LECTURE NOTES FOR 5114 (EQUIVARIANT METHODS IN SCHUBERT CALCULUS)

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Contents

1. Motivation: integration in (co)homology theories

One of the most important notions in mathematics is that of **integration**. For instance, if $V \subset \mathbb{R}^n$ is a *compact* subset then

$$\int_V \mathbb{1}_V dV = vol(V), \tag{1.1}$$

i.e. the volume of V. But one can also integrate more general functions and get other useful information. For instance, the divergence theorem states (roughly) that for a vector field $\overrightarrow{F}$ on V,

$$\int_V div(\overrightarrow{F}) dV = \int_S \overrightarrow{F} \cdot \overrightarrow{n} dS, \tag{1.2}$$

where $S := \partial V$ is the boundary of V. In many instances these integrals have meaningful physical interpretation.

Another source of integrals comes from *homology theories*. Roughly a homology theory is an assignment (or more fancy, a *functor*),

$$H : \mathcal{C} \to \{ \texttt{ Abelian Groups } \}$$

where $\mathcal{C}$ is a subcategory of $\{\texttt{(Topological spaces, Continuous maps)}\}$, such that if $f : X \to Y$ is an *allowable* continuous function then we have a group homomorphism $f_* : H(X) \to H(Y)$. Then if $\alpha \in H(X)$ and $f : X \to pt$ is an allowable function, the integral $\int_X \alpha$ is defined as

$$\int_X \alpha = f_*(\alpha) \in H(pt). \tag{1.3}$$

In other words, the integral takes values in $H(pt)$ - the homology of the point.

In the examples below I briefly introduce our main objects of study. If you have seen Algebraic Geometry, some may sound familiar; if not, do not worry about terminology at this time - all of these will become much more concrete as we move on. For now, if you are not familiar with algebraic varieties the think about compact manifolds and closed subspaces of them.

1.1. Examples of (co)homology theories.

1.1.1. *Chow homology.* The main reference is [Ful84], but see also [Bri05]. In this case $\mathcal{C}$ is the category of algebraic schemes, and the allowable functions are *proper* morphisms $f : X \to Y$. The associated homology theory is $A_*(X)$, the group of cycles modulo rational equivalence. This group is a graded $\mathbb{Z}$-module:

$$A_*(X) = \oplus_{i \geq 0} A_i(X),$$

and $A_i(X)$ is generated by classes of subvarieties of dimension i. We have $A_*(pt) = A_0(pt) = \mathbb{Z}$.

The objects are classes $[Z] \in A_i(X)$, where $Z \subset X$ is a closed subscheme of X of dimension i. Roughly, the addition is given by

$$[X_1] + [X_2] = [X_1 \cup X_2].$$

If $f : X \to Y$ is a proper map, then the push forward preserves this grading, i.e. $f_* : A_i(X) \to A_i(Y)$. For $[Z] \in A_i(X)$,

$$f_*[Z] = \begin{cases} \deg(f)[f(Z)] & \text{if } \dim Z = \dim f(Z); \\ 0 & \text{otherwise.} \end{cases}$$

Therefore if $Z \subset X$ is a 0-dimensional scheme,

$$\int_X [Z] = \#(Z),$$

i.e. it is the number of points in Z, counted with multiplicity.

If X is *smooth*, then we can also define Chow cohomology by

$$A^i(X) := A_{\dim X - i}(X).$$

This becomes a ring under the intersection product:

$$[X_1] \cdot [X_2] = [X_1 \cap X_2].$$

Defining rigorously, and in the highest generality, the intersection product $X_1 \cap X_2$, is highly non-trivial; see [Ful84]. However, in many 'nice' situations[1] $A^*(X) = H^*(X)$ - the usual integral (co)homology. In this situation, for $f : X \to Y$ a *flat* map[2], we also have *pull back maps*

$$f^* : A^i(Y) \to A^i(X); \quad [Z] \mapsto [f^{-1}Z].$$

1.1.2. *K-theory.* A good reference for what we need is [Bri05]. The category $\mathcal{C}$ is again the category of algebraic schemes, and the allowable functions are *proper* morphisms $f : X \to Y$. However, $K(X)$ is generated as a $\mathbb{Z}$-module by classes of coherent sheaves $[\mathcal{F}]$ modulo the Whitney relations $[\mathcal{F}] = [\mathcal{G}] + [\mathcal{H}]$ whenever we have a short exact sequence of sheaves $0 \to \mathcal{G} \to \mathcal{F} \to \mathcal{H} \to 0$. The K-theory group is filtered, and it is true (but non-trivial) to show that the associated graded $Gr(K(X)) = A_*(X)$. There are again push-forwards for proper morphisms and pull-backs for flat morphisms, and $K(pt) = \mathbb{Z}$. However, if $\mathcal{F}$ is a coherent sheaf and $[\mathcal{F}] \in K(X)$ is its associated K-theory class, then

$$\int_X [\mathcal{F}] = \chi(X; \mathcal{F}) = \sum (-1)^i \dim H^i(X; \mathcal{F}).$$

One can think of this as the (virtual) number of global sections of $\mathcal{F}$.

1.2. A classical example: the Gauss-Bonnet/Poincaré-Hopf theorem.

The classical theorem of Gauss-Bonnet (generalized by Poncaré-Hopf) states that if X is a compact manifold and T_X is its tangent bundle then its top-Chern class recovers the topological Euler characteristic:

$$\int_X c_{top}(T_X) = \chi(X) = \sum (-1)^i \dim_{\mathbb{Q}} H^i(X; \mathbb{Q}). \tag{1.4}$$

[1] For instance if X has a stratification by affine cells, see [Ful84, Examples 19.1.11].

[2] See [Har77] for precise definition.

(The integral of the left can also be written in terms of the function $\mathbb{1}_X$, and thus one can see this as an algebraic analogue of equation (1.1) above.) Let's illustrate this for $X = \mathbb{P}^1$, the *complex* projective line. Recall that

$$\mathbb{P}^1 = (\mathbb{C}^2 \setminus (0,0))/\mathbb{C}^*; \quad (z_0, z_1) \simeq (\lambda z_0, \lambda z_1), \quad \lambda \in \mathbb{C}^*.$$

From this description one can see that

$$\mathbb{P}^1 = \mathbb{C} \sqcup \{\infty\},$$

i.e. it is the one-point compactification of the plane $\mathbb{C} = \mathbb{R}^2$; in particular, it is homeomorphic to the real sphere S^2. From this description it follows that its Euler characteristic is $1 + 1 = 2$. *This is a topological calculation!* Algebraically, the tangent bundle of $\mathbb{P}^1$ is $\mathcal{O}_{\mathbb{P}^1}(2)$, and it has top Chern class $2[pt]$. Therefore

$$\int_{\mathbb{P}^1} 2[pt] = 2 \times 1 = 2.$$

Of course this calculation is easy to do in this simple example, but in general, understanding the tangent bundle, and its top Chern class can lead to very sophisticated mathematics. The idea of *equivariant (co)homology* is that in the presence of symmetries this calculation can be reduced to a calculation over a set which is much easier to understand. The price we need to pay is that we need to work in a more general context.

1.3. **Equivariant (co)homology theories.** Assume now that X admits a group action G. Most often $G = T = (\mathbb{C}^*)^k$ will be a complex torus. There are G-equivariant versions of both the Chow and K-theory. Everything is almost the same, except that all subschemes or coherent sheaves need to be equivariant with respect to the G-action and all maps are equivariant. However, one big difference is that the equivariant cohomology/K theory of the point is now quite big. For instance,

$$A_*^{(\mathbb{C}^*)^k}(pt) = \mathbb{Z}[t_1, \dots, t_k]; \quad K_*^{(\mathbb{C}^*)^k}(pt) = \mathbb{Z}[t_1^{\pm 1}, \dots, t_k^{\pm 1}].$$

In particular, taking integrals will now lead to either polynomials or Laurent polynomials! There are three main theorems about equivariant (co)homologies, and these will be considered in detail during this class. In all cases I assume that $G = T = (\mathbb{C}^*)^k$ is a complex torus acting on a projective (or compact) manifold X. We will identify homology and cohomology (although some care is needed to make this precise). Let

$$X^T := \{x \in X : t.x = x \ \forall t \in T\}.$$

We assume in addition that X^T is finite. (This will be the case in all our applications; but this hypothesis can be relaxed.)

Theorem 1.1 (Equivariant to classical)**.** *The usual (co)homology is a quotient of the equivariant (co)homology:*

$$A_*(X) = A_*^T(X)/I,$$

where $I = \langle t_1, \dots t_k \rangle$ is the ideal generated by the 'equivariant parameters' $t_1, \dots, t_k$.

Let $x \in X^T$ and consider the inclusion $i_x : \{x\} \to X$. This is an equivariant map, and it induces a ring homomorphism $i_X^* : A_T^i(X) \to A_T^i(pt) = \mathbb{Z}[t_1, \dots, t_k]$. This map is called *localization at* x, and if $\kappa \in A_T^i(X)$, then $\kappa|_x = i_x^*(\kappa)$ is called the localization of the class κ at x.

Theorem 1.2 (Goreski-Kottwitz-MacPherson (GKM) theorem)**.** *The inclusion $X^T \hookrightarrow X$ induces an* **injective** *ring homomorphism*

$$A_T^*(X) \to A_T^*(X^T) = \oplus_{x \in X^T} A_T^*(\{x\}) = \oplus_{x \in X^T} \mathbb{Z}[t_1, \dots, t_k].$$

In particular, every equivariant cohomology class is uniquely determined by its restrictions to fixed points. Image can be determined explicitly (by the GKM relations).

Theorem 1.3 (Atyiah-Bott (AB) localization)**.** *Let $\kappa \in A_T^*(X)$ be any equivariant cohomology class. Then*

$$\int_X^T \kappa = \sum_{x \in X^T} \frac{\kappa|_x}{c_{top}^T(T_x X)} \in \mathbb{Z}[t_1, \dots, t_k],$$

where $c_{top}^T(T_x X)$ is the product of all T-weight on the T-module $T_x X$, the tangent space of X at x.

1.4. **Equivariant Gauss-Bonnet.** The Gauss-Bonnet theorem also holds equivariantly, meaning that X is a G-variety, and all the classes on the left are replaced by the corresponding equivariant classes, then the equality in (1.4) holds. Let's see what happens in this case if one uses the AB localization theorem. We have

$$\int_X^T c_{top}^T(T_X) = \sum_{x \in X^T} \frac{c_{top}^T(T_X)|_x}{c_{top}^T(T_x X)}.$$

However, $i_x^*(c_{top}^T(T_X)) = c_{top}^T(T_x X)$, therefore

$$\int_X^T c_{top}^T(T_X) = \sum_{x \in X^T} 1 = \#(X^T).$$

We just proved that

$$\chi(X) = \#(X^T).$$

As an example, consider $\mathbb{P}^1$ with the action of $T = (\mathbb{C}^*)^2$ given by $(u, v).[x_0 : x_1] = [ux_0 : vx_1]$. Then

$$(\mathbb{P}^1)^T = \{[1:0], [0:1]\}.$$

(In fact one can show that $c_1^T(T_{\mathbb{P}^1}) \cap [\mathbb{P}^1]_T = [0:1]_T + [1:0]_T$, the sum of the equivariant fundamental classes of the torus fixed points. These classes are equivalent in the classical homology, thus giving $2[pt]$ in the example above.) Observe that if $X^T = \emptyset$, then $\chi(X) = 0$; for instance this happens if

$X = S^1$ (real circle) with the natural action of S^1. Therefore the topological data is encoded in the (algebraic) data of the fixed point set.[3]

1.5. **Goal of the class.** The goal is to study equivariant cohomology and equivariant K theory in the case when X is a projective space, a Grassmann manifold, or a flag manifold; understand the GKM theorem in this context; perform multiplications; and work out examples using the Atyiah-Bott localization. Some applications I have in mind:

(1) Applications of the Atiyah-Bott localization to enumerative geometry questions;
(2) Equivariant K theory and character formulas, such as the Weyl character formula.

2. Projective spaces, Grassmannians, and flag manifolds

A good reference for the facts presented in this section is Fulton's 'Young tableaux' [Ful97], especially Part III. Let V be a finite dimensional complex vector space. The following are the definitions of the main spaces used in this class. The **projective space** is

$$\mathbb{P}(V) = \{L \subset V : L \text{ is a linear subspace s.t. } \dim L = 1\}.$$

The **Grassmannian** is the set

$$\mathrm{Gr}(k, V) = \{F_k \subset V : F_k \text{ is a linear subspace s.t. } \dim F_k = k\}.$$

Observe that $\mathrm{Gr}(1, V) = \mathbb{P}(V)$.

The (complete) **flag manifold** is the set

$$\mathrm{Fl}(V) = \{(F_1 \subset F_2 \subset \ldots \subset F_{\dim V - 1} \subset V) : \dim F_k = k, \forall 1 \le k \le \dim V\}.$$

An element $F_\bullet := (F_1 \subset F_2 \subset \ldots \subset F_{\dim V-1} \subset V)$ is called a *flag*.

In the same family of spaces is any **partial flag manifold** $\mathrm{Fl}(i_1, \ldots, i_k; V)$. This parametrizes partial flags

$$F_{i_1} \subset F_{i_2} \subset \ldots \subset F_{i_k} \subset V,$$

where $\dim F_{i_j} = i_j$. Sometimes this is also called a **k-step flag manifold**. From this prospective, a 1-step flag manifold is a Grassmannian, and an n-step flag manifold is a complete flag manifold.

At this time these are just sets, but we will see later that we can endow them with a structure of complex manifolds, and also smooth projective algebraic varieties. For now, observe that these are **homogeneous** spaces, in the sense described in the next section.

[3] A warning: the Gauss-Bonnet theorem is usually stated for even (real-)dimensional varieties, so the proof above does not apply to S^1. However, it is true that in many cases the Euler characteristic of the fixed locus equals to the Euler characteristic of the variety; see e.g. [ByB73].

Remark 2.1. *A note about the field: in these notes, unless otherwise stated, we will work over the field of complex numbers. However, all the basic definitions, including those related to (equivariant) cohomology, work over any field. The difference arises when we actually describe the cohomology groups or rings. In fact, the (co)homology of the real projective space over the real numbers is much more subtle than that for the complex projective space. For instance, not all real projective spaces are orientable, while any complex manifold is automatically orientable.*

2.1. **Group actions.** Let $\mathrm{GL} := \mathrm{GL}(V;\mathbb{C})$ be the general linear group associated to the complex vector space V. It is clear that GL acts on each of the spaces above. For instance, fix a basis for F_k, i.e. $F_k = \langle v_1, \dots, v_k\rangle$ where $v_i \in V$. Then for $g \in \mathrm{GL}$,

$$g.\langle v_1, \dots, v_k\rangle := \langle gv_1, \dots, gv_k\rangle.$$

This gives an action on GL, in the sense that $id.F_k = F_k$ and $g_1.(g_2F_k) = (g_1g_2).F_k$. Obviously, this procedure also determines an action on $\mathrm{Fl}(V)$.

Definition 2.1. *A space X with a group action G is homogeneous if for any $x, y \in X$, there exists $g \in G$ such that $g.x = y$. Equivalently, G has a single orbit in X.*

Proposition 2.1. *The Grassmannian $\mathrm{Gr}(k, V)$ and the flag manifold $\mathrm{Fl}(V)$ are homogeneous under the action of $\mathrm{GL}(V)$.*

Proof. These claim follows from the familiar fact in linear algebra that any two bases of a vector space can be related to each other by some $g \in \mathrm{GL}$. □

Let $\dim V = n$, and fix the 'standard' basis $e_1, \dots, e_n$ of V, where $e_i = (0, \dots, 1, \dots, 0)$ and 1 is the in i-th position. Let $B \subset \mathrm{GL}$ be the subgroup of upper triangular matrices. The group B is called the *standard Borel subgroup* (its definition depends on the choice of a basis); any subgroup of GL conjugated to B is called a *Borel subgroup.* One example is the *opposite Borel subgroup* B^- defined as the subgroup of lower triangular matrices.

Exercise 1. *Show that B^- is a Borel subgroup.*

The intersection

$$T := B \cap B^- = \{ \begin{pmatrix} t_1 & 0 & \dots & 0 \\ 0 & t_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & t_n \end{pmatrix} : t_i \in \mathbb{C}^* \}$$

is called the *maximal torus.*

Exercise 2. *Let $N_{\mathrm{GL}}(T)$ be the normalizer of T in* GL.

(a) Prove that T is an abelian subgroup, and a normal subgroup of $N_{\mathrm{GL}}(T)$.

(b) Construct a group isomorphism $N_{\mathrm{GL}}(T)/T \simeq S_n$, the symmetric group of permutations in n letters.

In what follows we will define a complex manifold structure, and then analyze the torus fixed points and the Borel orbits in each of the spaces. We start by examining more concrete ways to describe the points and the actions on each space.

3. The projective space

Consider the identification

$$V = \mathbb{C}^{n+1} = \langle e_0, \dots, e_n \rangle.$$

Then the projective space can be written as the quotient

$$\mathbb{P}^n := \mathbb{P}(V) = (V \setminus 0)/\sim \quad ,$$

where the equivalence relation $\sim$ is given by $(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)$, with $\lambda \in \mathbb{C}^*$. This means that a point in $\mathbb{P}^n$ is given by a vector $v \in \mathbb{C}^{n+1} \setminus \{0\}$; this vector spans a line $\mathbb{C}^{n+1}$, with the ambiguity of multiplying by a nonzero complex scalar λ. For $v = (x_0, \dots, x_n) \in \mathbb{C}^{n+1} \setminus \{0\}$, we denote by $[x_0 : \dots : x_n]$ the equivalence class determined by v. The coordinates $x_0, \dots, x_n$ are called *projective coordinates*; these are defined only up to multiplying by a non-zero scalar.

We can use coordinates in order to describe the action of GL. If $(g_{i,j}) \in$ GL, then

$$(g_{i,j}).[x_0 : \dots : x_n] = [\sum_{i=1}^{n+1} g_{1,i}x_0 : \dots : \sum_{i=1}^{n+1} g_{n,i}x_n].$$

In other words, this is given by regarding $(x_0, \dots, x_n)$ as column vector, then multiplying on the left by the matrix $(g_{i,j})$. Note that this is independent of the choice of actual representatives.

One can also work with the row vector determined by v, and then multiplying on the right. This gives a different - right - action, but of course the two points of view are equivalent.

Define the subset $U_i := \{[x_0 : \dots : x_i : \dots : x_n] : x_i \neq 0\} \subset \mathbb{P}^n$; this is called a *distinguished open set*. Define the map $\varphi_i : U_i \to \mathbb{C}^n$ given by

$$\varphi_i[x_0 : \dots : x_i : \dots : x_n] = (\frac{x_0}{x_i}, \dots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \dots, \frac{x_n}{x_i}).$$

The map is clearly a bijection. The sets U_i cover $\mathbb{P}^n$, i.e.

$$\mathbb{P}^n = \bigcup_{i=0}^{n} U_i.$$

Proposition 3.1. *The collection $\{(U_i, \varphi_i), 0 \leq i \leq n\}$ is an atlas, giving $\mathbb{P}^n$ a structure of a complex manifold of dimension n. Further, the manifold $\mathbb{P}^n$ is compact.*

Proof. The definition of manifolds and the proof that $\mathbb{P}^n$ has the claimed complex manifold structure is explained in [GH94, p. 15]. It remains to show that $\mathbb{P}^n$ is compact. Let S^{2n+1} be the real sphere in $V = \mathbb{C}^{n+1}$. The projection $\mathbb{C}^{n+1} \setminus 0 \to \mathbb{P}^n$ is continuous and surjective, and it restricts to a surjective continuous map $S^{2n+1} \to \mathbb{P}^n$. Since S^{2n+1} is a compact manifold (because it is closed and bounded in the metric space $\mathbb{C}^{n+1}$), it follows that $\mathbb{P}^n$ is also compact. □

3.0.1. *B and T orbits.* We are interested in the 0 and 1 dimensional torus orbits, and in the Borel orbits.

Proposition 3.2. *(a) The torus fixed points are given by $\mathbb{P}(\mathbb{C}e_i) = [0 : \dots : 1 : \dots : 0]$ where $0 \le i \le n$.*

(b) Each Borel orbit is an orbit of one of the T-fixed points $\mathbb{P}(\mathbb{C}e_i)$.

(c) Each one-dimensional torus orbit is of the form

$$C_{i,j} := \mathbb{P}(\langle e_i, e_j\rangle) \setminus \{\mathbb{P}(\mathbb{C}e_i), \mathbb{P}(\mathbb{C}e_j)\},$$

for $0 \le i < j \le n$.

Proof. Consider a diagonal matrix $A = diag(t_1, \dots, t_{n+1})$. Then

$$A.[x_0 : \dots : x_n] = [t_1 x_0 : \dots : t_{n+1} x_n].$$

It follows from this description that a torus fixed point can contain only one nonzero projective coordinate. Part (a) follows from this.

To prove part (b) we calculate the B-orbits of the T-fixed points found in (a). We have

$$\begin{pmatrix} b_{1,1} & b_{1,2} & b_{1,3} & \dots & b_{1,i} & \dots & b_{1,n+1} \\ 0 & b_{2,2} & b_{2,3} & \dots & b_{2,i} & \dots & b_{2,n+1} \\ 0 & 0 & b_{3,3} & \dots & b_{3,i} & \dots & b_{3,n+1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & \dots & \dots & b_{n+1,n+1} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} = \begin{pmatrix} b_{1,i} \\ \vdots \\ b_{i,i} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

In terms of projective coordinates, this implies that $B.\mathbb{P}(\mathbb{C}e_i)$ consists of all points of the form $[* : \dots : * : 1 : 0 : \dots : 0]$ with 1 in position i. But these sets are non-intersecting, and they cover $\mathbb{P}^n$, therefore each orbit must be of this form. This proves (b).

For part (c), observe first that that $C_{i,j}$ is a one-dimensional torus orbit. Indeed, $\mathbb{P}(\langle e_i, e_j\rangle) \simeq \mathbb{P}^1$. If $[u : v]$ are the coordinates on the last $\mathbb{P}^1$, then the torus T acts by $diag(t_1, \dots, t_{n+1}).[u : v] = [t_{i+1}u : t_{j+1}v]$. The points $[1 : 0]$ and $[0 : 1]$ are fixed, and it is clear that T acts transitively on the complement. Since $\dim \mathbb{P}^1 = 1$ and $C_{i,j}$ is open in $\mathbb{P}^1$, it follows that $\dim C_{i,j} = 1$.

It remains to show that any one-dimensional T-orbit is of this form. To do that, observe that the closure $\overline{C_{i,j}} \simeq \mathbb{P}^1$ can also be described as the locus where all but 2 projective coordinates are zero: $x_k = 0$ for $k \neq i, j$. Since the open set of any one dimensional orbit has a point $[v]$ with at least two non-zero projective coordinates, we may assume $[v]$ has at least 3 nonzero coordinates. Then w.l.o.g. we can assume $[v]$ has *all* coordinates nonzero. (If not, just work in a smaller projective space.) But then the orbit $T.[v] = (\mathbb{C}^*)^{n+1}.[v]$ is just $\mathbb{P}^n$ minus all the hyperplanes $x_i = 0$, for $0 \leq i \leq n$, and this is an open set in $\mathbb{P}^n$. (It is open when intersected with each of the distinguished open sets U_i.) In particular, this orbit has dimension n, which contradicts that it must be one-dimensional. $\square$

The proposition and its proof has several consequences.

Corollary 3.3. *(a) For $0 \leq i \leq n$, the Borel orbit of $\mathbb{P}(\mathbb{C}e_i)$ is isomorphic to the affine space $\mathbb{C}^i$, and its closure is isomorphic to $\mathbb{P}^i$. This orbit, denoted by*

$$\Omega_i^\circ = \{[* : \ldots : * : 1 : 0 : \ldots 0]\},$$

is called a Schubert cell*; its closure, denoted by Ω_i, is called a* Schubert variety.[4]

(b) The Schubert cells provide a stratification of $\mathbb{P}^n$, i.e. the following hold:

$$\mathbb{P}^n = \bigsqcup_{k=0}^{n} \Omega_k^\circ; \quad \Omega_k = \bigsqcup_{i=0}^{k} \Omega_i^\circ.$$

In particular, part (b) shows that the Schubert variety Ω_i has a CW decomposition with even dimensional cells. General results imply that the closures of these cells give a basis for $H_*(\Omega_i)$, the homology group of Ω_i.

An analogue of this corollary is true for any Grassmannian or (partial) flag manifold. Something that is specific to $\mathbb{P}^n$ is that it is a *toric variety*, i.e. it has an *open dense* T-orbit. This was also observed in the proof: the orbit of $[1 : \ldots : 1]$ is open and dense in $\mathbb{P}^n$.

ADD Examples.

3.1. Tangent spaces for $\mathbb{P}^n$.

Lemma 3.4. *Fix $0 \leq i \leq n$ and consider the distinguished open set $U_i = \{[x] : x_i \neq 0\}$. Then the 'coordinate map' $\varphi_i : U_i \to \mathbb{C}^n$ sending $[x_0 : \ldots : 1 : \ldots : x_n] \to (x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ is T-equivariant, where T acts on $\mathbb{C}^n$ by*

$$(t_1, \ldots, t_{n+1}).(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) =$$
$$(\frac{t_1}{t_{i+1}}x_0, \ldots, \frac{t_i}{t_{i+1}}x_{i-1}, \frac{t_{i+2}}{t_{i+1}}x_{i+1}, \ldots, \frac{t_{n+1}}{t_{i+1}}x_{n+1}).$$

Further $\varphi(\mathbb{P}(\mathbb{C}e_i)) = 0 \in \mathbb{C}^n$ and both points are T-fixed.

The previous statements imply that $(d\varphi)_{\mathbb{P}(\mathbb{C}e_i)} : T_{\mathbb{P}(\mathbb{C}e_i)}\mathbb{P}^n \to \mathbb{C}^n$ is an isomorphism of T-modules. Observe that

$$T_0\mathbb{C}^n = \oplus_{j=0}^{n} \mathbb{C}_{\chi_j}$$

[4]The terminology *variety* means that Ω_i is an algebraic variety. This is obvious in this case (it is a projective space) but it is less so in general.

where $\chi_j(t) = \frac{t_{j+1}}{t_{i+1}}$ for and $0 \leq i, j \leq n$, and $i \neq j$. This implies that $T_{\mathbb{P}(\mathbb{C}e_i)}$ has a basis consisting of eigenvectors v_j each of which an eigenvector for one of the characters χ_j.

Exercise 3. *Use the homogeneity to describe the tangent spaces at any point in $\mathbb{P}^n$.*

3.2. **The moment graph.**

Definition 3.1. *Let X be a T-space. The moment graph is the graph with vertices corresponding to the 0-dimensional orbits, and edges xy corresponding to 1-dimensional orbits joining $x \in X^T$ and $y \in X^T$.*

The previous results imply the following description of the moment graph:

- The vertices correspond to points $\mathbb{P}(\mathbb{C}e_i) = [0 : \ldots : 1 : \ldots : 0]$ with $0 \leq i \leq n$.
- Any two vertices are connected by a T-stable curve. Therefore the moment graph is the complete graph K_{n+1} with $n+1$ vertices.
- To each edge $C_{i,j}$ we associate the weight $t_i - t_j$. This is the (additive) weight of the tangent space on $\overline{C_{i,j}} = \mathbb{P}(\langle e_i, e_j \rangle)$ at the point $\mathbb{P}(\mathbb{C}e_j)$.

Example 3.5. *Draw the moment graph for $\mathbb{P}^2$.*

3.3. **GKM theorem for the projective space.** As explained by Goresky-Kottwitz-MacPherson [GKM98], for *equivariatly formal spaces*, the moment graph, together with the weights associated to the 1-dimensional T-orbits, determines completely the T-equivariant cohomology of X. Equivariant formality is rather difficult to define in full generality, but for now we mention that the class of equivariantly formal spaces include any space with a T-invariant CW decomposition and even dimensional cells; in particular, all flag manifolds are included.

Let $\mathbb{Z}[t] := \mathbb{Z}[t_1, \ldots, t_{n+1}]$ be the polynomial ring in $n+1$ variables. This is a graded ring, with $\deg(t_i) = 1$.

Theorem 3.1 (GKM)**.** *There is a graded $\mathbb{Z}[t]$-algebra isomorphism*

$$H_T^*(\mathbb{P}^n) \simeq \{(P_i(t)) \in \oplus_{i=0}^n \mathbb{Z}[t] : (t_{i+1} - t_{j+1}) | (P_i(t) - P_j(t))\}.$$

Under this isomorphism, the equivariant Schubert class $[\Omega_i]_T$ is the unique class satisfying:

- *(support) $P_k(t) = 0$ for $k > i$;*
- *(degree) $\deg P_k(t) = n - i$;*
- *(normalization condition) $P_i(t) = \prod_{k > i+1} (t_k - t_{i+1})$.*

Example 3.6. *Work out the $\mathbb{P}^1$ case.*

Example 3.7. *Work out the Schubert classes and their multiplication for $\mathbb{P}^2$. The classes are $[1 : 0 : 0], [* : * : 0]$ and $[\mathbb{P}^2]$. The localizations are*

$$[1 : 0 : 0] = ((t_2 - t_1)(t_3 - t_1), 0, 0); \quad [* : * : 0] = (t_3 - t_1, t_3 - t_2, 0); \quad [\mathbb{P}^2] = (1, 1, 1).$$

Exercise 4. *Let ζ be the class in $H_T^*(\mathbb{P}^1)$ defined by $(-t_1, -t_2)$. Prove that there is an algebra isomorphism*

$$H_T^*(\mathbb{P}^1) \simeq \mathbb{Z}[t_1, t_2; \zeta]/\langle(\zeta + t_1)(\zeta + t_2)\rangle.$$

(The class $-\zeta$ is the equivariant first Chern class of the tautological line bundle $\mathcal{O}(-1)$.)

Remark 3.1 (A survivor introduction through (equivariant) cohomology)**.** *Let X be a complex manifold of (complex) dimension n. Assume that X has an action of a torus $T \simeq (\mathbb{C}^*)^k$. The the following hold:*

- *any closed, irreducible subspace $Y \subset X$ which is T-invariant and of codimension c determines an equivariant cohomology class*
$$[Y]_T \in H_T^{2p}(X);$$
- *$H_T^*(pt) = \mathbb{Z}[t_1, \dots, t_k]$;*
- *If Y_1, Y_2 are irreducible and invariant then $[Y_1]_T + [Y_2]_T = [Y_1 \cup Y_2]_T$;*
- *If Y_1, Y_2 are irreducuble, T-invariant, and suitably* **transversal** *then*
$$[Y_1]_T \cdot [Y_2]_T = [Y_1 \cap Y_2]_T.$$
- *If one is interested in ordinary cohomology addition, product etc, make all equivariant parameteres $t_i \mapsto 0$.*
- *If X_1, X_2 are complex manifolds with T-action and $f : X_1 \to X_2$ is a T-equivariant map, then there are pull-back maps $f^* : H_T^{2i}(X_2) \to H_T^{2i}(X_1)$; note that these maps are degree (or codimension) preserving. Roughly*
$$f^*[Y]_T = [f^{-1}(V)]_T.$$
Very often $X_1 = \{x\} \subset X_2$ is a fixed point in X_2.
- *If $f : X_1 \to X_2$ is a T-equivariant* **proper** *map, then we get push-forwards $f_* : H_T^{2i}(X_1) \to H_T^{2(\dim X_2 - \dim X_1 + i)}(X_2)$. These maps preserve* **dimension**. *Roughly,*
$$f_*[Z]_T = \deg(f)[f(Z)]_T,$$
where $\deg(f)$ is the number of points in a generic fibre over $f(Z)$. This degree is 0 if the fibre is positive dimensional.
- *For 'nice' manifolds X, we have*
$$H^*(X) = H_T^*(X)/\langle t_1, \dots, t_k\rangle.$$
All flag manifolds in this class will be 'nice'.

Remark 3.2. *The class $[\Omega_i]_T$ is the equivariant cohomology class of the Schubert variety of dimension i. In particular, $[\Omega_i]_T \in H_T^{2(n-i)}(\mathbb{P}^n)$, as the complex codimension equals to $n - i$. [Add normal bundle calculation.]*

4. Grassmannians

In this section we let $V \simeq \mathbb{C}^n = \langle e_1, \dots, e_n \rangle$. Fix $1 \leq k \leq n$. Recall that $\mathrm{Gr}(k,n) := \mathrm{Gr}(k,V)$ denotes the Grassmannian of subspaces of dimension k in $\mathbb{C}^n$. Each $V \in \mathrm{Gr}(k,n)$ can be written as the span of a $k \times n$ matrix

$$A_V := \begin{pmatrix} a_{1,1} & a_{1,2} & \dots & \dots & a_{1,n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{k,1} & a_{k,2} & \dots & \dots & a_{k,n} \end{pmatrix}$$

The condition that V is the row span of A_V implies that $rank(A_V) = k$. We will refer to A_V as a *matrix representing* V. Observe that A_V is *not* unique. In fact, the following transformations will not change the row-span of a matrix:

- exchange two rows;
- add a multiple of one row to another.

These can be summarized as 'row operations'; performing a row operation is equivalent to multiplying on the *left* by a matrix $h \in GL_k(\mathbb{C})$.

4.1. **Manifold structure.** In this section we indicate how one can define a complex manifold structure on $\mathrm{Gr}(k,n)$. The following is the key observation. Let A be a $k \times n$ matrix of rank k. Fix a set of k columns $1 \leq i_1 < i_2 < \dots < i_k \leq n$ and assume that $\det A_{i_1,\dots,i_k} \neq 0$, where $A_{i_1,\dots,i_k} \in \mathrm{GL}_k(\mathbb{C})$ is the minor formed by the given column.

Lemma 4.1. *Let M, N be two $k \times n$ matrices with minors of nonzero determinant on columns $i_1, \dots, i_k$. Then*

$$RowSpan(M) = RowSpan(N) \Longleftrightarrow M^{-1}_{i_1,\dots,i_k} M = N^{-1}_{i_1,\dots,i_k} N.$$

Proof. The lemma follows because row operations do not change the row span. More explicitly, $RowSpan(M) = RowSpan(M^{-1}_{i_1,\dots,i_k} M)$. $\square$

For $I := (i_1, i_2, \dots, i_k)$ define the set $U_I \subset \mathrm{Gr}(k,n)$ determined by those vector spaces V such that one (equivalently, any) matrix A_V representing V has $\det(A_V)_I \neq 0$. It is clear that $\mathrm{Gr}(k,n) = \bigcup_I U_I$, where the union is over k-tuples of columns. Define

$$\varphi_I : U_I \to \mathbb{C}^{k(n-k)}; \quad A \mapsto A^{-1}_{i_1,\dots,i_k} A.$$

The map φ_I is well defined because of the Lemma. Observe that

$$A^{-1}_{i_1,\dots,i_k} A = \begin{pmatrix} * & 1 & * & 0 & * & 0 & * & 0 & * \\ * & 0 & * & 1 & * & 0 & * & 0 & * \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ * & 0 & * & 0 & * & 0 & * & 1 & * \end{pmatrix}$$

where the identity matrix is on columns $(i_1, \dots, i_k)$ and there are $k(n-k)$ entries corresponding to $*$'s.

Proposition 4.2. *The collection* $\{(U_{(i_1,\dots,i_k)}, \varphi_{(i_1,\dots,i_k)}) : 1 \leq i_1 < \dots < i_k \leq n\}$ *forms an atlas, giving the Grassmannian* $\mathrm{Gr}(k,n)$ *a structure of a complex manifold of dimension* $k(n-k)$.

Proof. Exercise. □

4.2. **The Plücker embedding.** In this section we indicate (without proofs) that $\mathrm{Gr}(k,n)$ is a projective variety, i.e. it can be realized as the zero locus of several homogeneous polynomials in a projective space. In other words, we need to define a closed embedding $\iota : \mathrm{Gr}(k,n) \to \mathbb{P}^N$. One such embedding is the *Plücker embedding.* The easiest way to define it is to use the 'abstract' version of the Grassmannian $\mathrm{Gr}(k,V)$. Then define

$$\iota : \mathrm{Gr}(k,V) \to \mathbb{P}(\wedge^k V); \quad W \mapsto \wedge^k W.$$

However, it is not clear from this why the image is closed, and not even why this map is injective!

For vectors $v_1, \dots, v_k \in V$ we define by $v_1 \wedge \dots \wedge v_k \in \wedge^k V$ to be the *Plücker coordinate* of $v_1, \dots, v_k$. To make this more explicit, we fix a basis $e_1, \dots, e_n$ of V. Then $v_1, \dots, v_k$ determine $\binom{n}{k}$ Plücker coordinate obtained by expanding $v_1 \wedge \dots \wedge v_k$ in the basis $\{e_{i_1} \wedge \dots \wedge e_{i_k}\}$ of $\wedge^k V$.

Lemma 4.3. *The Plücker embedding* ι *is injective.*

Proof. Let $W_1, W_2 \in \mathrm{Gr}(k,V)$. Since the definition of the map ι does not depend on the choice of a basis for V, it follows that we may assume that $W_1 = \langle e_1, \dots, e_k \rangle$. The corresponding Plücker coordinates are $[1 : 0 : \dots : 0]$. We now argue by induction on $r \geq 1$ where $\dim W_1 \cap W_2 \geq k - r$. It is easy to see that the spaces W_1 and $W_2 = \langle e_1, \dots, e_{k-r}, e_{k+1}, \dots, e_{k+r} \rangle$ have different Plücker coordinates. □

We now turn to matrices. Let A be a $k \times n$ matrix of rank k. The *Plücker coordinates* of A are defined by

$$p_{i_1,\dots,i_k} = \det A_{(i_1,\dots,i_k)}.$$

(Of course this is the same as the Plücker coordinates for the wedge of the rows of A). If matrices A, A' have the same row span, then $A' = gA$ for some $g \in \mathrm{GL}_k$. Then $p_{i_1,\dots,i_k}(A') = \det g \cdot p_{i_1,\dots,i_k}(A)$. This shows that we have a well defined map

$$\iota : \mathrm{Gr}(k,n) \to \mathbb{P}^{\binom{n}{k}-1} : \quad V \mapsto [\dots : p_I(A_V) : \dots],$$

where A_V is any matrix representing V. This is clearly an algebraic morphism, as it is given by polynomial equations.

Lemma 4.4 (Sylvester, 1851)**.** *Let* M, N *be two* $k \times k$ *matrices, and let* $1 \leq r \leq k-1$ *be an integer. Then*

$$\det M \cdot \det N = \sum \det(M') \cdot \det(N'),$$

where the sum is over pairs of $k\times k$ matrices (M', N') obtained from (M, N) by exchanging a fixed set of r columns of N with any r columns of M, preserving the ordering of the columns.

Proof. See [Ful97, p. 108]. $\square$

Note that we may assume the fixed r columns are the first.

Theorem 4.1. *The image of the Plücker embedding ι is given by the quadratic relations*

$$p_{i_1,\dots,i_k}\cdot p_{j_1,\dots,j_k} - \sum p_{i'_1,\dots,i'_k}\cdot p_{j'_1,\dots,j'_k},$$

where the sum is over all exchanges of $i_1,\dots,i_k$ by a fixed set of r elements of $j_1,\dots,j_k$, maintaining order in each, and where $1\le r\le k-1$. Signs will show up upon rearranging indices in increasing order.

Proof. By Sylvester's lemma, any element in the image of ι must satisfy the quadratic relations. It remains to show that any point satisfying the quadratic relations is the image of some $V\in \mathrm{Gr}(k,n)$. For that, it suffices to show that given Plücker relations we can find a matrix giving them. First, we can rescale the Plücker coordinates by the same quantity and assume that one of them equals 1; w.l.o.g. $p_{1,2,\dots,k}=1$. But then we construct the matrix A having the first $k\times k$ minor equal to the identity, and the remaining coordinates defined by

$$a_{i,j} = (-1)^{k-i} p_{1,\dots,\hat{i},\dots,k,j}.$$

It remains to show that the Plücker coordinates of A equal to $p_{i_1,\dots,i_k}$. By construction, this is true for any $p_{j_1,\dots,j_k}$ such that $|\{1,\dots,k\}\cap\{j_1,\dots,j_k\}|\ge k-1$. Using the quadratic relations for the Plücker coordinates of A, one can find inductively all coordinates with overlap $k-2$, $k-3$, and so on; thus these must agree with the coordinates $p_{i_1,\dots,i_k}$. This finishes the proof. $\square$

Remark 4.1. *The proof actually shows that if $p_{1,\dots,k}\neq 0$, the coordinates $p_{i_1,\dots,i_k}$ are determined by the quadratic relations and the coordinates of the form $p_{1,\dots,\hat{i},\dots,k,s}$.*

Example 4.5. *The Grassmannian $\mathrm{Gr}(2,4)$ is a quadric in $\mathbb{P}^5$ given by equation*

$$p_{12}p_{34} - p_{13}p_{24} + p_{23}p_{14} = 0.$$

This is the only Grassmannian which is not a projective space and which is a complete intersection.

Example 4.6. *For the Grassmannian $\mathrm{Gr}(2,n)\subset \mathbb{P}^{\binom{n}{2}-1}$ there are $\binom{n}{4}$ Plücker relations. Observe that for $n\ge 5$,*

$$\binom{n}{2} - 1 - \binom{n}{4} < 2(n-2) = \dim \mathrm{Gr}(2,n).$$

Example 4.7. *In* $\mathrm{Gr}(3,6)$ *we have a relation*

$$p_{123}p_{145} - p_{423}p_{115} - p_{143}p_{125} - p_{124}p_{135} = 0.$$

This relation can be used to calculate p_{145} *from relations involving* p_{ijk} *where* $|\{i,j,k\} \cap \{1,2,3\} = 2$.

Remark 4.2. *One can prove that the ideal of the Plücker relations is a prime ideal, meaning that the Grassmannian is a reduced, irreducible, projective variety. For a proof, see* [Ful97, §8.4].

4.3. **Schubert cells and varieties.**

Lemma 4.8. *There is a bijection between the following sets:*

(a) 01 *words of length* n *with* k *labels equal to* 1*;*
(b) *Young diagrams (or* partitions*)* $\lambda = (\lambda_1, \dots, \lambda_k)$ *included in the* $k \times (n-k)$ *rectangle (i.e.* $n-k \geq \lambda_1 \geq \dots \geq \lambda_k \geq 0$*);*
(c) Grassmannian permutations, *i.e. permutations* $w \in S_n$ *such that* $w(1) < \dots (k)$ *and* $w(k+1) < \dots < w(n)$*;*
(d) *Weakly increasing sequences of non-negative integers* $0 \leq i_1 \leq i_2 \leq \dots \leq i_n = k$ *such that* $0 \leq i_s - i_{s-1} \leq 1$*. (The places where* $i_s - i_{s-1} = 1$ *are called jumps.)*
(e) *Strictly increasing sequences* $1 \leq j_1 < j_2 < \dots < j_k \leq n$*. (This is called the* **jumping sequence** *of* λ*.)*

Proof. The 1's correspond to the vertical steps, read NE-SW on the Young diagram, and also correspond to the jumps in the sequence from part (iv). □

We note that under the bijection above, the partition λ corresponds to the jumping sequence $(n-k+1-\lambda_1 < n-k+2-\lambda_2 < n-k+k-\lambda_k$. The corresponding Grassmannian permutation w_λ has values $w(i) = n-k+i-\lambda_i$ for $1 \leq i \leq k$.

We denote by $|\lambda| = \lambda_1 + \dots + \lambda_k$ the number of boxes of λ. We denote by $I(\lambda)$ the jumping sequence in (d).

Exercise 5. *Prove that* $\ell(w_\lambda) + |\lambda| = \dim \mathrm{Gr}(k,n)$.

Fix a flag $F_\bullet : F_1 \subset F_2 \subset \dots \subset F_n = \mathbb{C}^n$, and consider $V \in \mathrm{Gr}(k,n)$. Since $\dim V = k$, it follows that the the sequence

$$0 \leq \dim V \cap F_1 \leq \dots \leq \dim V \cap F_n = k$$

is a sequence with k jumps as in the part (d) of the Lemma.

Definition 4.1. *Let* λ *be a Young diagram included in the* $k \times (n-k)$ *rectangle.*

(a) The **Schubert cell** $\Omega^\circ_\lambda(F_\bullet)$ *is defined by as the subset of those* $V \in \mathrm{Gr}(k,n)$ *such that the sequence* $\dim(V \cap F_i)$ *has jumps in positions* $n-k+i-\lambda_i$*, for* $1 \leq i \leq k$.

(b) The **Schubert variety** *is defined by*

$$\Omega_\lambda(F_\bullet) := \{V \in \mathrm{Gr}(k,n) : \dim V \cap F_{n-k+i-\lambda_i} \geq i\}.$$

Proposition 4.9. *(a) The Schubert cell $\Omega^\circ_\lambda(F_\bullet)$ is isomorphic to $\mathbb{C}^{k(n-k)-|\lambda|}$.*

(b) Each Schubert variety $\Omega_\lambda(F_\bullet)$ is the disjoint union of Schubert cells:

$$\Omega_\lambda(F_\bullet) = \bigsqcup_{\lambda \subset \mu} \Omega^\circ_\mu(F_\bullet).$$

(c) The Schubert variety $\Omega_\lambda(F_\bullet)$ is the closure of the corresponding Schubert cell. In particular,

$$\dim \Omega_\lambda(F_\bullet) = \dim \Omega^\circ_\lambda(F_\bullet) = k(n-k) - |\lambda|.$$

Proof. The idea is to show that if $V \in \Omega^\circ_\lambda(F_\bullet)$ the conditions defining the Schubert cell determine a canonical 'reduced row-echelon' matrix A_V such that $rowspan(A_V) = V$. Let $(I_1, \dots, i_k)$ be the jumping sequence determined by the partition λ (determined from Lemma 4.8(d)). We are building a 'canonical' basis for V.

Since $\dim V \cap F_{i_1} = 1$ and $\dim V \cap F_s = 0$ for $s < i_1$, we obtain that

$$V \cap F_{i_1} = \langle e_{i_1} + \sum_{s<i_1} a_s e_s \rangle,$$

with a_s uniquely determined. In other words, we build the first row of A_v by placing a 1 in column i_1, and all 0's after that. The second row is built similarly, with a 1 on column i_2, 0's after that, and a 0 in column i_1, obtained because we are allowed to substract row 1. This procedure gives a unique matrix satisfying the following conditions:

- On column i_j there is a 'pivot' 1 in row j, and all other entries are 0;
- all row entries to the *right* of a pivot equal to 0.

The undetermined coordinates of this matrix give the isomorphism to $\mathbb{C}^{k(n-k)-|\lambda|}$, proving part (a).

Part (b) follows from the construction in (a). For part (c), it suffices to prove two things: that $\Omega_\lambda(F_\bullet)$ is closed, and that for any point $x \in \Omega^\circ_\mu(F_\bullet)$ there exists an open set U with $x \in U$ such that $U \cap \Omega^\circ_\lambda(F_\bullet) \neq \emptyset$.

To show that $\Omega_\lambda(F_\bullet)$ is closed, it suffices to show that the variety $\{V \in Gr(k,n) : \dim V \cap F_i \geq j\}$ is closed (possibly empty). Consider the sequence of vector spaces

$$\begin{tikzcd} 0 \arrow[r] & V \arrow[r] & \mathbb{C}^n \arrow[r] & \mathbb{C}^n/V \arrow[r] & 0 \\ & & F_i \arrow[u] \arrow[ur, "M_V"'] & & \end{tikzcd}$$

The condition that $\dim V \cap F_i \geq j$ implies that $\dim \ker M_V \geq j$, or, equivalently, that $\operatorname{rank}(M_V) \leq \dim(\mathbb{C}^n/V) - j = n-k-j$. In turn, this condition is equivalent to vanishing of the determinants of all $n-k-j+1$ minors. Since this is a polynomial condition, it is closed, and it will hold for coordinates arising from any affine neighborhood of V. This implies that $\Omega_\lambda(F_\bullet)$ is closed.

Let now $x \in \Omega_\lambda(F_\bullet)$. W.l.o.g. we may assume that that $x = e_\mu$ is the rowspan of a 01 matrix in a Schubert cell $\Omega_\mu^\circ(F_\bullet)$. (This will be clear once we realize Schubert cells as B^--orbits; cf. Exercise 6 below.) The standard open neighborhood U_μ of e_μ consists of coordinates outside the columns given by the jumps of the partition μ. Since $\mu \subset \lambda$, it follows that the space given by $rowspan(e_\mu + e_\lambda)$ is in the Schubert cell $\Omega_\lambda^\circ(F_\bullet)$. By rescaling the 1's coming from the e_λ we see that any neighborhoods of e_μ intersects the Schubert cell $\Omega_\lambda^\circ(F_\bullet)$. This finishes the proof. $\square$

A particular case of (a) is that the Grassmannian is the disjoint union of the Schubert cells $\Omega_\lambda^\circ(F_\bullet)$. This implies that $\mathrm{Gr}(k,n)$ is a CW complex with even (real)-dimensional cells. A consequence is that the fundamental classes of the closures of these cells form a basis for the usual homology $H_*(\mathrm{Gr}(k,n))$:

$$H^*(\mathrm{Gr}(k,n)) = \oplus_\lambda \mathbb{Z}[\Omega_\lambda],$$

where the sum is over all partitions $\lambda \subset k \times (n-k)$.

Remark 4.3. *Any two flags $F_\bullet, G_\bullet$ can be related by some $g \in \mathrm{GL}_n$, i.e $g.F_\bullet = G_\bullet$. Since GL_n is path connected, the multiplication by g is homotopy equivalent to the identity, therefore the Schubert varieties $\Omega_\lambda(F_\bullet)$ and $\Omega_\mu(G_\bullet)$ have the same* **non-equivariant** *fundamental class: $[\Omega_\lambda(F_\bullet)] = [\Omega_\mu(G_\bullet)] \in H^{2|\lambda|}(\mathrm{Gr}(k,n))$.*

This is not true equivariantly: for instance two T-fixed points have different cohomology classes, e.g. by the GKM theorem.

Example 4.10. *Work out the Schubert cell decomposition for* $\mathrm{Gr}(2,4)$.

4.4. **The moment graph.** We need to describe the T-fixed points, the weights of the tangent space at each of these points, and the 1-dimensional T-orbits. As before we fix the standard basis $e_1, \dots, e_n$ of $\mathbb{C}^n$, and $F_\bullet$ will be the standard flag

$$F_\bullet : \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \mathbb{C}^n.$$

We will omit the dependence on $F_\bullet$ from the notation.

Lemma 4.11. *Let λ be a partition in the $k \times (n-k)$ rectangle given by the jump sequence $(i_1, \dots, i_k)$. Each Schubert cell Ω_λ° contains exactly one T-fixed point $e_\lambda := \langle e_{i_1}, \dots, e_{i_k} \rangle$.*

Proof. Follows from the description of the reduced row echelon form matrix for the spaces V in a fixed Schubert cell. $\square$

Exercise 6. *Show that the Schubert cells defined above are the B^--orbits of the T-fixed points.*

Proposition 4.12. *Let λ be a partition in the $k \times (n-k)$ rectangle, and let $J(\lambda) = \{j_1, \dots, j_k\}$ the corresponding set of jumps. Then the weights of the tangent space at e_λ (written additively) are $t_i - t_j$, where $j \in J(\lambda)$ and $i \in \{1, 2, \dots, n\} \setminus J(\lambda)$.*

Proof. W.l.o.g. we assume that λ is the full $k \times (n-k)$ rectangle, so $I(\lambda) = (1, 2, \dots, k)$. Let $\varphi := \varphi_{J(\lambda)}$ be the coordinate map corresponding to λ. This is a T-equivariant isomorphism $\varphi : U_{J(\lambda)} \to \mathbb{C}^{k(n-k)}$ such that $\varphi(e_\lambda) = 0$. By Proposition 8.1(f) there is a T-equivariant isomorphism $d(\varphi)_{e_\lambda} : T_{e_\lambda}(\mathrm{Gr}(k,n)) \to T_0(\mathbb{C}^{k(n-k)}) \simeq \mathbb{C}^{k(n-k)}$. Thus we may identify equivariantly the tangent space $T_{e_\lambda}(\mathrm{Gr}(k,n))$ to the coordinates on $U_{J(\lambda)}$. It follows that in order to calculate the action of T on $T_0(\mathbb{C}^{k(n-k)})$, we need to calculate the action of T in terms of the coordinates on $U_{J(\lambda)}$. Using the canonical reduced row echelon form, we obtain that

$$rowspan(Id_k|(a_{i,j})) \times (diag(t_1, \dots, t_n)) = rowspan(Id_k|(\frac{t_j}{t_i} a_{i,j})).$$

The claim follows from this. ☐

Given a partition λ a **rim hook** is a connected subset of the boxes having at least one point in common to the SE boundary of λ. For instance (2) is obtained by removing a maximal rim hook from the partition $(3,3,1)$.

Lemma 4.13. *Let $\lambda \neq \mu$ be two partitions included in the $k \times (n-k)$ rectangle, let $w(\lambda), w(\mu)$ be the associated Grassmannian permutations, and let w_λ and w_μ be the 01 words. Then the following are equivalent:*

(a) λ is obtained from μ by removing a nonempty rim hook.

(b) Let $w(\lambda)$ and $w(\mu)$ be the Grassmannian permutations associated respectively to λ and μ. There exists a transposition (i,j) such that $w(\lambda) = (i,j)w(\mu)$.

(c) There exists a transposition (i',j') such that

$$w(\lambda) = w(\mu) \cdot (i',j')$$

as cosets of $S_n/(S_k \times S_{n-k})$.

(d) Let w_λ and w_μ be the 01 words associated respectively to λ and μ. There exists a transposition (i,j) such that $w(\lambda) = (i,j)w(\mu)$.

The pairs (i,j) from (b),(d) are the same, and (i,j) is uniquely determined by the condition

$$w(\lambda) = (i,j)w(\mu) \mod (S_k \times S_{n-k}).$$

Further, any pair $(i',j)'$ from (c) satisfies $1 \leq i' \leq k < j' \leq n$ and all such transpositions occur for a fixed $w(\mu)$.

The pairs (i,j) and (i',j') are related by $w(\mu)(i',j') = (i,j)$.

Proof. Homework. ☐

Example 4.14. *Take $\lambda = (5,5,3,3,1)$, $\mu = (4,2,2,1,1)$ in the $5 \times (10-5)$ rectangle. Then*

$$w(\lambda) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 1 & 2 & 5 & 6 & 9 & 3 & 4 & 7 & 8 & 10 \end{pmatrix};$$

$$w(\mu) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 2 & 5 & 6 & 8 & 9 & 1 & 3 & 4 & 7 & 10 \end{pmatrix}.$$

Then $w(\lambda) = (1,8)w(\mu)$. Observe that labels 1 and 8 are precisely the labels which determine the rim hook removed from λ.

Proposition 4.15. *(a) Every T-invariant curve is isomorphic to $\mathbb{P}^1$, and it joins two T-fixed points e_λ and e_μ. Further,*

(b) Let w_λ, w_μ be the 01 words determined by the two partitions λ, μ. The points e_λ and e_μ are joined by a T-stable curve if and only if $w_\lambda = (i,j)w_\mu$ where (i,j) is a transposition.

Proof. By Proposition 4.12, there is a T-equivariant isomorphism of the standard open set U_λ and the tangent space $T_{e_\lambda}(\mathrm{Gr}(k,n))$. The T-invariant curves on the tangent space are the coordinate lines. Under this isomorphism, each such line gives a T-invariant curve C in U_λ, isomorphic to $\mathbb{A}^1$ and such that $e_\lambda \in C$. For instance, if we work in $\mathrm{Gr}(4,7)$ and $J(\lambda) = \{1,2,\dots,k\}$, the curve C is given in terms of coordinates by

$$C = rowspan \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} = \langle e_1, e_2, e_3 + a_1 e_6, e_4 \rangle,$$

where the isomorphism to $\mathbb{A}^1$ is given by the parameter a_1. Observe that this curve can also be described as the T-orbit

$$\langle e_1, e_2, e_3 + a_1 e_6, e_4 \rangle.T = \langle e_1, e_2, t_3 e_3 + t_6 e_6, e_4 \rangle = \langle e_1, e_2, e_3 + \frac{t_6}{t_3} e_6, e_4 \rangle.$$

The closure of this curve is

$$\overline{C} = \langle e_1, e_2, a_0 e_3 + a_1 e_6, e_4 \rangle = rowspan \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_0 & 0 & 0 & a_1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix} \subset \mathrm{Gr}(4,7)$$

giving the isomorphism to $\mathbb{P}^1 = \{[a_0 : a_1]\}$. Observe that $\overline{C} \setminus C$ is given by coordinate $a_1 = 0$, and this corresponds to a partition μ. In the example above $\mu = (3,3,2,1)$.

The words for λ and μ only differ in 2 positions; this follows by reading off the columns corresponding to λ and μ. (In the previous case it is $(3,6)$.) The second claim follows from this. $\square$

Exercise 7. *Describe all T-stable curves in $\mathrm{Gr}(2,5)$ passing through the T-fixed point e_λ where $\lambda = \emptyset$.*

Exercise 8. *Let C be a T-stable curve. Let $w \in S_n$ be a permutation, and consider its associated permutation matrix. Prove that the curve $w.C$ is T-stable.*

Corollary 4.16. *(a) Two points e_λ and e_μ are joined by a T-stable curve C if and only if one of the two partitions is obtained from the other by the removal of a rim hook.*

(b) The weight of T on C is $\pm(t_i - t_j)$, where $w(\lambda)$ and $(i,j)w(\lambda)$ is guaranteed from Lemma 4.13.

Proof. The first assertion follows from Lemma 4.13.

For part (b) one first assumes that $w(\mu) = id$; then C relates the fixed points given by id and (i,j) and the statement follows from the construction of C. In general, if C joins the fixed points $w(\mu)$ and $w(\mu)(i',j')$ then the translated curve $w(\mu)^{-1}.C$ has weight $\pm(t_{i'} - t_{j'})$, thus C has weight $\pm w(\mu)(t_{i'} - t_{j'}) = \pm(t_i - t_j)$. □

This corollary completes the description of the moment graph for $\mathrm{Gr}(k,n)$.

- Vertices: e_λ for $\lambda \subset k \times (n-k)$ rectangle;
- Edges: There exists an edge $C_{\lambda,\mu} : \lambda - \mu$ whenever λ is obtained from μ by removing a rim hook.
- Weights: the weight of $C_{\lambda,\mu}$ is $t_i - t_j$ where $w_\lambda = (i,j)w_\mu$ holds between the 01 words associated to λ and μ.

For a partition λ with 01 word w_λ, it is convenient to define the **inversion set**:

$$inv(\lambda) := \{(i > j) : (w_\lambda)_i = 1, (w_\lambda)_j = 0\}.$$

(This is just the set of pairs of positions (i,j) where a 1 occurs before a 0 in the 01 word.) The the following is an equivalent description of the moment graph:

- Vertices: 01 words with k 1's;
- Edges: There exists an edge C_{ij} whenever two 01 words are related by an inversion (i,j).
- Weights: the weight of $C_{i,j}$ is $t_i - t_j$.

Example 4.17. *Draw the octahedron for* $\mathrm{Gr}(2,4)$.

We can now restate the GKM theorem for the Grassmannian; see [GKM98, Thm. 1.2.2]. Let $\Lambda := \mathbb{Z}[t_1, \dots, t_n]$.

Theorem 4.2 (GKM)**.** *There is a graded Λ-algebra isomorphism*

$$H_T^*(\mathrm{Gr}(k,n)) \simeq \{(P_\lambda(t)) \in \oplus_{\lambda \subset k\times(n-k)} \Lambda : wt(C_{\lambda,\mu}) | (P_\lambda(t) - P_\mu(t))\}.$$

Our next goal is to define and study the *Schubert basis* for this ring.

5. Equivariant cohomology of the Grassmannian

A good reference for this section is the wonderful paper by Knutson and Tao [KT03].

5.1. Schubert classes.

For a cohomology class $\kappa \in H_T^*(\mathrm{Gr}(k,n))$ denote by $\kappa|_\mu$ the *restriction (or localization)* of κ at μ. If all restrictions are homogeneous polynomials of degree k then we say that κ has degree k; equivalently, $\kappa \in H_T^{2j}(\mathrm{Gr}(k,n))$. We denote this by $\deg(\kappa) = k$.

Theorem 5.1. *Let $\lambda \subset k \times (n-k)$. Then there exists a unique cohomology class $\sigma_\lambda \in H_T^*(\mathrm{Gr}(k,n))$ such that the following hold:*

- *(Support) The restriction* $(\sigma_\lambda)|_\mu = 0$ *unless* $\lambda \subset \mu$*;*
- *(Degree)* $\deg(\sigma_\lambda) = |\lambda|$*;*
- *(Normalization)* $(\sigma_\lambda)|_\lambda = \prod_{(i,j)\in inv(\lambda)}(t_j - t_i)$*.*

Proof. Assume that there are two classes A, B satisfying these conditions. Then the support of $A - B$ is included in the partitions μ such that $\lambda \subset \mu$ and $\lambda \neq \mu$. Then by the GKM conditions $(t_i - t_j)|(A|_\mu - B|_\mu)$ for any $(i,j) \in inv(\mu)$. Since $\deg(A) = \deg(B) = |\lambda|$, this forces $A = B$. The existence follows from geometric reasons: the Schubert class

$$\sigma_\lambda = [\Omega_\lambda(F_\bullet)] \in H^{2|\lambda|}(\mathrm{Gr}(k,n))$$

satisfies these conditions, where $F_\bullet$ is the standard flag. The main calculation is the check of the normalization condition. For that one uses the fact that the localization $([\Omega_\lambda(F_\bullet)]_T)|_\lambda$ is the product of the weights of the normal bundle at the point e_λ in the Schubert variety $\Omega_\lambda(F_\bullet)$. □

Exercise 9. *Prove that the normal space at* e_λ *to* $\Omega_\lambda(F_\bullet)$ *has indeed the weights* $t_j - t_i$ *for* $(i,j) \in Inv(\lambda)$*. (Draw a picture for* $\mathrm{Gr}(2,4)$ *and the Schubert divisor.)*

Example 5.1. *For* $\mathrm{Gr}(2,4)$*, using the order* $\emptyset, (1), (1,1), (2), (2,1), (2,2)$ *we have*

$$\sigma_\emptyset = (1,1,1,1,1), \sigma_{(1)} = (0, t_3 - t_2, t_4 - t_2, t_3 - t_1, t_4 - t_1, t_4 + t_3 - t_2 - t_1).$$

$$\sigma_{(1,1)} = (0, 0, (t_4 - t_2)(t_4 - t_3), 0, (t_4 - t_1)(t_4 - t_3), (t_4 - t_2)(t_4 - t_1).$$

Exercise 10. *Work this out for the remaining Schubert classes in* $\mathrm{Gr}(2,4)$*.*

Proposition 5.2. *Schubert classes* σ_λ *form a* Λ*-basis for* $H_T^*(\mathrm{Gr}(k,n))$*.*

Proof. Consider a combination

$$\sum a_\lambda \sigma_\lambda = 0, \tag{5.1}$$

with $a_\lambda \in \Lambda$. Let $S := \{\mu : a_\mu \neq 0\}$. If S is nonempty, pick any minimal $\lambda \in S$. Then by the support condition it follows that $(\sigma_\mu)|_\lambda = 0$ for all $\lambda \neq \mu \in S$. Then we localize both sides of the equation (5.1) to λ to get that

$$a_\lambda(\sigma_\lambda)|_\lambda = 0.$$

Since $(\sigma_\lambda)|_\lambda \neq 0$ and Λ is a domain, it follows that $a_\lambda = 0$. This shows the Schubert classes are linearly independent over Λ.

To prove they generate the equivariant cohomology ring, we pick a class $\kappa \in H_T^*(\mathrm{Gr}(k,n))$. W.l.o.g. we may assume that κ is homogeneous of some degree a. We wish to write

$$\kappa = \sum a_\lambda \sigma_\lambda.$$

We find coefficients by Gauss elimination, using the partial order given by the inclusions of partitions. Observe that by the support condition $(\sigma_\lambda)|_\mu = 0$ unless $\lambda \subset \mu$. The minimal element with respect to this order is $\emptyset$. By the support condition it follows that $a_\emptyset = \kappa|_\emptyset$. The general case follows by induction on the number of boxes. Indeed, assume that a_λ is known for all partitons with $\leq p-1$ boxes, where $p \geq 1$. Let λ be a partition with p boxes. By localization

$$\kappa|_\lambda = \sum_\mu a_\mu (\sigma_\mu)|_\lambda.$$

By the support condition, the only non-zero localizations $(\sigma_\mu)|_\lambda$ may occur when $\mu \subset \lambda$. By the induction hypothesis, this gives a recursion

$$a_\lambda = \frac{1}{(\sigma_\lambda)|_\lambda}(\kappa|_\lambda - \sum_{\mu \neq \lambda} a_\mu (\sigma_\mu)|_\lambda).$$

This finishes the proof. □

5.2. **Schubert Calculus.** We proved that the Schubert classes $\sigma_\lambda \in H_T^*(\mathrm{Gr}(k,n))$ form a Λ-basis for the equivariant cohomology ring. Therefore one can write multiplications

$$\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda,\mu}^\nu \sigma_\nu,$$

where $c_{\lambda,\mu}^\nu$ are polynomials in Λ.

Example 5.3. *Calculate $\sigma_{(1)} \cdot \sigma_{(1)}$. Recall that* (1) *corresponds to the word* 0101*.*

$$\sigma_{(1)} \cdot \sigma_{(1)} = \sigma_{(2)} + \sigma_{(1,1)} + (t_3 - t_2)\sigma_{(1)}.$$

Software to calculate products of Schubert classes is available. For instance, one can use the *Equivariant Schubert Calculator* - a Maple program by Anders Buch available at

`https://sites.math.rutgers.edu/~asbuch/equivcalc/`

(Warning: the conventions used to define Schubert classes used in this program are different from those in our class. The difference can only be seen if one works equivariantly. As far as I see one needs to make the substitution $T_i \mapsto t_{n+1-i} - t_{n-i}$.) We list some properties of these polynomials:

- $c_{\lambda,\mu}^\nu$ is a homogeneous polynomial of degree $|\lambda| + |\mu| - |\nu|$;
- $c_{\lambda,\mu}^\nu = 0$ unless $\lambda, \mu \subset \nu$.
- $c_{\lambda,\mu}^\mu = (\sigma_\lambda)|_\mu$ (this follows from localization).

If $\deg c_{\lambda,\mu}^\nu = 0$ then $c_{\lambda,\mu}^\nu$ is an integer. In fact one can use the Kleiman-Bertini theorem ADDREF to show that in this case

$$c_{\lambda,\mu}^\nu = \#(\Omega_\lambda(F_\bullet) \cap \Omega_\mu(G_\bullet) \cap \Omega_{\nu^\vee}(H_\bullet)) \in \mathbb{Z}_{\geq 0}, \tag{5.2}$$

where $F_\bullet, G_\bullet, H_\bullet$ are flags in general position and $\nu^\vee$ is the dual of ν (the complement of ν in the $k \times (n-k)$ rectangle). This is the source of the applications of Schubert Calculus to *Enumerattive Geometry.*

A conjecture by D. Peterson, proved by W. Graham states the following:

Theorem 5.2. *The structure coefficient $c_{\lambda,\mu}^\nu$ is a polynomial in variables $t_2 - t_1, t_3 - t_2, \ldots, t_n - t_{n-1}$ with* non-negative *coefficients.*

This theorem generalizes to any flag manifold. A *manifestly positive* formula to calculate these coefficients was first obtained by Knutson and Tao [KT03] in terms of combinatorial objects called *puzzles*.

5.3. **The Pieri-Chevalley formula and recursions for the structure constants.** In what follows I will explain how we can use equivariant techniques to obtain algorithms to calculate the structure constants. Of course, one (rather blunt - but very effective) tool is to reduce everthing to polynomial multiplication:

(1) identify each Schubert class σ_λ by a sequence of polynomials ($P_\lambda(\mu) = (\sigma_\lambda)|_\mu$);
(2) multiply two sequences of polynomials component-wise;
(3) use triangularity of the Schubert classes (i.e. Gauss elimination) to expand the result in terms of Schubert classes.

Implementing this method for $H_T^*(\mathrm{Gr}(2,4))$ was one of the assigned problems in homework 2.

Another efficient method to calculate the structure constants is based on *(Pieri-)Chevalley recursions.* The *Chevalley rule* is a combinatorial rule to perform the multiplication $\sigma_\lambda \circ \sigma_\Box$. By homogeneity, and since $\lambda \subset \nu$, the degree

$$\deg c_{\lambda,\Box}^\nu = |\lambda| + 1 - |\nu| \in \{0,1\}.$$

Therefore there are two types of contributions to this product:

$$\sigma_\lambda \circ \sigma_\Box = c_{\lambda,\Box}^\lambda \sigma_\lambda + \sum_\mu c_{\lambda,\Box}^\mu \sigma_\mu.$$

The coefficients in the sum have degree 0 (i.e. they are integers).

Lemma 5.4. *Let $\underline{w}$ be the* 01 *word associated to λ. Then*

$$(\sigma_\Box)|_\lambda = \sum_{i=n-k+1}^n t_i - \sum_{j=1}^n \underline{w}_j t_j.\text{[5]}$$

Proof. The class is clearly of degree 1 and it satisfies the normalization condition (according to our conventions to associate 01 words to partitions). If the words for two partitions λ and μ differ by a transposition (i,j), then the difference

$$\sum_{j=1}^n \underline{w(\lambda)}_j t_j - \sum_{j=1}^n \underline{w(\mu)}_j t_j$$

is divisible by $t_i - t_j$. By the uniqueness of Schubert classes from Theorem 5.1 this finishes the proof. $\Box$

[5] I believe there is a typo in [KT03, Lemma 3]; at least the coefficients from that formula do not add up to 0 as they should.

Theorem 5.3 (Equivariant Chevalley formula)**.** *Let σ_λ be a Schubert class in $H_T^*(\mathrm{Gr}(k,n))$. Then the following holds:*

$$\sigma_\lambda \cdot \sigma_\Box = (\sigma_\Box)|_\lambda \sigma_\lambda + \sum_{\mu \to \lambda} \sigma_\mu,$$

where $\mu \to \lambda$ means that μ is obtained from λ by adding one box.

Proof. The coefficient of σ_λ is obtained by localizing both sides at λ. The 'classical' coefficients can be found using e.g. the method from [Ful97]. The idea of the proof is the following. By the geometric interpretation of the coefficients,

$$c_{\lambda,\Box}^{\mu} = \#(\Omega_\lambda(F_\bullet) \cap \Omega_\Box(G_\bullet) \cap \Omega_{\mu^\vee}(H_\bullet)).$$

The condition that $\mu \to \lambda$ implies that $C := \Omega_\lambda(F_\bullet) \cap \Omega_{\mu^\vee}(H_\bullet)$ is a curve. Further, since the flags $F_\bullet$ and $H_\bullet$ are in general position, one may assume that $H_\bullet = F_\bullet^{opp}$ is the opposite flag of $F_\bullet$. Then C is a T-stable curve, and therefore it must be one of the curves described in Proposition 4.15. Then one needs to intersect $\Omega_\Box(G_\bullet) \cap C$. Under the Plücker embedding this is the intersection of a hyperplane by a line in $\mathbb{P}(\wedge^k \mathbb{C}^n)$, and it consists of a single point, therefore $c_{\lambda,\Box}^{\mu} = 1$. [6] □

The Chevalley formula gives a remarkable recursive formula to calculate the structure constants $c_{\lambda,\mu}^{\nu}$. Variants of this were noticed by Okounkov, Molev-Sagan, and finally Knutson and Tao. Nowadays such recursions exist for the equivariant quantum K ring of any flag manifold, see e.g. [Mih06, BCMP18] and references within.

Theorem 5.4. *Let λ, μ, ν be three partitions included in the $k \times (n-k)$ rectangle. Then the following holds:*

$$((\sigma_\Box)|_\nu - (\sigma_\Box)|_\lambda) c_{\lambda,\mu}^{\nu} = \sum_{\delta \to \lambda} c_{\delta,\mu}^{\nu} - \sum_{\nu \to \epsilon} c_{\lambda,\mu}^{\epsilon}.$$

Proof. By associativity of the product in the equivariant ring,

$$\sigma_\Box \cdot (\sigma_\lambda \cdot \sigma_\mu) = (\sigma_\Box \cdot \sigma_\lambda) \cdot \sigma_\mu.$$

We now expand both sides using the equivariant Chevalley formula from Theorem 5.3, and collect the coefficient of σ_ν. We have

$$\sigma_\Box \cdot (\sigma_\lambda \cdot \sigma_\mu) = \sigma_\Box \cdot \sum_\rho c_{\lambda,\mu}^{\rho} \sigma_\rho = \sum_\rho c_{\lambda,\mu}^{\rho} (c_{\rho,\Box}^{\rho} \sigma_\rho + \sum_{\theta \to \rho} \sigma_\theta)$$

For the second formula,

$$(\sigma_\Box \cdot \sigma_\lambda) \cdot \sigma_\mu = (\sum_{\delta \to \lambda} \sigma_\delta + c_{\lambda,\Box}^{\lambda} \sigma_\lambda) \cdot \sigma_\mu = \sum_{\delta \to \lambda} \sum_\alpha c_{\delta,\mu}^{\alpha} \sigma_\alpha + c_{\lambda,\Box}^{\lambda} \sum_\gamma c_{\lambda,\mu}^{\gamma} \sigma_\gamma.$$

[6] An argument following this line of thought can be found in [Bri05, Prop. 1.4.3].

We now identify the coefficient of σ_ν in both expressions. In the first we obtain
$$c_{\lambda,\mu}^{\nu} c_{\nu,\Box}^{\nu} + \sum_{\nu \to \epsilon} c_{\lambda,\mu}^{\epsilon},$$
while in the second
$$\sum_{\delta \to \lambda} c_{\delta,\mu}^{\nu} + c_{\lambda,\Box}^{\lambda} c_{\lambda,\mu}^{\nu}.$$
Rearranging these expressions gives the desired formula. $\Box$

We explain next in what sense the formula from Theorem 5.4 gives a recursion for calculating equivariant Schubert constants. First, for partitions λ, ν, denote by
$$F_{\nu,\lambda} := (\sigma_\Box)|_\nu - (\sigma_\Box)|_\lambda = \sum_{j=1}^{n} \underline{w(\lambda)}_j t_j - \sum_{j=1}^{n} \underline{w(\nu)}_j t_j.$$
Observe that
$$\lambda \neq \nu \Longleftrightarrow F_{\nu,\lambda} \neq 0.$$
Then for $\lambda \neq \nu$, we can rewrite the formula as
$$c_{\lambda,\mu}^{\nu} = \frac{1}{F_{\nu,\lambda}} \Big(\sum_{\delta \to \lambda} c_{\delta,\mu}^{\nu} - \sum_{\nu \to \epsilon} c_{\lambda,\mu}^{\epsilon} \Big). \tag{5.3}$$
The key observation is that the coefficients appearing in the two sums have polynomial degree *larger by* 1. This gives the following recursive formula to calculate any structure constant $c_{\lambda,\mu}^{\nu}$:

- If $\lambda = \mu = \nu$ then $c_{\lambda,\lambda}^{\lambda} = (\sigma_\lambda)|_\lambda = \prod_{(i,j) \in I(\lambda)} (t_j - t_i)$;
- If $\lambda = \nu \neq \mu$ then use $c_{\lambda,\mu}^{\nu} = c_{\mu,\lambda}^{\nu}$;
- If $\lambda \neq \nu$ then apply formula (5.3).

Applying this procedure repeatedly leads to the following initial cases: calculate $c_{\lambda,\lambda}^{\lambda}$, or calculate the coefficients $c_{\lambda,\mu}^{\nu}$ of *maximal* degree. The maximal degree condition implies that $\lambda = \mu = (n-k)^k$ and $\nu = \emptyset$. But $c_{(n-k)^k,(n-k)^k}^{\emptyset} = 0$, since $(n-k)^k \not\subseteq \emptyset$. This proves that the equation (5.3) gives indeed a recursion.

This recursion, based on the Chevalley formula, is remarkable, as the divisor class $\sigma_\Box$ does *not generate* the classical or the equivariant cohomology ring. (One can easily see this in the non-equivariant case; for instance, non-equivariantly,
$$\sigma_\Box \cdot \sigma_\Box = \sigma_{(2)} + \sigma_{(1,1)}$$
and it is impossible to generate either $\sigma_{(2)}$ or $\sigma_{(1,1)}$. However, this class generates the *localized ring* $H_T^*(\mathrm{Gr}(k,n))_{\Lambda^+}$, where Λ^+ denotes the prime ideal of Λ generated by $t_1, \dots, t_n$. The following has been proved (in a more general setting) in [BCMP18].

Theorem 5.5. *The classes* $1, \sigma_\Box, \sigma_\Box^2, \dots, \sigma_\Box^{\binom{n}{k}-1}$ *form a* Λ*-basis for the localized ring* $H_T^*(\mathrm{Gr}(k,n))_{\Lambda^+}$.

Proof. We know that as a Λ-module, the localized ring $H_T^*(\mathrm{Gr}(k,n))_{\Lambda^+}$ has rank $\binom{n}{k}$. Therefore it suffices to show that the classes in the statement are linearly independent over Λ. Consider Ψ, the Λ-linear operator given by multiplication by $\sigma_\Box$. The matrix of this operator is upper triangular (by the Chevalley rule) and the coefficients on the diagonal are $(\sigma_\Box)|_\lambda$. Observe that these coefficients are *distinct*, therefore the minimal polynomial of Ψ coincides with the characteristic polynomial. Since the characteristic polynomial has degree $\binom{n}{k}$, this finishes the proof. $\Box$

Example 5.5. *Homework: apply this recursion to calculate again the multiplication table of $H_T^*(\mathrm{Gr}(2,4))$. You may assume that you already know the Chevalley formula.*

6. Equivariant vector bundles

Most of this section, except for the examples, are discussed [Bri07] and [CG97, Ch.5.1]. Let X be manifold/algebraic variety, and let $\pi : E \to X$ be a vector bundle. (If we work in the algebraic variety, then we use Zariski topology, so restrictions of vector bundles are trivial over large open sets.) Denote by E_x the fibre $\pi^{-1}(x)$.

Definition 6.1. *Let G be a group. A vector bundle $\pi : E \to X$ is G-equivariant if the following hold:*

- *The spaces X, E admit G-actions such that π is G-equivariant;*
- *For each $g \in G$, the natural map $E_x \to E_{gx}$ is a linear transformation of vector spaces.*

In our case $X = \mathrm{Gr}(k,n)$ and $G = \mathrm{SL}_n$. Observe that X is homogeneous under the G action, and that in fact $\mathrm{Gr}(k,n) = G/P$, where P is the stabilizer of the point $W_k := \langle e_1, \dots, e_k \rangle$. In particular, W_k is a representation of P. (One can actually write down this group explicitly: it is the subgroup of SL_n where the coordinates in the upper right $k \times (n-k)$ rectangle equal to 0.) A rich source of homogeneous bundles is given by P-representations; we explain this next.

Let G be any simple Lie group and let $P \le G$ be a parabolic subgroup. (One may take (G,P) the groups above.) Let V be a representation of P. Define the space

$$G \times^P V = (G \times V)/\sim; \quad (g,v) \sim (gp^{-1}, pv)$$

where $p \in V$. There is a well defined projection $\pi : G \times^P V \to G/P$ sending $[g,v] \mapsto gP$. One can define a G-action on $G \times^P V$ by left multiplication: $g.[g',v] = [gg',v]$, and π becomes G-equivariant. It is not hard to show that $\pi : G \times^P V \to G/P$ gives a structure of a G-equivariant vector bundle on G/P. We will call this process *induction from P*. One can recover the representation V by taking the fibre over $1.P$; that is sometimes called *restriction.* We illustrate this with several examples, all in the case when $G/P = \mathrm{Gr}(k,n)$.

Example 6.1 (The tautological sequence)**.** *Consider the exact sequence of P-modules*

$$0 \longrightarrow \langle e_1, \dots, e_k \rangle \longrightarrow \mathbb{C}^n \longrightarrow \mathbb{C}^n/\langle e_1, \dots, e_k \rangle \longrightarrow 0$$

Performing induction gives a short exact sequence of vector bundles:

$$0 \longrightarrow G \times^P \langle e_1, \dots, e_k \rangle \longrightarrow G \times^P \mathbb{C}^n \longrightarrow G \times^P \mathbb{C}^n/\langle e_1, \dots, e_k \rangle \longrightarrow 0$$

One can show that this is precisely the tautological sequence on the Grassmannian:

$$0 \longrightarrow S \longrightarrow \mathrm{Gr}(k,n) \times \mathbb{C}^n \longrightarrow Q \longrightarrow 0$$

Here the fibre of S over $W \in \mathrm{Gr}(k,n)$ is W, and the fibre of Q is $\mathbb{C}^n/W$.

Example 6.2 (The tangent bundle)**.** *Recall that P denotes the stabilizer of the point $W = rowspan(Id_k|O_{k,n-k})$. Since $P.W = W$, it follows that the tangent space $T_W \mathrm{Gr}(k,n)$ is a P-module, called the* **isotropy representation***. The coordinates of the tangent space correspond to the right $k \times (n-k)$ matrix; equivalently,*

$$T_W \mathrm{Gr}(k,n) = Hom(W, \mathbb{C}^n/W).$$

We induct using this representation to obtain that

$$G \times^P T_W \mathrm{Gr}(k,n) = G \times^P Hom(W, \mathbb{C}^n/W) = Hom(S,Q),$$

where the first equality follows from the previous example. Of course, this is just the tangent bundle of $\mathrm{Gr}(k,n)$.

Example 6.3 (The Euler sequence for projective spaces)**.** *Let $X = \mathbb{P}^n$, regarded as SL_n/P. Consider $V = \langle e_1 \rangle \subset \mathbb{C}^{n+1}$. There is an exact sequence of P-representations*

$$0 \longrightarrow V \longrightarrow \mathbb{C}^{n+1} \longrightarrow \mathbb{C}^{n+1}/V \longrightarrow 0\ .$$

Applying the exact functor $Hom(V,-)$ we obtain

$$0 \longrightarrow Hom(V,V) \longrightarrow Hom(V,\mathbb{C}^{n+1}) \longrightarrow Hom(V,\mathbb{C}^{n+1}/V) \longrightarrow 0\ .$$

Observe that $Hom(V,V) \simeq V \otimes V^*$ is the trivial P-representation of dimension 1. The P-module $Hom(V,\mathbb{C}^{n+1}) \simeq (V^*)^{\oplus(n+1)}$. By inducting using $G \times^P -$ and using examples 6.1 and 6.2 above one obtains the Euler sequence for the tangent bundle of $\mathbb{P}^n$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0\ .$$

(Here we also used that $\mathcal{O}_{\mathbb{P}^n}(-1)$ is the tautological subbundle on $\mathbb{P}^n$.

6.1. **Formalism of Chern classes.** Let X be an algebraic variety of dimension n, and let E be an algebraic vector bundle. The Chern classes $c_i(E)$ of E $(0 \le i)$ may be regarded as linear operators

$$c_i \cap : H_k(X) \to H_{k-i}(X),$$

which satisfy the following properties:

- (Vanishing) If $i > rank(E)$ then $c_i(E) = 0$;
- (Commutativity) $c_i(E) \cap (c_j(F) \cap \alpha) = c_j(F) \cap (c_i(E) \cap \alpha)$, for any $\alpha \in H_*(X)$.
- (Projection formula) Let $f : X' \to X$ be a proper[7] morphism. Then

  $$f_*(c_i(f^*E) \cap \alpha) = c_i(E) \cap f_*(\alpha),$$

  for all $\alpha \in H_*(X')$ and all i. (The precise meaning of $H_*(X)$ is the Chow group of X, but for most practical purposes this may be replaced by the homology group.)
- (Pull back property) Let $f : X' \to X$ be a flat[8] morphism. Then

  $$c_i(f^*E) \cap f^*(\alpha) = f^*(c_i(E) \cap \alpha).$$

- (Whitney sum) For any short exact sequence of vector bundles

  $$0 \longrightarrow E' \longrightarrow E \longrightarrow E'' \longrightarrow 0$$

  Then

  $$c_t(E) = c_t(E')c_t(E'')$$

  where $c_t(E)$ is the generating function

  $$c_t(E) = \sum c_i(E)t^i.$$

- (Normalization) If E is a line bundle (i.e. $rank(E) = 1$) and D is a Cartier divisor such that $E = \mathcal{O}_X(D)$ then

  $$c_1(E) \cap [X] = [D].$$

  More generally, if $H^0(E) \neq 0$ and $s \in H^0(E)$ is a general section with zero locus $Z(s) \subset X$, then

$$c_e(E) \cap [X] = [Z(s)]. \tag{6.1}$$

The class

$$c(E) = 1 + c_1(E) + \ldots + c_e(E)$$

is called the *total Chern class* of E. It can also be regarded as the Chern polynomial $c_t(E)$ evaluated at $t \mapsto 1$.

One can actually show that these properties determine the Chern classes. The idea is as follows. The *splitting principle* states that if $E \to X$ is a vector bundle, then there exists a flat map $f : X' \to X$ such that:

- The pull back $f^* : H^*(X) \to H^*(X')$ is injective;

[7]The properness is only required in the algebraic category.

[8]Flatness is only required in the algebraic category.

- The pull back bundle f^*E has a filtration
$$0 \subset E_1 \subset E_2 \subset \ldots \subset E_e = f^*E$$
such that $E_i \to X'$ is a vector bundle on X' of rank i.

By the Whitney formula, it follows that
$$c(E_e) = c(E_1) \cdot c(E_2/E_1) \cdot \ldots \cdot c(E_e/E_{e-1}).$$
Since each quotient E_i/E_{i-1} is a line bundle, the normalization formula (applied only for line bundles) calculates each class $c(E_i/E_{i-1})$. Therefore the linear operator $c(f^*E)$ is calculated. Now by injectivity the class $c(E) \cap \alpha$ may be identified by $c(f^*E) \cap f^*(\alpha)$. (A subtlety: in the topological category one needs to ensure that $f^*(\alpha)$ is well defined. For instance this happens if the Poincaré dual of α exists, e.g. if X is non-singular. This is not needed if one works in the algebraic category, and uses Chow groups; in this case *flatness* ensures that pull-backs exist.) To construct $f : X' \to X$ with the required properties, one may take $X' = \mathrm{Fl}(E)$. This is the variety parametrizing complete flags of vector bundles $E_1 \subset E_2 \subset \ldots \subset E_e = E$, equipped with its natural projection to X. For instance, if $X = pt$, then E is just a vector space, and this is the complete flag variety $\mathrm{Fl}(E)$. For details, we refer to [Ful84, §3.2].

The splitting principle allows one to formally define the *Chern roots* of a vector bundle. If $rank(E) = e$, these are formal indeterminates $x_1, \ldots, x_e$ such that
$$c(E) = (1 + x_1)(1 + x_2) \cdot \ldots \cdot (1 + x_e) = \sum_{i \geq 0} e_i(x_1, \ldots, x_e).$$
Then the Chern class $c_i(E) = e_i(x_1, \ldots, x_e)$. The idea is that x_i's are only formal, but their symmetrizations are actual classes. By the splitting principle, one may actually identify $x_i = c_1(E_i/E_{i-1})$.

The Chern roots are very useful tools relating Chern classes of vector bundles. I will list several properties below - all follow from the judicious use of the splitting principle.

Lemma 6.4. *In all statements $x_1, \ldots, x_e$ are the Chern roots of E. Then the following hold:*

- *(Dual bundles) The Chern roots of the dual bundle $E^\vee$ are $-x_1, \ldots, -x_e$. In particular,*
$$c_i(E^\vee) = (-1)^i c_i(E).$$
- *(Tensor products) Let $F \to X$ be a vector bundle with Chern roots $y_1, \ldots, y_f$. Then the Chern roots of $E \otimes F$ are $x_i + y_j$, for $1 \leq i \leq e, 1 \leq j \leq f$.*
- *(Symmetric powers) The Chern roots of $Sym^p E$ are $x_{i_1} + x_{i_2} + \ldots + x_{i_p}$ where $1 \leq i_1 \leq i_2 \leq \ldots \leq i_p \leq e$.*
- *(Exterior powers) The Chern roots of $\bigwedge^p E$ are $x_{i_1} + x_{i_2} + \ldots + x_{i_p}$ where $1 \leq i_1 < i_2 < \ldots < i_p \leq e$.*

We illustrate this by some examples.

Example 6.5 (The Chern classes of the tangent bundle of $\mathbb{P}^n$)**.** *We use the Euler sequence*

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n} \longrightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \longrightarrow T_{\mathbb{P}^n} \longrightarrow 0\ .$$

to calculate the Chern classes of the tangent bundle on $\mathbb{P}^n$*. Let* $H = c_1(\mathcal{O}_{\mathbb{P}^n}(1)$*. By the normalization property this is the class of a hyperplane in* $\mathbb{P}^n$*. From the Whitney formula we obtain that the total Chern class*

$$c(\mathcal{O})c(T_{\mathbb{P}^n}) = c(\mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} = c(\mathcal{O}_{\mathbb{P}^n}(1))^{n+1} = (1+H)^{n+1}.$$

Since $c(\mathcal{O}) = 1$*, it follows that*

$$c(T_{\mathbb{P}^n}) = (1+H)^{n+1} = \sum_{k=0}^{n} \binom{n+1}{k} H^k.$$

For instance, $c_n(T_{\mathbb{P}^n}) = (n+1)[pt]$*, reflecting the fact that the topological Euler characteristic of* $\mathbb{P}^n$ *is* $n+1$*. (This is an instance of the Gauss-Bonnet theorem.)*

Example 6.6 (Lines on cubic surfaces)**.** *Let* $\Sigma \subset \mathbb{P}^3$ *be a general surface, i.e.* $\Sigma = Z(F)$ *where* F *is a general homogeneous polynomial of degree* 3 *in variables* x_0, x_1, x_2, x_3*. We are interested in how many lines are included in* $\mathbb{P}^3$*. The space of lines in* $\mathbb{P}^3$ *is the same as the Grassmannian* $\mathrm{Gr}(2,4)$*. Let* S *be the tautological subbundle. The polynomial* F *is a section of* $Sym^3(\mathbb{C}^4)^*$*, and the condition that a line* $\mathbb{P}(V) \in \mathrm{Gr}(2,4)$ *is included in* $Z(F)$ *means that* $F|_V \equiv 0$*. In other words,* F *gives a (general) section*

$$s \in H^0(\mathrm{Gr}(2,4), Sym^3(S^*)), \quad V \mapsto F|_V.$$

We are interested in the zero locus of this section. By the normalization property (6.1)

$$\Sigma = Z(s) = c_r(Sym^3(S^*)),$$

where $r = rank(Sym^3 S^*) = 4$*. Let* x_1, x_2 *be the Chern roots of* S^**. Then by Lemma 6.4 the Chern roots of* $Sym^3 S^*$ *are* $3x_1, 2x_1 + x_2, x_1 + 2x_2, 3x_2$*. It follows that*

$$\begin{aligned} c_4(Sym^3 S^*) &= 9x_1x_2(x_1+2x_2)(2x_1+x_2) \\ &= 9x_1x_2(2x_1^2 + 5x_1x_2 + 2x_2^2) \\ &= 9x_1x_2(2(x_1+x_2)^2 + x_1x_2) \\ &= 18x_1x_2(x_1+x_2)^2 + 9(x_1x_2)^2 \\ &= 18c_2(S^*)c_1(S^*)^2 + 9c_2(S^*)^2. \end{aligned}$$

In the next homework you will check that

$$c_1(S^*) = \sigma_{(1)}; \quad c_2(S^*) = \sigma_{(1,1)}.$$

Therefore

$$c_4(Sym^3 S^*) = 18\sigma_{(1,1)}\sigma_1^2 + 9\sigma_{(1,1)}^2 = 18\sigma_{(2,2)} + 9\sigma_{(2,2)} = 27\sigma_{(2,2)}.$$

This shows that a general cubic surface will contain 27 *lines.*

7. Equivariant cohomology

Our main reference for this section is [Bri98, §1]. Let X be a topological space with an action of a group G. We say that the G-action is *free* if $gx = x$ for all x implies that $g = 1$.

The idea behind the definition of equivariant cohomology $H^*_G(X)$ is to give a cohomology theory associated to the orbit space X/G. In some special cases (e.g. X is a manifold, and G acts freely and properly discontinuously, cf. [dC92, p.23], [AB84]) this works. But in general the orbit space may be badly behaved, and the idea is to replace X by another space, homotopically equivalent to it, but where G acts freely. To explain this we need some facts from topology; we refer to [Hus94, Ch. IV] for details.

In what follows, we say that X is a G-space if X admits a left action. For any topological group G one has a principal G-bundle $p : E_G \to B_G$ where E_G is contractible, and which is universal among principal G bundles. The space EG comes endowed with a *right* G-action. By definition, G acts freely on E_G, and $B_G = E_G/G$ is the space of orbits; this implies that fibers of p are all copies of G. The space B_G is called the *classifying space* and E_G is the *universal* G*-bundle*. These spaces are only unique up to homotopy. (In a nutshell, a principal G-bundle $p : E \to B$ is a locally trivial free G-space E such that p factors through a homeomorphism $P/G \to B$, where P/G is the space of G-orbits on P. We refer to [Hus94, Ch.IV] or the nice lecture notes [Mit] for the precise definitions. We will also construct below the examples relevant for this class.)

We let G act on the right on $E_G \times X$ by

$$(e, x).g = (eg, g^{-1}x).$$

The **Borel mixing space** (sometimes also called the *homotopic quotient*) is defined by

$$X_G := (E_G \times X)/ \sim .$$

We have the following commutative diagram:

$$\begin{tikzcd} E_G \times X \arrow[r, "pr"] \arrow[d] & E_G \arrow[d, "p"] \\ X_G \arrow[r, "\pi"] & B_G \end{tikzcd}$$

The projection π is defined by $[e, x] \mapsto p(e)$, and all its fibers are copies of X. (I.e., it is an X-fibration.) Observe that with this definition

$$(e, g.x) \sim (e.g, g^{-1}.(g.x)) = (e.g, x),$$

showing that the fibre $X \simeq \pi^{-1}(p(e))$ has the natural left G-action.

The G-equivariant cohomology of X is defined by

$$H_G^*(X) := H^*(X_G).$$

In particular, $H_G^*(pt) = H^*(B_G)$. The natural morphism $\pi^* : H^*(B_G) \to H^*(X_G)$ gives $H_G^*(X)$ a structure of a $H_G^*(pt)$-algebra. By picking a base point in $b_0 \in BG$ there is a diagram

$$\begin{tikzcd}
\pi^{-1}(b_0) \simeq X \arrow[r] \arrow[d] & X_G \arrow[d, "\pi"] \\
b_0 \arrow[r] & BG
\end{tikzcd}$$

The restriction to the fibre $\pi^{-1}(b_0) \simeq X$ induces a natural map $\pi^* : H_G(X) \to H^*(X)$, and it is known (e.g. restricting to an open neighborhood of b_0, then using Künneth theorem) that $H^+(BG) = H_G^+(pt) \subset \ker \pi^*$, where $H_G^+(pt)$ denotes the ideal of images of elements in $H_G^*(pt)$ of positive degree. Therefore we obtain a map

$$\rho : H_G^*(X)/H_G^+(pt) \to H^*(X).$$

The map ρ *may not* be an isomorphism, but *it is* for flag manifolds; see [Bri98, Prop.2].

Remark 7.1. (1) *The equivariant cohomology is independent of the choice the the universal bundle $E_G \to B_G$.*

(2) *If G acts trivially on X, then*

$$X_G = EG \times_G X = EG \times X$$

and by Künneth isomorphism we obtain that as a $H_G^*(pt)$-algebra,

$$H_G^*(X) = H_G^*(pt) \otimes H^*(X).$$

(3) *In general EG will be an infinite dimensional space. But we can calculate the G-equivariant cohomology using finite dimensional 'approximations' of EG. For n a positive integer, assume we are given $EG_n \subset EG$ such that EG_n is n-connected (i.e. the homotopy groups $\pi_i(EG_n) = 0$ for $i \leq n$) and $EG_n \to EG_n/G$ is a principal G-bundle. Then*

$$H_G^i(X) = H^i(EG_n \times_G X) \quad , \quad 0 \leq i \leq n.$$

This will allow to reduce calculations to the finite dimensional setting.

7.1. **Examples.**

7.1.1. $G = \mathbb{C}^*$. In this case one may take $E\mathbb{C}^* = \mathbb{C}^\infty \setminus 0$ with the $\mathbb{C}^*$-action by left multiplication. The orbit space is

$$B\mathbb{C}^* = E\mathbb{C}^*/\mathbb{C}^* = \mathbb{P}^\infty.$$

A finite dimensional approxmation can be taken to be $(E\mathbb{C}^*)_n := \mathbb{C}^{n+1} \setminus 0$. Then $(B\mathbb{C}^*)_n = \mathbb{P}^n$, and it follows that

$$H^*_{\mathbb{C}^*}(pt) = H^*(\mathbb{P}^\infty) = \lim_n H^*(\mathbb{P}^n) = \lim_n \mathbb{C}[x]/\langle x^{n+1}\rangle = \mathbb{C}[x].$$

Here we identify $x = c_1(\mathcal{O}_{\mathbb{P}^\infty}(1))$.

7.1.2. $G = T = (\mathbb{C}^*)^n$. Then one can take $ET = (E\mathbb{C}^*)^n$ and as before one obtains that

$$H^*_T(pt) = \mathbb{C}[x_1, x_2, \dots, x_n].$$

As before, $x_i = c_1(\mathcal{O}_{\mathbb{P}^\infty_{(i)}}(1)$.

7.1.3. $G = \mathrm{GL}_n$. Let $N \geq n$ be an integer, and consider the space $M_{N,n}(\mathbb{C})$ of $N \times n$ matrices. Consider the open subset $M^{\max}_{N,n}$ of those matrices of maximal rank (sometimes this is called the *Stiefel variety*). The group GL_n acts by left multiplication on $M^{\max}_{N.n}$, and the orbit space is the Grassmannian $\mathrm{Gr}(n, \mathbb{C}^N)$. By analyzing the codimension of $M_{N,n} \setminus M^{\max}_{N.n}$ one may prove that $M^{\max}_{N.n}$ is $N-n$-connected. It follows that $BG = \mathrm{Gr}(n, \mathbb{C}^\infty)$ and that

$$H^*_{\mathrm{GL}_n}(pt) = H^*(\mathrm{Gr}(n, \mathbb{C}^\infty) = \lim_N H^*(\mathrm{Gr}(n, \mathbb{C}^N).$$

One may actually prove that if $T = (\mathbb{C}^*)^n$ is a maximal torus, then

$$H^*_{\mathrm{GL}_n}(pt) = H^*_T(pt)^{S_n} = \mathbb{C}[x_1, \dots, x_n]^{S_n}.$$

7.2. **Equivariant Chern classes.** If X is a G-space and V is a G-equivariant vector bundle, then $V_G \to X_G$ is a vector bundle on the homotopic quotient. We define the G-equivariant Chern classes by

$$c_i^G(V) := c_i(V_G) \in H^{2i}(X_G).$$

We illustrate this using some examples.

7.2.1. Take $X = pt$, $G = \mathbb{C}^*$ and let $V = \mathbb{C}_a$ be the 1-dimensional $\mathbb{C}^*$-module with action $z.v = z^a v$. This may be regarded as a $\mathbb{C}^*$-equivariant vector bundle over a point, and it gives a line bundle

$$(\mathbb{C}_a)_{\mathbb{C}^*} \to B\mathbb{C}^* = \mathbb{P}^\infty.$$

In other words V is a trivial vector bundle but not *equivariantly trivial.*

By definition the equivalence is given by

$$(e, z^a v) = (e, z.v) \sim (e.z, v).$$

On one side this gives that the action on $\mathbb{C}^*$ on the fibre over $[e]$ is $z.v = z^a v$. On the other, recall that the fibers of the tautological subbundle $\mathcal{O}_{\mathbb{P}^\infty}(-1)$ have the $\mathbb{C}^*$-action given by $v.z = zv$. It follows that $(\mathbb{C}_a)_{\mathbb{C}^*} = \mathcal{O}_{\mathbb{P}^\infty}(-a)$. In particular

$$c_1^{\mathbb{C}^*}(\mathbb{C}_a) = -ax = \mathcal{O}_{\mathbb{P}^\infty}(-a).$$

More generally, let $T = (\mathbb{C}^*)^n$ and consider the representation $V = \mathbb{C}^n$, where T acts by multiplication by t_i on the ith component. Then V may be regarded as a T-equivariant vector bundle over a point. Further, the identity map $V_{t_i} \to V_{t_i}$ is equivariant with respect to the projection $T \to \mathbb{C}^*$, giving the diagram

$$\begin{tikzcd} (V_{t_i})_T \arrow[r] \arrow[d] & (V_{t_i})_{\mathbb{C}^*} \arrow[d] \\ BT \arrow[r] & B\mathbb{C}^* \end{tikzcd}$$

using the previous example this diagram may be identified with

$$\begin{tikzcd} \mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1) \arrow[r] \arrow[d] & \mathcal{O}_{(\mathbb{P}^\infty)}(-1) \arrow[d] \\ (\mathbb{P}^\infty)^n \arrow[r] & \mathbb{P}^\infty \end{tikzcd}$$

This shows that the vector bundle $V_T \to BT$ is

$$\oplus_i \mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1) \to (\mathbb{P}^\infty)^n.$$

Then we may identify

$$x_i = c_1(\mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(1)) = c_1^T(V_{-t_i}).$$

7.2.2. Take $G = (\mathbb{C}^*)^2$ and $X = \mathbb{P}^1$ Consider the action given by $(t_1, t_2).[x : y] = [t_1 x : t_2 y]$. In other words, we write $\mathbb{P}^1 = \mathbb{P}(\mathbb{C}_{t_1} \oplus \mathbb{C}_{t_2})$. Then by the previous example

$$(\mathbb{P}^1)_{(\mathbb{C}^*)^2} = \mathbb{P}(\mathbb{C}_{t_1} \oplus \mathbb{C}_{t_2})_{(\mathbb{C}^*)^2} = \mathbb{P}(\mathcal{O}(-1)_{(1)} \oplus \mathcal{O}(-1)_{(2)}) \to (\mathbb{P}^\infty)^2$$

is the projectivization of a rank two bundle.

More generally, take $T = (\mathbb{C}^*)^{n+1}$ and $X = \mathbb{P}^n$. Let T act by

$$(t_0, \dots, t_n)[x_0 : \dots : x_n] = [\dots : t_i x_i : \dots].$$

In other words, $\mathbb{P}^n = \mathbb{P}(V)$, where V is the T-representation with weight decomposition

$$V = \oplus V_{t_i}.$$

As before one may regard V as a vector bundle over a point, and $V_T = \oplus_i \mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1) \to BT$. Then $\mathbb{P}(V)_T = \mathbb{P}(V_T)$ as bundles over BT, i.e.

$$\mathbb{P}(V)_T = \mathbb{P}(\oplus_i \mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1)). \tag{7.1}$$

7.2.3. Take now $G = \mathrm{GL}_n$ and $X = \mathrm{Gr}(p, \mathbb{C}^n)$. In this case one regards $V := \mathbb{C}^n$ as a GL_n representation. As before one identifies the associated vector bundle $V_G \to BG = \mathrm{Gr}(n, \mathbb{C}^\infty)$ to the (rank n) tautological subbundle $\mathcal{S} \to \mathrm{Gr}(n, \mathbb{C}^\infty)$. Then one may identify $X_G \to BG$ with the Grassmann bundle

$$\mathbb{G}(p, \mathcal{S}) \to \mathrm{Gr}(n, \mathbb{C}^\infty).$$

(The fibre over W of the Grassmann bundle is the Grassmannian $\mathrm{Gr}(p, W) \simeq X$.) If one takes the $T := (\mathbb{C}^*)^n$-equivariant cohomology, still based on $\mathrm{Gr}(n, \mathbb{C}^\infty)$, then $\mathcal{S}$ comes with a splitting of line bundles $V_{\chi_1} \oplus \ldots \oplus V_{\chi_n}$ corresponding to the restriction of V from a G-representation to a T-representation.

7.2.4. One can use the previous examples to calculate the equivariant cohomology rings for projective spaces and Grassmannians. We illustrate this for the projective space $\mathbb{P}^n$. We have seen that $H_T^*(\mathbb{P}^n) = H^*(\mathbb{P}(V_T))$ (with notation from the example above).

We need to understand the cohomology of a projective bundle $p : \mathbb{P}(E) \to B$, where $E \to B$ is a rank e vector bundle. There is a tautological sequence over $\mathbb{P}(E)$:

$$0 \longrightarrow \mathcal{O}_E(-1) \longrightarrow p^*E \longrightarrow Q \longrightarrow 0$$

Multiplying by $\mathcal{O}_E(1)$ shows that the trivial bundle $\mathcal{O}_{\mathbb{P}(E)}$ is a subbundle of $p^*E \otimes \mathcal{O}_E(1)$. This shows that

$$c_e(p^*E \otimes \mathcal{O}_E(1)) = 0.$$

Let now $\xi := c_1(\mathcal{O}_E(1))$ and let $a_1, \ldots, a_e$ be the Chern roots of E. Then

$$H^*(\mathbb{P}(E)) = \frac{H^*(B)[\xi]}{\langle \prod(\xi + a_i) \rangle} = \frac{H^*(B)[\xi]}{\langle \xi^e + \sum c_i(E)\xi^{e-i} \rangle}.$$

Here $H^*(X)$ is embedded as a subalgebra of $H^*(\mathbb{P}(E))$ via the (injective!) pull back $p^* : H^*(B) \to H^*(\mathbb{P}(E))$.

We now apply this formula to calculate $H_T^*(\mathbb{P}^n)$. We obtain

$$H_T^*(\mathbb{P}^n) = \frac{H^*((\mathbb{P}^\infty)^n)[\xi]}{\langle \prod(\xi + t_i) \rangle} = \frac{H^*(\mathbb{P}(\oplus_i \mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1)))[\xi]}{\langle \prod(\xi + t_i) \rangle}$$

where $t_i = c_1^T(V_{t_i})$ (cf. (7.1) above).

One may check (homework!) that an equivalent presentation is given as follows. Consider the tautological sequence on $\mathbb{P}^n$:

$$0 \longrightarrow \mathcal{O}(-1) \longrightarrow \mathbb{C}^{n+1} \longrightarrow Q \longrightarrow 0$$

As before let $\xi = c_1^T(\mathcal{O}(1))$. Then

$$H_T^*(\mathbb{P}^n) = \frac{\mathbb{C}[\xi]}{\langle c^T(\mathcal{O}(-1)) \cdot c^T(Q) = c^T(\mathbb{C}^{n+1}) \rangle}.$$

7.2.5. A similar discussion applies to calculate the T-equivariant cohomology of a Grassmannian $\mathrm{Gr}(k, n)$. In that case

$$\mathrm{Gr}(k, n)_T = \mathbb{G}(\oplus_i \mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1)),$$

and the equivariant cohomology is follows from the calculation of the cohomology of Grassmann bundles.

As before, if $E \to B$ is a vector bundle, then we can consider the Grassmann bundle $p : \mathbb{G}(k, E) \to B$; the fibre over $b \in B$ is the Grassmannian $\mathrm{Gr}(k, E_b)$. This comes equipped with a tautological sequence

$$0 \longrightarrow S \longrightarrow p^*E \longrightarrow Q \longrightarrow 0 .$$

Its cohomology is given by

$$H^*(\mathbb{G}(k,E)) = \frac{H^*(B)[c_1(S), \dots, c_e(S); c_1(Q), \dots, c_{e-k}(Q)]}{\langle c(S)c(Q) = c(p^*E)\rangle}.$$

If we apply this to the homotopic quotient of the Grassmannian, then we obtain

$$H_T^*(\mathrm{Gr}(k,n)) = \frac{\mathbb{C}[t_1, \dots, t_n; c_1^T(S), \dots, c_k^T(S); c_1^T(Q), \dots, c_{n-k}^T(Q)]}{\langle c^T(S)c^T(Q) = e_i(t_1, \dots, t_n)\rangle}.$$

Here we used that the tautological subbundle S has homotopic quotient $S_T = \mathcal{S}$, the tautological subbundle on $\mathbb{G}(k, \oplus_i \mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1))$, therefore $c^T(S) = c(\mathcal{S})$. The same property holds for the quotient bundle. We also used that with the identifications above $t_i = c_1(\mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1))$, thus

$$c(\oplus_i \mathcal{O}_{(\mathbb{P}^\infty)_{(i)}}(-1)) = (1+t_1) \cdot \ldots \cdot (1+t_n) = \sum e_i(t_1, \dots, t_n).$$

7.3. **Functorial properties.** Let X, Y be compact (e.g projective) manifolds. Assume that X is a G-space and Y is an H-space. Let $f : X \to Y$ be a morphism and $\varphi : G \to H$ be a group homomorphism. We say that f is φ-equivariant if

$$f(g.x) = \varphi(g).f(x).$$

Almost all the time $G = H$ and $\varphi = id_G$; in this case we simply say that f is G-equivariant.

For simplicity we will focus on G-equivariant morphisms, but everything can be stated in terms of φ-equivariant morphisms. The idea is that a G-equivariant morphism $f : X \to Y$ induces a morphism $F : X_G \to Y_G$ between the homotopic quotients, and of their finite dimensional approximations. We have a commutative 'compatibility diagram':

$$\begin{tikzcd} X_{G,n} \arrow[r] \arrow[d, "F_n"] & X_G \arrow[d, "F"] \\ Y_{G,n} \arrow[r] & Y_G \end{tikzcd}$$

where the horizontal maps are inclusions and the vertical maps are induced by f. Then if f is proper/flat/closed embedding so will be the finite dimensional approximations F_n. Further, all these maps are also compatible with the inclusions $BG_n \subset BG$. In algebraic geometry, this was made rigorous by Edidin and Graham ADDREF.

This implies that the usual formalism of in Chow group/ (co)homology, involving push-forwads, pull-back, and various compatibilities extends *verbatim* to the equivariant setting. We recall below the most important constructions.

Remark 7.2. *A note on Poincaré duality: for smooth spaces X there is an isomorphism $H_i(X) \simeq H^{2\dim X - i}(X)$. For Chow groups this isomorphism holds by definition. This isomorphism extends to the equivariant setting, although some care is required. For instance $H^G_{-i}(X)$ will be non-zero (just take the class $\kappa \cap [X]_G$, where κ has sufficiently high cohomological degree. We will tacitly assume the equivariant Poincaré duality in the constructions below.*

7.3.1. *The normal bundle and the Euler class.* Let Y be a manifold and consider a submanifold $j : X \hookrightarrow Y$. The inclusion of tangent spaces gives an exact sequence of vector bundles on X:

$$0 \longrightarrow TX \longrightarrow j^*TY \longrightarrow N_X(Y) \longrightarrow 0$$

The quotient bundle $N_X(Y)$ is called the *normal bundle* of X in Y. It is a vector bundle on X or rank $\dim Y - \dim X$. The equivariant *Euler class* $e^T(N_X(Y))$ is defined to be the equivariant top Chern class

$$c_{top}(N_X(Y)) \in H^*(X).$$

If all manifolds and maps are equivariant then so is the normal bundle, and the equivariant Euler class is $c^G_{\dim Y - \dim X}(N_X(Y)) \in H^*_G(X)$.

7.3.2. *Push forward.* Let $f : X \to Y$ be a G-equivariant morphism compact oriented manifolds (projective manifolds satisfy this property.) Then there is a push-forward $f_* : H^G_*(X) \to H^G_*(Y)$ is a morphism of $H^*_G(pt)$-modules.

In the context of algebraic varieties and (equivariant) Chow groups and algebraic this morphism is determined by the following. Let $W \subset X$ be a G-invariant irreducible subvariety. The inclusion $W \subset X$ determines a subvariety of the homotopic quotient $W_G \subset X_G$; we denote by $[W]_G \in H^G_{2k}(X)$ the fundamental class. After identifying the G-equivariant cohomology to the G-equivariant homology by Poincaré duality, this may also be regarded as a class in $H_G^{2\mathrm{codim} W}(X)$. Then

$$f_*[W]_G = \begin{cases} d_W[f(W)]_G & \text{if } \dim f(W) = \dim W \\ 0 & \text{otherwise} \end{cases} \tag{7.2}$$

Here d_W is the degree of the restriction $f : W \to f(W)$, i.e. the number of points in a generic fibre of this restriction.

An important special case is when $Y = pt$. Then the push-forward is the *integration map*

$$\int_X^G : H^G_*(X) \to H^G_*(pt) = H^*_G(pt)$$

obtained again after using Poincaré duality. Observe that $\int_X^G \alpha = 0$ unless $\alpha \in H_0^G(X)$.

The push forwards also induce *Gysin morphisms* in cohomology. As before $f : X \to Y$ is a proper morphism, and Y is smooth (smoothness of X is not needed here). Then one has a push-forward $f_* : H^i_G(X) \to H_G^{i-2(\dim X - \dim Y)}(Y)$ obtained by compositions

$$H^i_G(X) \xrightarrow{\cap [X]_G} H^G_{2\dim X - i}(X) \xrightarrow{f_*} H^G_{2\dim X - i}(Y) \xrightarrow{PD} H_G^{2\dim Y - 2\dim X + i}(Y) \ ,$$

where the last arrow is given by (equivariant) Poincaré duality.

7.3.3. *Pull-back and localization.* As before let $f : X \to Y$ be a G-equivariant morphism of compact oriented manifolds. There are two situations when we define a pull-back: f is flat and f is a closed embedding. In both situations there is a pull-back map

$$f^* : H^i_G(Y) \to H^i_G(X).$$

If f is flat then $f^*[W]_G = [f^{-1}(W)]_G$, where $W \subset Y$ is closed and irreducible.

If $f = i : X \hookrightarrow Y$ is a closed embedding of smooth manifolds then $N_X(Y)$ is a G-equivariant vector bundle on Y, and the Gysin-morphism is related to the pull-back via the self intersection formula:

$$i^* i_*([W]_G) = c^G_{top}(N_X(Y)) \cap [W]_G \in H^G_*(X). \tag{7.3}$$

An important special case is when we consider a component $F = X \subset Y$ which is fixed by G (i.e. G acts trivially on F). A theorem of Fogarty ADDREF implies that if G is a linear algebraic group, then its fixed scheme Y^G is smooth, thus F is smooth. Consider the inclusion $\iota : F \hookrightarrow X$. Since G acts trivially on F, it follows that $H^*_G(F) = H^*(F) \otimes H^*_G(pt)$. Then the pull-back map gives a $H^*_G(pt)$-algebra homomorphism

$$\iota^* : H^i_G(X) \to H^i_G(F) = (H^*(F) \otimes H^*_G(pt))_{\deg i},$$

called the *localization map.* The situation of most concern to us is when $F = pt$ and $\iota = \iota_x : \{x\} \hookrightarrow X$ is the inclusion. Then the localization map is $\iota_x^* : H^*_G(X) \to H^*_G(pt)$, and by the self intersection formula (7.3),

$$\iota_x^*[x]_G = e^G(T_x Y). \tag{7.4}$$

Further, assume that $G = T$ is a torus, and $E \to X$ is a T-equivariant complex vector bundle. Let $x \in X^T$ be a fixed point. Then the fibre E_x is a T-module, thus it has a weight decomposition $E_x = \oplus \mathbb{C}_{\chi_i}$. By the pull-back property of Chern classes, it follows that

$$\iota_x^*(c^T(E)) = c^T(E_x) = \prod c^T(\mathbb{C}_{\chi_i}) = \prod (1 + \chi_i).$$

Here we regard the characters χ_i as elements of $H^*_T(pt)$ by writing each character as $\chi_i = \sum a_{ij} t_j$. In other words, *the Chern roots of the localization* *at x of an equivariant vector bundle are the weights of the fibre over x.* (Applying this in a concrete example is a homework problem.)

LM:051320: Add an explanation for this, based on the identification of equivariant parameters with Chern roots.

Example 7.1. *Work out localizations of $\mathcal{O}(-1)$ and on $\mathbb{P}^n$. Then work out the localizations of the tangent bundle.*

The pull-back and push-forward are related by the *projection formula*: if $f : X \to Y$ is a proper morphism, then

$$f_*(\alpha \cap f^*(\beta)) = f_*(\alpha) \cap \beta; \quad \forall \alpha \in H_*(X), \beta \in H^*(Y).$$

7.4. **The localization theorem.** The following is the Atyiah-Bott localization theorem ADDREF; this version follows [Bri98, Thm.3].

Theorem 7.1. *Let X be a compact oriented manifold with a T-action and let $X^T = \bigcup F_i$ the decomposition into irreducible components. Let $\iota : X^T \hookrightarrow X$ and $\iota_j : F_j \to X$ be the inclusion maps. Then the restriction map*

$$\iota^* : H_T^*(X) \to H_T^*(X^T) = \oplus_{F_i \subset X^T} H_T^*(pt) \otimes H^*(F_i),$$

is an isomorphism of $H_T^*(pt)$-algebras after inverting finitely many non-trivial characters of T.

Further, after localization, for any $\alpha^T \in H_T^*(X)$,

$$\alpha^T = \sum_j (\iota_j)_* \frac{\iota_j^*(\alpha^T)}{e^T(N_{F_j}(X))}. \tag{7.5}$$

As stated the theorem does not account for possible torsion, i.e. the image of α^T in the localization of $H_T^*(X)$ may actually be 0. One needs additional conditions on X to ensure that the localization map is injective. One such condition is that X is *equivariant formal* (EF); this condition was introduced in [GKM98]. We list below several situation when a T-variety X (possibly singular) is equivariantly formal:

- X has a T-invariant CW decomposition;
- equivariant cohomology of X vanishes in odd degree;
- X is a symplectic manifold with a Hamiltonian T-action.

The original condition (EF) has to do with the degeneration of the cohomology spectral sequence associated to the projection $X_T \to BT$. By a theorem by Leray-Hirsch, (EF) also holds if the following is satisfied:

- (Leray-Hirsch condition) $H_T^*(X)$ is a free module over $H_T^*(pt)$, and it has a basis which restricts to a $\mathbb{Z}$-basis of $H^*(X)$.

This last condition holds for any (generalized) flag manifold G/P, such as the Grassmannian. The basis of the equivariant cohomology is given by the fundamental classes of the Schubert varieties.

We have the following more precise theorem; cf. [Bri98, Thm. 6] and [GKM98, Thm. 1.2.2].

Theorem 7.2. *(a) Let X be a compact manifold. If the $H^*_T(pt)$-module $H^*_T(X)$ is free, then the localization map*

$$\iota^* : H^*_T(X) \to H^*_T(X^T)$$

is injective, and the image is the intersection of the images of the maps

$$\iota_{T,T'} : H^*_T(X^{T'}) \to H^*_T(X^T)$$

over all codimension 1 subtori $T' \subset T$.

(b) (GKM theorem) Let X be an equivariantly formal T-variety (possibly singular), and assume that X has finitely many T-fixed points $x_1, \dots, x_k$ and finitely many 1-dimensional orbits $E_1, E_2, \dots E_s$. Then the localization map

$$\iota^* : H^*_T(X) \to H^*_T(X^T)$$

is injective, and the image is the subalgebra

$$\{(P_1, \dots, P_k) \in \oplus_{i=1}^k H^*_T(pt) : GKM\, equivalences\ hold\ \}.$$

The GKM equivalences are obtained as follows. Each 1-dimensional orbit E is isomorphic to $\mathbb{P}^1$, and it has exactly two T-fixed points, 0 and ∞. T acts by a character χ on $T_0(E)$ and $-\chi$ on $T_\infty(E)$. Then the set of GKM restrictions consists of $\chi|(P_i - Q_j)$ for each one-dimensional orbit E joining x_i, x_j with associated character $\pm\chi$.

7.5. **Integration formula and the fixed point basis.** In this section we assume that X is a projective T-manifold with finitely many fixed points $x_1, \dots, x_k$ and finitely many one-dimensional T-orbits; our main example is a (generalized) flag variety. By [ByB73, Theorem §4], the ordinary cohomology is a free module with a basis given by the closure of BB cells. Therefore the equivariant cohomology is also free over $H^*_T(pt)$. In this case we may consider the prime ideal S consisting of homogeneous polynomials of positive degree. Then S is a multiplicative set and

$$rank_{H^*_T(pt)} H^*_T(X) = rank_{S^{-1}H^*_T(pt)} S^{-1} H^*_T(X) = rank_{S^{-1}H^*_T(pt)} S^{-1} H^*_T(X^T).$$

with all modules free of rank $k = \#(X^T)$. Let and $\iota : X^T \hookrightarrow X$ and $\iota_j : x_j \hookrightarrow X$ be the inclusions. In this case the localization map ι^* is injective, and we have the following Atyiah-Bott localization formula:

Corollary 7.2 (Integration formula)**.** *Let $\alpha^T \in H^*_T(X)$. Then the following hold:*

$$\alpha^T = \sum \frac{\iota_j^*(\alpha^T)}{e^T(T_{x_j}X)} [x_j]_T.$$

In particular, we have the **integration formula**

$$\int_X^T \alpha^T = \sum_{j=1}^k \frac{\iota_j^*(\alpha^T)}{e^T(T_{x_j}X)}.$$

Proof. Localize both sides at each fixed point, and observe that $\iota_j^*[x_j] = \delta_{j,k} e^T(T_{x_j} X)$. $\square$

The hypothesis on X also implies that the fundamental classes of fixed points form a basis for the localized ring $H_T^*(X)_{loc} := S^{-1} H_T^*(X)$. The multiplication in this basis is particularly easy:

$$[x_i]_T \cdot [x_j]_T = \delta_{i,j} e^T(T_{x_j} X)[x_i]_T.$$

Example 7.3. *Let $X = \mathbb{P}^2$, and let $\alpha^T = [\mathbb{P}^1]_T$ where $\mathbb{P}^1 = \{[* : * : 0]\}$. Then*

$$[\mathbb{P}^1]_T = a[1 : 0 : 0]_T + b[0 : 1 : 0]_T + c[0 : 0 : 1]_T.$$

By localization it follows that

$$a := \frac{\iota^*_{[1:0:0]}[\mathbb{P}^1]_T}{e^T(T_{[1:0:0]}},$$

and similarly for B, C. One calculates that

$$e^T(T_{[1:0:0]}) = (t_2 - t_1)(t_3 - t_1).$$

To calculate $\iota^*_{[1:0:0]}[\mathbb{P}^1]_T$ one may use the recursions from the GKM relations, and one obtains:

$$\iota^*_{[1:0:0]}[\mathbb{P}^1]_T = t_3 - t_1.$$

Example 7.4. *Example for $\mathbb{P}^n$ and tautological sequence. Let $\xi := \mathcal{O}_{\mathbb{P}^n}(1)$. This is a T-equivariant bundle and we know that*

$$c_1^T(\xi) = a[\mathbb{P}^{n-1}]_T + b[\mathbb{P}^n]_T,$$

where $\mathbb{P}^{n-1}$ is embedded by $x_n = 0$. By localization at $[0 : 0 : \ldots : 0 : 1]$ we find that

$$b = \iota^*_{[0:\ldots:0:1]}(c_1^T(\xi)) = -t_{n+1}.$$

After localizing at $[0 : \ldots : 1 : 0]$ we obtain

$$\iota^*_{[0:\ldots:1:0]}(c_1^T(\xi)) = a\iota^*_{[0:\ldots:1:0]}[\mathbb{P}^{n-1}]_T + b,$$

thus

$$a = \frac{\iota^*_{[0:\ldots:1:0]}(c_1^T(\xi)) + t_{n+1}}{\iota^*_{[0:\ldots:1:0]}[\mathbb{P}^{n-1}]_T} = \frac{-t_n + t_{n+1}}{t_{n+1} - t_n} = 1.$$

If one works non-equivariantly, this implies that

$$c_1(\xi) = [\mathbb{P}^{n-1}].$$

We now illustrate the localization formula by integrating $c_1^T(\xi)^{n-1}$. Then on one side, from the non-equivariant calculation we know that

$$\int_{\mathbb{P}^n}^T c_1^T(\xi)^n = 1.$$

On the other, by the localization formula,

$$1 = \int_{\mathbb{P}^n}^T c_1^T(\xi)^n = \sum_{i=1}^{n+1} \frac{(-t_i)^n}{\prod_{j \neq i}(t_j - t_i)}.$$

7.6. **Characters of irreducible** $\mathrm{SL}_2(\mathbb{C})$ **modules via Grothendieck-Riemann-Roch.**

7.6.1. *The Grothendieck-Rimemann-Roch theorem.* Let X be a projective manifold and $E \to X$ a vector bundle. Denote by

$$\chi(X;E) = \sum (-1)^i \dim H^i(X;E),$$

the sheaf Euler characteristic of E. The GRR theorem states that

$$\chi(X;E) = \int_X ch(E)Td(T_X).$$

Here $ch(E)$ is the *Chern character* of E. If E has Chern roots $x_1, \ldots, x_e$ then

$$ch(E) = e^{x_1} + e^{x_2} + \ldots + e^{x_e} = rk(E) + c_1(E) + \frac{1}{2}(c_1^2(E) - c_2(E)) + \ldots.$$

The Chern character is additive for short exact sequences: if

$$0 \longrightarrow E \longrightarrow G \longrightarrow F \longrightarrow 0 \tag{7.6}$$

then

$$ch(G) = ch(E) + ch(F).$$

From this and the splitting principle it follows that the Chern character is multiplicative for tensor products of vector bundles:

$$ch(E \otimes F) = ch(E)ch(F).$$

The class $Td(E)$ is called the *Todd class*. If E is a line bundle with Chern root x then

$$Td(E) = \frac{x}{1 - e^{-x}} = 1 + \frac{1}{2}x + \ldots.$$

The Todd class is multiplicative with respect to exact sequences, i.e. for the sequence (7.6),

$$Td(G) = Td(E)Td(F).$$

From this it follows that

$$Td(E) = 1 + \frac{1}{2}c_1(E) + \frac{1}{12}(c_1(E)^2 + c_2(E)) + \frac{1}{24}c_1(E)c_2(E) + \ldots$$

We refer to [Ful84, §3.2] for more properties of the Chern character and the Todd class.

The GRR theorem extends equivariantly using the equivariant Chow groups of Edidin and Graham; cf. [EG00]. We illustrate the localization formula and the GRR theorem in the example below. Before we start, we observe that by definition, both the Todd class and the Chern character commute with localization. This means that if E is a T-equivariant vector bundle, and if $x \in X^T$, then

$$\iota_x^* ch^T(E) = ch^T(E_x); \quad \iota_x^* Td^T(E) = Td^T(E_x), \tag{7.7}$$

where E_x is the fibre over x. This follows from the multiplicativity properties.

7.6.2. *Characters of $\mathcal{O}_{\mathbb{P}^1}(n)$.* Consider the group $G := \mathrm{SL}_2(\mathbb{C})$ and its maximal torus

$$T = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix} \subset G.$$

This is a simple Lie group, and any G-representation W is completely reducible. This means that $W = \oplus W_i$, where W_i is an irreducible representation. A famous result classifies the irreducible representations as follows. For each $n \geq 0$ there is exactly one irreducible representation V^n such that $\dim V^n = n+1$. Then V^n is also a T-module and it has a decomposition

$$V = V_n \oplus V_{n-2} \oplus \ldots \oplus V_{-n},$$

where

$$V_i := \{v \in V : t.v = t^i v, \forall t \in T \simeq \mathbb{C}^*\}$$

are the weight spaces. The *character* of V^n is

$$ch(V^n) = \sum ch(V_i) = e^{nt} + e^{(n-2)t} + \ldots + e^{-nt} = h_n(e^t, e^{-t}), \tag{7.8}$$

where h_n denotes the complete homogeneous symmetric function. We want to realize this character by the localization formula.

Let $\mathbb{C}^*$ act on $\mathbb{P}^1$ by $t.[x_0 : x_1] = [tx_0 : t^{-1}x_1]$. This is the natural action induced from SL_2:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{bmatrix} x_0 \\ x_1 \end{bmatrix}$$

Consider the SL_2-equivariant vector bundle $\mathcal{O}_{\mathbb{P}^1}(n)$. This is the bundle associated to the representation $(\langle e_1 \rangle^*)^{\otimes n}$.

Let $0 := [0 : 1]$ and $\infty := [1 : 0]$. The localizations of $c_1^T(\mathcal{O}_{\mathbb{P}^1}(n))$ at $0, \infty$ are respectively $nt, -nt$. The weights of the tangent spaces are

$$T_0\mathbb{P}^1 = \mathbb{C}_{2t}; \quad T_\infty(\mathbb{P}^1) = \mathbb{C}_{-2t}.$$

The Todd class has localizations

$$\iota_0^*(Td(T_{\mathbb{P}^1}) = \frac{2t}{1 - e^{-2t}}; \quad \iota_\infty^*(Td(T_{\mathbb{P}^1}) = \frac{-2t}{1 - e^{2t}}.$$

We now apply the equivariant GRR theorem

$$\chi^{\mathbb{C}^*}(\mathcal{O}_{\mathbb{P}^1}(n)) = \int_{\mathbb{P}^1}^{\mathbb{C}^*} ch^{\mathbb{C}^*}(\mathcal{O}_{\mathbb{P}^1}(n)) Td^{\mathbb{C}^*}(T_{\mathbb{P}^1}).$$

By localization theorem this equals to

$$\begin{aligned}\frac{e^{nt}\frac{2t}{1-e^{-2t}}}{2t}+\frac{e^{-nt}\frac{-2t}{1-e^{2t}}}{-2t} &= \frac{e^{nt}}{1-e^{-2t}}+\frac{e^{-nt}}{1-e^{2t}}\\ &= \frac{e^{(n+1)t}}{e^{t}-e^{-t}}+\frac{e^{-(n+1)t}}{e^{-t}-e^{t}}\\ &= \frac{e^{(n+1)t}-e^{-(n+1)t}}{e^{t}-e^{-t}}.\end{aligned}$$

After making $x := e^t$ this gives

$$\frac{x^{n+1}-1/x^{n+1}}{x-1/x} = \frac{(x^2)^{n+1}-1}{x^2-1}\cdot\frac{1}{x^n} = x^n + x^{n-2} + \ldots + x^{-n},$$

giving the promised character formula from (7.8).

8. Appendix: Basics on tangent spaces and group actions

Let G be a topological group, and let X be any topological space. We say that G acts on X, or that X is a G-space, if there exists a continuous morphism $G\times X \to X$ such that $id.x = x$ and $g_1.(g_2.x) = (g_1g_2).x$ for all $x \in X$. For each $g \in G$ we have an automorphism $\varphi_g : X \to X$ defined by $x \mapsto g.x$, and the definition of a group action can be rephrased as the existence of a group homomorphism $\Psi : G \to \mathrm{Aut}(X)$. If $f : X \to Y$ is a continuous map of G-spaces, we say that f is G-equivariant if $f(g.x) = g.f(x)$. There is a category with objects G-spaces and morphisms G-equivariant morphisms.

Remark 8.1. *If one works in the analytic or algebraic context, then one also requires that the spaces G, X, the morphism $G \times X \to X$, and any other definition, make sense in that category. In the case discussed in this class, both G and X will be algebraic varieties, G will be a complex manifold, and X will be a G-invariant subspace of a complex G-manifold M. For instance, $G = \mathrm{GL}$, and X could be a Schubert variety in the Grassmann manifold $M = \mathrm{Gr}(k, V)$.*

The fixed point locus is

$$X^G := \{x \in X : g.x = x\ \forall\ g \in G\}.$$

For a smooth point $x \in X$ (provided this makes sense), one can define a *tangent space* T_xX. This is a vector space. In what follows we collect the following standard facts about tangent spaces and differentials.

Proposition 8.1. *Let X, Y be two G-spaces, and let $f : X \to Y$ be a G-equivariant morphism.*

(a) If X is smooth at x then T_xX is a vector space of dimension $\dim X$.

(b) If X is smooth at x and $U \subset X$ is an open set such that $x \in U$, then $T_xU = T_xX$. In particular, $\dim X = \dim U$*, giving a way to define dimension for possibly singular spaces X.*

(c) Let $x \in X$ and $y := f(x)$ be smooth points. Then f induces a linear map (called the differential*) $df_x : T_xX \to T_yY$.*

(d) If $x \in X^G$, then the tangent space T_xM has a structure of a G-module, i.e. a G-representation.

(e) Let $x \in X$ and $y := f(x)$ be smooth points. If $x \in X^G$ then $y \in Y^G$, and df_x is a map of G-modules. If $X = Y = U \subset \mathbb{C}^n$ is an open set, then the linear map df_x is given by $(\partial/\partial z_1, \dots, \partial/\partial z_n)$. In particular, if f is a linear map (in terms of coordinates) then $f = df_x$ for any $x \in X$.

(f) Assume that $f : M \to U \subset \mathbb{C}^n$ is a G-equivariant isomorphism to U, a G-invariant open set of $\mathbb{C}^n$, sending $x \in X^G$ to 0. Then f induces an isomorphism of G-modules $df_x : T_xM \to T_0U = \mathbb{C}^n$.

Sketch of the proof. We assume that $x \in X$ has a neighborhood isomorphic to an open set $U \subset \mathbb{C}^n$; we may also assume that under this isomorphism $x \mapsto 0$. The notion of the tangent space makes sense in $\mathbb{C}^n$. In fact, $T_0\mathbb{C}^n$ is the set of vectors $v \in \mathbb{C}^n$ such that $v = \frac{d}{dt}\gamma(t)_{t=0}$, where $\gamma : (-1,1) \to U$ is a differentiable map such that $\gamma(0) = 0$. Since this is a local definition (it uses U, not X), part (b) follows. The dimension statement from (a) follows from the fact that $T_0\mathbb{C}^n$ has a basis of *derivations* $\partial/\partial z_i$ for $1 \le i \le n$. Under the identification $e_i \leftrightarrow \partial/\partial z_i$ identifies $\mathbb{C}^n$ with $T_0\mathbb{C}^n$ as vector spaces.

Part (c) follows from the local construction.

If in addition $x \in X^G$, then $g.x = x$ for any $g \in G$. Consider the map $\varphi_g : X \to X$ given by $x \mapsto g.x$. The associated differential is linear. Since $\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1g_2}$, it follows that $d(\varphi_{g_1})d(\varphi_{g_2}) = d\varphi_{g_1g_2}$. This shows that $g.(v+w) = g.v + g.w$ (as $d\varphi_g$ is linear) and that $g_1.(g_2.v) = (g_1g_2).v$ for any $v \in T_xX$, proving that T_xX is a G-module. Part (d) follows.

Part (e) follows from standard multivariable calculus (or complex analysis). Part (f) follows from the chain rule (as f has an inverse g, and we calculate $d(f \circ g)$ and $d(g \circ f)$). $\square$

Often $G = T = (\mathbb{C}^*)^r$ is a torus and V is a complex T-module. It is useful to recall some basic facts in character theory.

An (algebraic) character of T is a group homomorphism $\chi : T \to \mathbb{C}^*$ of the form $(t_1, \dots, t_r) \mapsto t_1^{a_1} \cdots t_r^{a_r}$ where $a_i \in \mathbb{Z}$. In particular the group $\widehat{T}$ of characters of T is naturally isomorphic to the additive group $\mathbb{Z}^r$. If $V \simeq \mathbb{C}$ is a one-dimensional module and $\chi(t) = t_1^{a_1} \cdots t_r^{a_r}$ is a character we denote by $\mathbb{C}_\chi$ the T-module given by $t.z = \chi(t)z$.

More generally, fix a basis $e_1, \dots, e_n$ of V. A standard fact is that any T-module V is *diagonalizable* i.e.

$$V \simeq \mathbb{C}_{\chi_1} \oplus \ldots \oplus \mathbb{C}_{\chi_r}$$

is isomorphic to a direct sum of one dimensional T-modules for some characters $\chi_1, \dots, \chi_r$. ADDREF

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