HOMEWORK

1. Lecture 1 homework (K theory)

1. Use the intersection property in K theory to work out the multiplication table for $K(\mathbb{P}^n)$.

2. Utilize the short exact sequence $0 \to \mathcal{O}_{\mathbb{P}^n}(-1) \to \mathcal{O}_{\mathbb{P}^n} \to \mathcal{O}_{\mathbb{P}^{n-1}} \to 0$ to prove that $K(\mathbb{P}^n)$ has a presentation by generators and relations given by $\mathbb{Z}[x]/\langle(1-x)^{n+1}\rangle$. (*) Can you say how is this related to the Whitney relations:

$$\lambda_y(\mathcal{O}(-1)) \cdot \lambda_y(\mathbb{C}^{n+1}/\mathcal{O}(-1)) = \lambda_y(\mathbb{C}^{n+1}) \quad ?$$

3. Find the Schubert class expansion of $[\mathcal{O}_{\mathbb{P}^3}(\pm k)] \in K(\mathbb{P}^3)$.

4. Consider the Chern character $ch : K(\mathbb{P}^n) \to H^*(\mathbb{P}^n)$.
(a) Find the Schubert expansion of $ch(\mathcal{O}_{\mathbb{P}^n}(\pm 1))$.
(b) Find the image of each K-theoretic Schubert class through the Chern character.

5.(*) Let $\mathcal{Q}$ be the (rank $n-k$) tautological quotient bundle over $\mathrm{Gr}(k,n)$. Prove that $\det \mathcal{Q} = \sum \mathcal{O}_\lambda$. (Hint: multiply by the duals of Schubert classes and take Euler characteristic.)

6. (The Demazure operator) Let $\mathrm{Fl}(\hat{i}, n)$ denote the partial flag manifold parametrizing $F_1 \subset \ldots \subset \widehat{F_i} \subset \ldots \subset \mathbb{C}^n$, and let $p_i : \mathrm{Fl}(n) \to \mathrm{Fl}(\hat{i}; n)$ be the natural projection. Define the endomorphism of $K(\mathrm{Fl}(n))$

$$\partial_i = p_i^*(p_i)_*.$$

Prove that

$$\partial_i(\mathcal{O}^w) = \begin{cases} \mathcal{O}^{ws_i} & \text{if } ws_i < w; \\ \mathcal{O}^w & \text{otherwise} \end{cases}$$

Utilize this to show that ∂_i's satisfy $\partial_i^2 = \partial_i$, and the usual commutation and braid relations.

2. Lectures 2 and 3: Quantum K theory

1. Describe all lines (i.e., degree 1 rational curves) in $\mathbb{P}^n$.

2. Draw the moment graphs, and utilize them to calculate the curve neighborhoods (of all degrees) in $\mathbb{P}^n$ and in $\mathrm{Gr}(2,4)$.

3*. Use the formula for the distance $d_{min}(\lambda, \mu)$ to calculate the quantum K metric $((\mathcal{O}^{(1)}, \mathcal{O}^\lambda))$ for each λ included in the $2 \times (4-2)$ rectangle. (Hint: use that the quantum cohomology ring $\mathrm{QH}^*(\mathrm{Gr}(k,n))$ is graded with $\deg q = n$ to deduce that $d_{min}((1),(2,2)) < 2$.)

4. Use the recursion formulae from the notes to calculate the curve neighborhood elements z_d (i.e., $\Gamma_d(pt) = X_{z_d}$) in the case $X = \mathrm{Fl}(3)$.

5. Utilize the 'quantum = classical' to prove that for any $d > 0$,

$$\langle \mathcal{O}^{(1)}, \mathcal{O}^\lambda, \mathcal{O}^\mu \rangle_d = \langle \mathcal{O}^\lambda, \mathcal{O}^\mu \rangle_d.$$

6. Utilize problems 1 and 4 in set 2 to work out the quantum K theory of $\mathbb{P}^n$. Check the quantized Whitney relations:

$$\lambda_y(\mathcal{O}(-1)) \circ \lambda_y(\mathbb{C}^n/\mathcal{O}(-1)) = \lambda_y(\mathbb{C}^n) - y^n \frac{q}{1-q} \mathcal{O}(-1) \circ \det(\mathbb{C}^n/\mathcal{O}(-1)).$$

(Problem 5 in homework set 1 should be helpful.)