi h\rangle = \theta_1$ for all $h \in H\backslash 1$ and $\langle \theta\pi 1, \pi 1\rangle = 0$, hence

$$\begin{aligned} &\langle (\theta - \alpha^{-1}\theta\alpha)h, h\rangle = 0 \quad \text{if } h \in H\backslash 1, \\ &\langle (\theta - \alpha^{-1}\theta\alpha)1, 1\rangle = \theta_1. \end{aligned}$$

Since H is an $L^2(A)$-basis for $L^2(G)$, we can calculate $\operatorname{tr}_A \theta$ with respect to this basis and the result follows. $\square$

Let G be a group, let $n \in \mathbb{P}$, let $\theta \in \mathrm{M}_n(\mathbb{C}G)$, and let $X \subseteq G$. If $\theta = \sum_{g \in G} \theta_g g$ where $\theta_g \in \mathrm{M}_n(\mathbb{C})$, then $\operatorname{supp}\theta$ is defined to be $\{g \in G \mid \theta_g \neq 0\}$, a finite subset of G. Also $L^2(X)$ will indicate the closed subspace of $L^2(G)$ with Hilbert basis X.

Lemma 12.2. *Let H be the free group of rank two, let A be a group, let $G = H \times A$, let $n \in \mathbb{P}$, and let $\theta \in \mathrm{M}_n(\mathbb{C}G)$. If $\operatorname{rank}_A \phi \in \mathbb{Z}$ for all $\phi \in \mathrm{M}_r(\mathbb{C}A)$ and for all $r \in \mathbb{P}$, then $\operatorname{rank}_G \theta \in \mathbb{Z}$.*

Proof. Let $\pi\colon G \to G \cup G \cup A$ be the bijection given by (12.3), and let $\beta\colon L^2(G)^n \to L^2(G)^n \oplus L^2(G)^n \oplus L^2(A)^n$ be the unitary operator given by (12.4). Let

$$K = \{k \in H \mid g(\pi k) = \pi(gk) \quad \text{for all } g \in \operatorname{supp}\theta\},$$

let $J = H \backslash K$, let $L_1 = \theta L^2(G)^n$, let $L_2 = \beta^{-1}\theta\beta L^2(G)^n$, and let λ denote the projection of $L^2(G)^n$ onto $\overline{L_1}$. Then $|J| < \infty$ by (12.2), and $\beta^{-1}\lambda\beta$ is the projection of $L^2(G)^n$ onto $\overline{L_2}$. Since $\operatorname{rank}_G \theta = \operatorname{tr}_G \lambda$, we want to prove that $\operatorname{tr}_G \lambda \in \mathbb{Z}$.

Let $M = \theta L^2(KA)^n$, and let μ denote the projection of $L^2(G)^n$ onto $\overline{M}$. Note that $M = \beta^{-1}\theta\beta L^2(KA)^n$ because $\beta^{-1}\theta\beta u = \theta u$ for all $u \in L^2(KA)^n$. Let N_1 and N_2 denote the orthogonal complements of $\overline{M}$ in $\overline{L_1}$ and $\overline{L_2}$ respectively, and let η_1 and η_2 denote the projections of $L^2(G)^n$ onto N_1 and N_2 respectively. Let $P_1 = \theta L^2(JA)^n$, let $P_2 = \beta^{-1}\theta\beta L^2(JA)^n$, and for $i = 1, 2$, let Q_i denote the orthogonal complement of $\overline{P_i} \cap \overline{M}$ in $\overline{P_i}$. Note that $\overline{M} \cap Q_i = 0$ and $\overline{M} + Q_i$ is dense in $\overline{L_i}$ $(i = 1, 2)$. Thus if π_i is the projection of $\overline{L_i}$ onto N_i, then the restriction of π_i to Q_i is an $L^2(A)$-monomorphism with dense image, so $N_i \cong Q_i$ by Lemma 11.1 $(i = 1, 2)$. Therefore

$$N_i \oplus (\overline{P_i} \cap \overline{M}) \cong \overline{P_i}. \tag{12.5}$$

Using Lemma 11.4, we see that N_i is finitely generated, hence $\eta_1 - \eta_2$ has finite $L^2(A)$-rank. Also $\lambda = \mu + \eta_1$ and $\beta^{-1}\lambda\beta = \mu + \eta_2$, hence $\lambda - \beta^{-1}\lambda\beta = \eta_1 - \eta_2$. Therefore $\operatorname{tr}_G \lambda = \operatorname{tr}_A(\lambda - \beta^{-1}\lambda\beta)$ by Lemma 12.1, and since $\operatorname{tr}_A(\eta_1 - \eta_2) = \operatorname{tr}_A \eta_1 - \operatorname{tr}_A \eta_2$, it will suffice to prove that $\operatorname{tr}_A \eta_1$ and $\operatorname{tr}_A \eta_2 \in \mathbb{Z}$. Now $\operatorname{tr}_A \eta_i = \dim_A N_i$ so in view of (12.5), we require that $\dim_A \overline{P_i} \cap \overline{M}$ and $\dim_A \overline{P_i} \in \mathbb{Z}$.

We apply Lemma 11.7: note that with respect to the standard $L^2(A)$-basis H^n of $L^2(G)^n$, the matrices of θ and $\beta^{-1}\theta\beta$ have the required form for this lemma. But $P_1 = \theta L^2(JA)^n$, $P_2 = \beta^{-1}\theta\beta L^2(JA)^n$, and $M = \theta L^2(KA)^n = \beta^{-1}\theta\beta L^2(KA)^n$, and the result follows. $\square$

The proof of the following lemma is identical to the proof of [41, lemma 3.7].

Lemma 12.3. *Let G be a group. If $\operatorname{rank}_G \theta \in \mathbb{Z}$ for all $\theta \in \mathrm{M}_n(\mathbb{C}G)$ and for all $n \in \mathbb{P}$, then $D(G)$ is a division ring.*

Proof. We shall use the theory of [15, section 7.1]. Let R denote the rational closure [15, p. 382] of $\mathbb{C}G$ in $U(G)$, and let $\alpha \in D(G) \backslash 0$. By [15, exercise 7.1.4] $D(G) \subseteq R$, so we can apply Cramer's rule [15, proposition 7.1.3] to deduce that α is stably associated over R to a matrix in $\mathrm{M}_m(\mathbb{C}G)$ for some $m \in \mathbb{P}$. Therefore there exists $n \geq m$ such that $\operatorname{diag}(\alpha, 1_{n-1})$ is associated over R to a matrix $\theta \in \mathrm{M}_n(\mathbb{C}G)$, which means that there exist $X, Y \in \mathrm{GL}_n(U(G))$ such that $X \operatorname{diag}(\alpha, 1_{n-1}) Y = \theta$.

Suppose α is not invertible in $U(G)$. Using Lemma 8.3, we see that $0 < \operatorname{rank}_G \alpha < 1$ and thus $n - 1 < \operatorname{rank}_G \theta < n$. This contradicts Lemma 12.2, hence α is invertible in $U(G)$. Since $D(G)$ is closed under taking inverses, $D(G)$ must be a division ring. $\square$

Lemma 12.4. *Let $n \in \mathbb{P}$, and let $G = H_1 \times \cdots \times H_n$ where H_i is isomorphic to the free group of rank two for all i. Then $D(G)$ is a division ring.*

Proof. By induction on n and Lemma 12.2, $\operatorname{rank}_G \theta \in \mathbb{Z}$ for all $\theta \in \mathrm{M}_n(\mathbb{C}G)$ and for all $n \in \mathbb{P}$. Now use Lemma 12.3. $\square$

Lemma 12.5. *Let $H \lhd G$ be groups such that G/H is free, let Φ denote the matrices over $\mathbb{C}H$ which become invertible over $D(H)$, and let Σ denote the matrices over $\mathbb{C}G$ which become invertible over $D(G)$. Suppose $D(G)$ is a division ring. If the identity map on $\mathbb{C}H$ extends to an isomorphism $\phi \colon \mathbb{C}H_\Phi \to D(H)$, then the identity map on $\mathbb{C}G$ extends to an isomorphism $\sigma \colon \mathbb{C}G_\Sigma \to D(G)$.*

Proof. By Lemma 9.3, we may view $\sum_{g \in G} D(H)g$ as $D(H) * [G/H]$. Suppose $H \subseteq N \lhd K \subseteq G$ and $K/N \cong \mathbb{Z}$ with generator Nt where $t \in K$. Then if $d_i \in D(N)$ and $\sum_i d_i t^i = 0$, it follows that $d_i = 0$ for all i, which means in the terminology of [31, §2] that $D(G)$ is a free division ring of fractions for $D(H) * [G/H]$. Therefore $D(G)$ is the universal field of fractions for $D(H) * [G/H]$ by the theorem of [31] and the proof of [37, proposition 6]. Since $D(H) * [G/H]$ is a free ideal ring [14, theorem 3.2], the results of [15, §7.5] show that $D(G) = D(H) * [G/H]_\Psi$ for a suitable set of matrices Ψ with entries in $D(H) * [G/H]$. The proof is completed by applying [56, proof of theorem 4.6] and [15, exercise 7.2.8]. $\square$

Lemma 12.6. *Let $G \leqslant F$ be groups such that F is a direct product of finitely generated free groups, and let Σ denote the set of matrices over $\mathbb{C}G$ which become invertible over $D(G)$. Then $D(G)$ is a division ring, and the identity map on $\mathbb{C}G$ extends to an isomorphism $\mathbb{C}G_\Sigma \to D(G)$.*

Proof. We may write $F = F_1 \times \cdots \times F_n$ where $n \in \mathbb{P}$ and the F_i are finitely generated free groups, and since any finitely generated free group is isomorphic to a subgroup of the free group of rank 2, we may assume that each F_i is free of rank 2. Then $D(F)$ is a division ring by Lemma 12.4, hence $D(G)$ is a division ring. Write $H_i = F_1 \times \cdots \times F_i$ for $0 \leq i \leq n$ (so $H_0 = 1$). Then $(G \cap H_i)/(G \cap H_{i-1})$ is isomorphic to a free group for all i, so we can now use Lemma 12.5 and induction on n to complete the proof. $\square$

13. Proof of Theorem 10.3

To simplify the notation in the following lemma, we assume that $1, 2, \ldots \in \mathcal{I}$.

Lemma 13.1. *Let $\{H_i \mid i \in \mathcal{I}\}$ be a family of nonabelian free groups, let $G = H_1 \times H_2 \times \cdots$, and let θ be an automorphism of G. Then $\theta H_1 = H_i$ for some $i \in \mathcal{I}$.*

Proof. Suppose $g = (g_1, g_2, \dots) \in G$ where $g_i \in H_i$ for all i. Then

$$C_G(g) = C_{H_1}(g_1) \times C_{H_2}(g_2) \times \cdots$$

and $C_{H_i}(g_i) \cong \mathbb{Z}$ if $g_i \neq 1$, and $C_{H_i}(g_i) = H_i$ if $g_i = 1$. It follows that $Z(C_G(g)) \cong \mathbb{Z}^r$, where $Z(C_G(g))$ denotes the center of $C_G(g)$ and $r = |\{i \mid g_i \neq 1\}|$.

Let $x, y \in H_1 \backslash 1$. Then by the above we have $\theta x \in H_i$ and $\theta y \in H_j$ for some $i, j \in \mathcal{I}$. If $i \neq j$, then $\langle x, y\rangle \cong \mathbb{Z} \times \mathbb{Z}$ which is not possible. Therefore $i = j$ and the result follows. $\square$

Lemma 13.2. *Let $H \lhd G$ be groups such that H is a direct product of nonabelian free groups and G/H is finite. Let X be a finite subset of G. Then there exists* *a finitely generated subgroup G_0 of G such that $X \subseteq G_0$ and $G_0 \cap H$ is a direct product of nonabelian free groups.*

Proof. By enlarging X if necessary, we may assume that $HX = G$. Let H be the direct product of the nonabelian free groups H_i. Using Lemma 13.1 we see that G permutes the H_i by conjugation, so we may write $H = \times_i K_i$ where $K_i = K_{i1} \times \cdots \times K_{im_i}$ with the K_{ij} nonabelian free groups (so each K_{ij} is an H_k for some k), and for each i the set $\{K_{i1}, \dots, K_{im_i}\}$ is permuted transitively by conjugation by G. For each i, let N_i denote the normalizer of K_{i1} in G, and then choose right transversals $S_i \subseteq X$ for H in N_i, and $T_i \subseteq X$ for N_i in G; thus $|T_i| = m_i$ and we may write $T_i = \{t_{i1}, \dots, t_{im_i}\}$ where $t_{ij}^{-1} K_{i1} t_{ij} = K_{ij}$. Set $H_0 = H \cap \langle X \rangle$ and note that since it is a subgroup of finite index in a finitely generated group, it is also finitely generated, so we may write $H_0 \subseteq K_1 \times \cdots \times K_n$ for some $n \in \mathbb{P}$, and then there are finite subsets $Y_{ij} \subseteq K_{ij}$ ($1 \le i \le n$ and $1 \le j \le m_i$) such that $H_0 \subseteq \langle \bigcup_{i,j} Y_{ij} \rangle$. Then we may choose finitely generated nonabelian free subgroups $\tilde{K}_{i1}$ of K_{i1} such that

$$Y_{ij} \subseteq t_{ij}^{-1} \tilde{K}_{i1} t_{ij} \quad \text{for } j = 1, \dots, m_i.$$

Set $L_{i1} = \langle s^{-1} \tilde{K}_{i1} s \mid s \in S_i \rangle$ and

$$L_i = t_{i1}^{-1} L_{i1} t_{i1} \times t_{i2}^{-1} L_{i1} t_{i2} \times \cdots \times t_{im_i}^{-1} L_{i1} t_{im_i}.$$

Then L_{i1} and hence also L_i is a finitely generated subgroup. Also if $i, j \in \mathbb{P}$ with $i \le n$, $j \le m_i$ and $x \in X$, then we may write $t_{ij} x = h s t_{ik}$ for some $h \in H_0$, $s \in S_i$ and $k \in \mathbb{P}$, and then $x^{-1} t_{ij}^{-1} L_{i1} t_{ij} x = t_{ik}^{-1} L_{i1} t_{ik}$ and we deduce that X normalizes L_i. Therefore if $L = L_1 \times \cdots \times L_n$, then L is a finitely generated subgroup and X normalizes L. Moreover L_{i1} is a free group because it is a subgroup of the free group K_{i1}, and it is nonabelian because it contains the nonabelian subgroup $\tilde{K}_{i1}$, hence L is a direct product of nonabelian free groups. Thus we may set $G_0 = L\langle X \rangle$ for the required subgroup. $\square$

For the purposes of the next two lemmas, given a group G and $n \in \mathbb{P}$, we shall define $\mathcal{S}_n G$ to be the intersection of normal subgroups of index at most n in G. Note that $\mathcal{S}_n G$ is a characteristic subgroup of G and that $\mathcal{S}_n G \supseteq \mathcal{S}_{n+1} G$ for all $n \in \mathbb{P}$. Furthermore if G is finitely generated, then $G/\mathcal{S}_n G$ is finite.

Lemma 13.3. *Let $F \lhd G$ be groups such that F is finitely generated free and G/F is finite. Suppose for all $n \in \mathbb{P}$, there exists $H_n \leqslant G$ such that $H_n F = G$ and $H_n \cap F = \mathcal{S}_n F$. Then there exists $H \leqslant G$ such that $HF = G$ and $H \cap F = 1$.*

Proof. Since $\mathcal{S}_n F$ is a normal subgroup of finite index in G, there are only finitely many subgroups of G which contain $\mathcal{S}_n F$, hence an application of the König graph theorem shows we may assume that $H_n \supseteq H_{n+1}$ for all $n \in \mathbb{P}$. It follows that if $\hat{G}$ denotes the profinite completion of G, then $\hat{G}$ has a subgroup K isomorphic to G/F.

We shall now use the notation and results of [60]. Since G has a free subgroup of finite index, we see from [17, theorem IV.3.2] that G is isomorphic to the fundamental group of a graph of groups $\pi_1(\mathcal{G}, \Gamma)$ with respect to some tree T, where Γ is a finite graph of groups, and the vertex groups $G(v)$ are finite for all vertices v of Γ. Then we can form the fundamental group $\Pi_1(\mathcal{G}, \Gamma, T)$ in the category of profinite groups, and by construction, $\Pi_1(\mathcal{G}, \Gamma, T) \cong \hat{G}$ [60, p. 418]. Of course the vertex groups $G(v)$ are the same as the vertex groups $\hat{G}(v)$. By [60, theorem 3.10] and the fact that K is a finite subgroup, we see that $K \subseteq g\hat{G}(v)g^{-1}$ for some vertex v of Γ and some $g \in \hat{G}$. Thus G has a subgroup isomorphic to G/F and the result follows. $\square$

Lemma 13.4. *Let $l \in \mathbb{P}$, and let $H \lhd G$ be groups such that G is finitely generated, G/H is finite, and H is a direct product of nonabelian free groups. Assume that whenever $K \lhd G$ such that $K \subseteq H$ and G/K is abelian-by-finite, then G/K has a subgroup of order l. Then G has a subgroup of order l.*

Proof. Write $H = H_1 \times \cdots \times H_t$ where the H_i are nonabelian free groups, and set $H_{(n)} = \mathcal{S}_n H_1 \times \cdots \times \mathcal{S}_n H_t$ for $n \in \mathbb{P}$. Note that if $K \lhd G$ and G/K is abelian-by-finite, then $H'_{(n)} \subseteq K$ for some $n \in \mathbb{P}$ and that $H/H'_{(n)}$ is torsion free.

First we reduce to the case $|G/H| = l$. We know by hypothesis that $G/H'_{(n)}$ has a subgroup $L_n/H'_{(n)}$ of order l for all $n \in \mathbb{P}$. Since $H/H'_{(n)}$ is torsion free, we see that $L_n \cap H = H'_{(n)}$ and therefore $|L_n H/H| = l$. Now G/H has only finitely many subgroups of order l, hence there exists a subgroup G_0/H of order l in G/H with $L_n \subseteq G_0$ for infinitely many n. Thus replacing G with G_0, we may assume that $|G/H| = l$.

We now use induction on t, the case $t = 1$ being a consequence of Lemma 13.3. Suppose we can write $H = F_1 \times F_2$ where $F_1, F_2 \lhd G$ and each F_i is a direct product of a proper subset of $\{H_1, \ldots, H_t\}$. Then by induction on t there exists $G_1 \leqslant G$ such that $F_1 \subseteq G_1$ and $|G_1/F_1| = l$. The natural injection $G_1 \hookrightarrow G$ induces an isomorphism $G_1 \hookrightarrow G/F_2$, so again using induction we see that G_1, and hence also G, has a subgroup of order l. Therefore we may assume that no such decomposition $H = F_1 \times F_2$ as above exists. It now follows from Lemma 13.1 that G permutes the H_i transitively by conjugation. Let D_n be the normalizer of H_1 in L_n, and let $Z = H_2 \times \cdots \times H_t$. Then $D_n H$ is the normalizer of both H_1 and Z in G for all $n \in \mathbb{P}$, so we may set $D = D_n H$ for all n. Since $D_n H = D$ and $D_n \cap H = H_{(n)}$ for all $n \in \mathbb{P}$, we have from the case $t = 1$ that D/Z has a subgroup of order $|D/H|$. Thus D/Z is isomorphic to a semidirect product of H_1 and D/H, so we may apply [30, theorem 3] to obtain a subgroup of G isomorphic to G/H, which is what is required. $\square$

Lemma 13.5. *Let $G = \bigcup_{i \in \mathcal{I}} G_i$ be groups such that given $i, j \in \mathcal{I}$, there exists $l \in \mathcal{I}$ such that $G_i, G_j \subseteq G_l$, let Σ denote the matrices with entries in $\mathbb{C}G$ which become invertible over $D(G)$, and let Σ_i denote the matrices with entries in $\mathbb{C}G_i$ which become invertible over $D(G_i)$. Assume that the orders of the finite subgroups of G are bounded, and that $D(G_i)$ is an Artinian ring for all $i \in \mathcal{I}$. Suppose $\operatorname{lcm}(G_i) \operatorname{tr}_{G_i} e \in \mathbb{Z}$ whenever e is a projection in $D(G_i)$, for all $i \in \mathcal{I}$. Then*

(i) *$D(G) = \bigcup_{i \in \mathcal{I}} D(G_i)$ and $\operatorname{lcm}(G) \operatorname{tr}_G e \in \mathbb{Z}$ for all projections $e \in D(G)$.*
(ii) *$D(G)$ is a semisimple Artinian ring.*
(iii) *Suppose the identity map on $\mathbb{C}G$ extends to an isomorphism $\lambda_i \colon \mathbb{C}G_{i\Sigma_i} \to D(G_i)$ for all $i \in \mathcal{I}$. Then the identity map on $\mathbb{C}G$ extends to an isomorphism $\lambda \colon \mathbb{C}G_\Sigma \to D(G)$.*

Proof. (i) This is obvious.

(ii) If $I_0 > I_1 > \cdots > I_r$ is a strictly descending sequence of right ideals in $D(G)$, then

$$I_0 \cap D(G_i) > I_1 \cap D(G_i) > I_2 \cap D(G_i) > \cdots > I_r \cap D(G_i)$$

is a strictly descending sequence of right ideals in $D(G_i)$ for some $i \in \mathcal{I}$, hence $r \leq \operatorname{lcm}(G)$ by (i) and Lemma 9.6. This shows that $D(G)$ is Artinian, and the result now follows from Lemma 9.4.

(iii) Since every matrix in Σ_i becomes invertible over $\mathbb{C}G_\Sigma$, we see that there are maps $\mu_i \colon \mathbb{C}G_{i\Sigma_i} \to \mathbb{C}G_\Sigma$ which extend the inclusion map $\mathbb{C}G_i \to \mathbb{C}G$. Now λ_i is an isomorphism for all $i \in \mathcal{I}$, hence there are maps $\nu_i \colon D(G_i) \to \mathbb{C}G_\Sigma$ defined by $\nu_i = \mu_i \lambda_i^{-1}$, which extend the inclusion map $\mathbb{C}G_i \to \mathbb{C}G$. If $G_i \subseteq G_j$ and $\psi_{ij} \colon D(G_i) \to D(G_j)$ is the inclusion map, then $\nu_j \psi_{ij} = \nu_i$ and it follows that the ν_i fit together to give a map $\nu \colon \bigcup_{i \in \mathcal{I}} D(G_i) \to \mathbb{C}G_\Sigma$ such that $\nu\psi_i = \nu_i$, where $\psi_i \colon D(G_i) \to \bigcup_{i\in\mathcal{I}} D(G_i)$ is the natural inclusion. But $\bigcup_{i\in\mathcal{I}} D(G_i) = D(G)$ by (i), and we deduce that $\nu \colon D(G) \to \mathbb{C}G_\Sigma$ is a map which extends the identity on $\mathbb{C}G$. By the universal property of $\mathbb{C}G_\Sigma$, there is a map $\lambda \colon \mathbb{C}G_\Sigma \to D(G)$ which also extends the identity on $\mathbb{C}G$. Then $\nu\lambda$ is the identity on $\mathbb{C}G_\Sigma$ and $\lambda\nu$ is the identity on $D(G)$, and we deduce that λ is an isomorphism, as required. □

We need the following three technical lemmas.

Lemma 13.6. *(cf.* [41, lemma 4.4]*.) Let Q be a semisimple Artinian ring, let $G = \langle x \rangle$ be an infinite cyclic group, let $Q * G$ be a crossed product, and let S be the set of nonzero divisors of $Q * G$. Let R be a ring containing $Q * G$, and let D be the division closure of $Q * G$ in R. Suppose every element of $Q * G$ of the form $1 + q_1 x + \cdots + q_t x^t$ with $q_i \in Q$ and $t \in \mathbb{P}$ is invertible in R. Then $Q * G$ is a semiprime Noetherian ring and D is an Artinian ring. Furthermore every element of S is invertible in D, and the identity map on $Q * G$ extends to an isomorphism $Q * G_S \to D$.*

Lemma 13.7. *Let $H \lhd G$ be groups, let $D(H)G$ denote the subring generated by $D(H)$ and G in $D(G)$, and let Σ denote the matrices with entries in $\mathbb{C}H$ which become invertible over $D(H)$. If the identity map on $\mathbb{C}H$ extends to an isomorphism $\mathbb{C}H_\Sigma \to D(H)$, then the identity map on $\mathbb{C}G$ extends to an isomorphism $\mathbb{C}G_\Sigma \to D(H)G$.*

Proof. This follows from Lemma 9.3 and [41, 4.5] □

Lemma 13.8. *(See* [41, lemma 4.7]*.) Let D be a $*$-ring, let $\mathcal{R}$ be a set of subrings of D, let $n \in \mathbb{P}$, and let $e \in \mathrm{M}_n(D)$ be an idempotent. Assume that whenever $R \in \mathcal{R}$ and P is a finitely generated projective R-module, there exist projections $f_i \in R$ such that $P \cong \bigoplus f_i R$. If the natural induction map*

$$\bigoplus_{R\in\mathcal{R}} K_0(R) \to K_0(D)$$

is onto, then there exist $r, s \in \mathbb{P}$, $R_1, \dots, R_s \in \mathcal{R}$, and projections $f_i \in R_i$ ($1 \leq i \leq s$) such that

$$\operatorname{diag}(e, 1_r, 0_s) = u \operatorname{diag}(f_1, \dots, f_s, 0_{n+r}) u^{-1},$$

where $u \in \mathrm{GL}_{n+r+s}(D)$.

The essence of the next two lemmas is to show that if Theorem 10.3 holds for the group G_0 and G/G_0 is finitely generated abelian-by-finite, then it also holds for G. This is to prepare for an induction argument to follow.

Lemma 13.9. *Let $H \lhd G$ be groups such that G/H is free abelian of finite rank, let $D(H)G$ denote the subring of $D(G)$ generated by $D(H)$ and G, and let S denote* *the nonzero divisors in $D(H)G$. Suppose $D(H)$ is an Artinian ring. Then $D(H)G$ is a semiprime Noetherian ring and $D(G)$ is an Artinian ring. Furthermore every element of S is invertible in $D(G)$, and the identity map on $D(H)G$ extends to an isomorphism from $D(H)G_S$ to $D(G)$.*

Proof. By induction on the rank of G/H, we immediately reduce to the case G/H is infinite cyclic, say $G = \langle Hx \rangle$ where $x \in G$. Since $D(H)$ is semisimple by Lemma 9.4 and $D(H)G \cong D(H)*G/H$ by Lemma 9.3, we are in a position to apply Lemma 13.6. If $\alpha = 1+q_1x+\cdots+q_tx^t \in D(H)G$ where $t \in \mathbb{P}$ and $q_i \in D(H)$, then by Proposition 9.5(ii) there is a nonzero divisor β in $W(H)$ such that $\beta q_i \in W(H)$ for all i. Using [40, theorem 4], we see that $\beta\alpha\gamma \neq 0$ for all $\gamma \in W(G)\backslash 0$, and we deduce that α is invertible in $U(G)$. The result now follows from Lemma 13.6. $\square$

Lemma 13.10. *Let $N \lhd H \lhd G$ be groups such that $N \lhd G$, H/N is free abelian of finite rank, and G/H is finite. Let $D(N)G$ denote the subring of $D(G)$ generated by $D(N)$ and G, and let S denote the nonzero divisors of $D(N)G$. Suppose $D(N)$ is an Artinian ring. Then*

(i) *$D(N)G$ is a semiprime Noetherian ring and $D(G)$ is a semisimple Artinian ring. Furthermore every element of S is invertible in $D(G)$, and the identity map on $D(N)G$ extends to an isomorphism from $D(N)G_S$ to $D(G)$.*

(ii) *Let Φ denote the matrices of $\mathbb{C}N$ which become invertible over $D(N)$, and let Σ denote the matrices of $\mathbb{C}G$ which become invertible over $D(G)$. If the identity map on $\mathbb{C}N$ extends to an isomorphism $\mathbb{C}N_\Phi \to D(N)$, then the identity map on $\mathbb{C}G$ extends to an isomorphism $\mathbb{C}G_\Sigma \to D(G)$.*

(iii) *Suppose $m, n \in \mathbb{P}$ and the orders of the finite subgroups of G are bounded. If $m \operatorname{lcm}(F) \operatorname{tr}_F e \in \mathbb{Z}$ whenever $F/N \in \mathcal{F}(G/N)$ and e is a projection in $D(F)$, then $m \operatorname{lcm}(G) \operatorname{tr}_G e \in \mathbb{Z}$ for all projections e in $\mathrm{M}_n(D(G))$.*

Proof. (i) This follows from Lemmas 9.4 and 13.9.

(ii) Lemma 13.7 shows that the identity map on $\mathbb{C}G$ extends to an isomorphism $D(N)G \to \mathbb{C}G_\Phi$. We now see from (i) and the proof of [56, theorem 4.6] that $D(G)$ is $\mathbb{C}G_\Psi$ for a suitable set of matrices Ψ with entries in $\mathbb{C}G$. An application of [15, exercise 7.2.8] completes the proof.

(iii) Using (i), we see that $D(N)G$ is Noetherian and that $D(G) \cong D(N)GS^{-1}$, so it follows from [34, lemma 2.2] that the natural inclusion $D(N)G \to D(G)$ induces an epimorphism $G_0(D(N)G) \to G_0(D(G))$. Now $D(F) \cong D(N) * F/N$ whenever $F/N \in \mathcal{F}(G/N)$ by Lemma 9.4, and $D(N)G \cong D(N) * G/N$ by Lemma 9.3, so we can apply Moody's induction theorem (Lemma 4.4) to deduce that the natural map

$$\bigoplus_{F/N \in \mathcal{F}(G/N)} G_0(D(F)) \longrightarrow G_0(D(G))$$

is also onto. Since $D(G)$ and $D(F)$ are semisimple Artinian by (i), we have natural isomorphisms $K_0(D(G)) \cong G_0(D(G))$ and $K_0(D(F)) \cong G_0(D(F))$ for all F such that $F/N \in \mathcal{F}(G/N)$, and we conclude that the natural induction map

$$\bigoplus_{F/N \in \mathcal{F}(G/N)} K_0(D(F)) \longrightarrow K_0(D(G))$$

is onto. When $F/N \in \mathcal{F}(G/N)$, we see from Lemma 9.4, that $D(F)$ is semisimple Artinian, hence every indecomposable $D(F)$-module is of the form $eD(F)$ for some idempotent $e \in D(F)$ and in view of Proposition 9.5(i), we may assume that e is a projection. We are now in a position to apply Lemma 13.8, so we obtain $r, s \in \mathbb{P}$, $F_1/N, \dots, F_s/N \in \mathcal{F}(G/N)$, and projections $f_i \in D(F_i)$ such that

$$\operatorname{diag}(e, 1_r, 0_s) = u \operatorname{diag}(f_1, \dots, f_s, 0_{n+r}) u^{-1}$$

where $u \in \mathrm{GL}_{n+r+s}(D(G))$. Applying Lemma 8.4, we may assume that $u \in \mathrm{GL}_{n+r+s}(W(G))$, hence

$$\operatorname{tr}_G e + r = \operatorname{tr}_G f_1 + \cdots + \operatorname{tr}_G f_s$$

and the result follows. □

The following result could easily be proved directly, but is also an immediate consequence of the above Lemma 13.10(iii) (use the case $G = N$ and note that the orders of the finite subgroups of G all divide l).

Corollary 13.11. *Let G be a group such that $D(G)$ is Artinian, and let $l, n \in \mathbb{P}$. If $l \operatorname{tr}_G e \in \mathbb{Z}$ for all projections $e \in D(G)$, then $l \operatorname{tr}_G e \in \mathbb{Z}$ for all projections $e \in \mathrm{M}_n(D(G))$.*

Lemma 13.12. *Let $H \lhd G$ be groups such that $|G/H| < \infty$ and H is a direct product of nonabelian free groups, let $l = \operatorname{lcm}(G)$, and let Σ denote the set of matrices with entries in $\mathbb{C}G$ which become invertible over $D(G)$. Then*

(i) *$D(G)$ is a semisimple Artinian ring.*
(ii) *The identity map on $\mathbb{C}G$ extends to an isomorphism $\mathbb{C}G_\Sigma \to D(G)$.*
(iii) *If $e \in D(G)$ is a projection, then $l \operatorname{tr}_G e \in \mathbb{Z}$.*

Proof. Let $\{X_i \mid i \in \mathcal{I}\}$ denote the family of finite subsets of G. For each $i \in \mathcal{I}$, there is by Lemma 13.2 a finitely generated subgroup G_i containing X_i such that $G_i \cap H$ is a direct product of nonabelian free groups. Let Σ_i denote the matrices over $\mathbb{C}G_i$ which become invertible over $D(G_i)$.

If (i), (ii) and (iii) are all true for all $i \in \mathcal{I}$ when G is replaced by G_i and Σ by Σ_i, then the result follows from Lemma 13.5 so we may assume that G is finitely generated.

Lemma 13.4 now shows that there exists $K \lhd G$ such that $K \subseteq H$, G/K is abelian-by-finite, and $\operatorname{lcm}(G/K) = l$. Using Lemma 12.6, we see that $D(K)$ is a division ring and that the identity map on $\mathbb{C}K$ extends to an isomorphism $\mathbb{C}K_\Phi \to D(K)$, where Φ denotes the matrices with entries in $\mathbb{C}K$ which become invertible over $D(K)$. Therefore the only projections of $D(K)$ are 0 and 1, so $\operatorname{tr}_K e \in \mathbb{Z}$ for all projections $e \in D(K)$.

Let $F/K \in \mathcal{F}(G/K)$, let $[F : K] = f$, let $\{x_1, \dots, x_f\}$ be a transversal for K in F, let $e \in D(F)$ be a projection, and let $\hat{}\colon W(F) \to \mathrm{M}_f(W(K))$ denote the monomorphism of Lemma 8.6. In view of the previous paragraph, Corollary 13.11 tells us that $\operatorname{tr}_K h \in \mathbb{Z}$ for all projections $h \in \mathrm{M}_f(D(K))$. Since $e \in W(F)$, we may write $e = \sum \epsilon_i x_i$ where $\epsilon_i \in W(K)$ for all i. Using Lemma 9.3, we deduce that $\epsilon_i \in D(K)$ for all i, and it is now not difficult to see that $\hat{e} \in \mathrm{M}_f(D(K))$. Therefore $\operatorname{tr}_K \hat{e} \in \mathbb{Z}$ by Corollary 13.11, and we conclude from Lemma 8.6 that $f \operatorname{tr}_F e \in \mathbb{Z}$. But $f | l$ and the result follows from Lemma 13.10. □

Proof of Theorem 10.3. Replacing F with F', we may assume that F is a direct product of *nonabelian* free groups. We now use a transfinite induction argument, and since this is standard when dealing with elementary amenable groups, we will only sketch the details. If $\mathcal{Y}$ is a class of groups, then $H \in \mathrm{L}\,\mathcal{Y}$ means that every finite subset of the group H is contained in a $\mathcal{Y}$-subgroup, and $\mathcal{B}$ denotes the class of finitely generated abelian-by-finite groups. For each ordinal α, define $\mathcal{X}_\alpha$ inductively as follows:

$$\begin{aligned} \mathcal{X}_0 &= \text{all finite groups,} \\ \mathcal{X}_\alpha &= (\mathrm{L}\,\mathcal{X}_{\alpha-1})\mathcal{B} \quad \text{if } \alpha \text{ is a successor ordinal,} \\ \mathcal{X}_\alpha &= \bigcup_{\beta<\alpha} \mathcal{X}_\beta \quad \text{if } \alpha \text{ is a limit ordinal.} \end{aligned}$$

Then $\bigcup_{\alpha \geq 0} \mathcal{X}_\alpha$ is the class of elementary amenable groups [34, lemma 3.1(i)]. Let α be the least ordinal such that $G/F \in \mathcal{X}_\alpha$. If $\alpha = 0$, the result follows from Lemma 13.12. The use of transfinite induction now means that we have two cases to consider.

Case (i) The result is true with H in place of G whenever H/F is a finitely generated subgroup of G/F. Here we use Lemma 13.5.

Case (ii) There exists $H \lhd G$ such that $F \subseteq H$ and G/H is finitely generated abelian-by-finite, and the result is true with E in place of G whenever E/H is a finite subgroup of G/H. Here we use Lemma 13.10. □

Proof of Corollary 10.4. By Theorem 10.3, we know that $D(G)$ is semisimple Artinian so if $D(G)$ is not simple Artinian, then there is a central idempotent $e \in D(G)$ such that $0 \neq e \neq 1$. Using Proposition 9.5(i), we deduce that $e \in W(G)$. Since $geg^{-1} = e$ for all $g \in G$, we see that $\{gxg^{-1} \mid g \in G\}$ is finite whenever $x \in G$ and $e_x \neq 0$, hence $e \in D(\Delta(G))$ where $\Delta(G)$ denotes the finite conjugate center of G [47, §5]. But $\Delta^+(G) = 1$, hence $\Delta(G)$ is torsion free abelian by [47, lemma 5.1(ii)] and it now follows from Theorem 10.3 that $\operatorname{tr}_G e \in \mathbb{Z}$. Therefore $e = 0$ or 1 by Kaplansky's theorem (§8), a contradiction, thus $D(G)$ is simple Artinian and we may write $D(G) = \mathrm{M}_m(D)$ for some $m \in \mathbb{P}$ and some division ring D.

It remains to prove that $m = l$. Using Lemma 9.6 and Theorem 10.3, we see that $m \leq l$. Now let $F \in \mathcal{F}(G)$ and set $f = \frac{1}{|F|}\sum_{g \in F} g$, a projection in $\mathbb{C}F$. Write $1 = e_1 + \cdots + e_r + \cdots + e_m$ where the e_i are primitive idempotents of $D(G)$, $1 \leq r \leq m$, and $f = e_1 + \cdots + e_r$. By Lemma 9.5(i), there are projections $f_i \in D(G)$ such that $f_i D(G) = e_i D(G)$ $(1 \leq i \leq m)$, and then application of Lemma 8.5 shows that $\operatorname{tr}_G f_1 + \cdots + \operatorname{tr}_G f_m = 1$. Also for each i, there exists a unit $u_i \in D(G)$ such that $u_i f_i u_i^{-1} = f_1$, and by Lemma 8.4 we may assume that $u_i \in W(G)$ for all i. Therefore $\operatorname{tr}_G f_i = \operatorname{tr}_G f_1$ for all i and we deduce that $\operatorname{tr}_G f_i = 1/m$ for all i. Another application of Lemma 8.5 shows that $\operatorname{tr}_G f = \operatorname{tr}_G f_1 + \cdots + \operatorname{tr}_G f_r$ and we conclude that $1/|F| = r/m$. Therefore $|F|$ divides m for all $F \in \mathcal{F}(G)$, hence $l|m$ and we have proven the result in the case $n = 1$. The case for general n follows from Lemma 9.1 and Corollary 13.11. □

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