Chern-Schwartz-MacPherson classes for Schubert varieties in the Grassmannian

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Chern classes of Schubert cells and varieties
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Smooth varieties; the Grassmannian

Let X be a smooth variety, TX its tangent bundle, $c(TX)$ its total Chern class.
Chern class of X in $H_*(X)$ or $A_*(X)$:

$$c(X) = c(TX) \cap [X].$$

If X is a Grassmannian $Gr(p, m)$:

$$c(X) = \sum_{\mu \subset \lambda} d_{\lambda,\mu}(\sigma_\mu \cdot \sigma_{\lambda^\vee}) \cap [X]$$

where $\lambda = (m - p \geqslant \lambda_1 \geqslant \cdots \geqslant \lambda_p \geqslant 0)$, $\mu = (m - p \geqslant \mu_1 \geqslant \cdots \geqslant \mu_p \geqslant 0)$ are partitions; $\lambda^\vee$ conjugate of λ;
$\sigma_{\lambda^\vee}$,σ_μ the associated Schubert classes in $A^*(X)$.

Positivity for the Grassmannian

$$c(X) = \sum_{\mu \subset \lambda} d_{\lambda,\mu} (\sigma_\mu \cdot \sigma_{\lambda^\vee}) \cap [X],$$

$$d_{\lambda,\mu} = \det \left(\binom{\lambda_i + p - i}{\mu_j + p - j} \right)_{1 \leqslant i,j \leqslant p}.$$

Fact: $c(X)$ is an effective cycle:

- *geometrically* since TX is ample;
- *combinatorially* since $d_{\lambda,\mu}$ counts non-intersecting $n-$tuples of plane lattice paths (Gessel-Viennot, Lindstrom, etc).

Questions: 1. Extend the construction of $c(X)$ to singular varieties (enter CSM classes).
2. Compute these classes for Schubert varieties in the Grassmannian.
3. Any (hope of) positivity ?

CSM classes of constructible functions

Let X be a complete variety. $F(X)$ denotes the additive group of constructible functions on X:

$$\varphi : X \longrightarrow \mathbb{Z}$$

$$\varphi = \sum n_i \cdot char_{W_i}.$$

where $n_i \in \mathbb{Z}$, W_i constructible (or locally closed) subset of X and $char_{W_i}$ is the characteristic function. If $f : X \to Y$ is a proper morphism, then get $f_* : F(X) \to F(Y)$, defined by

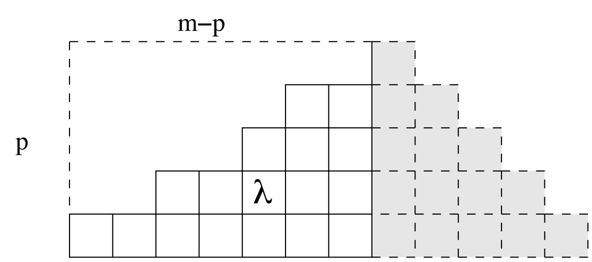
$$f_*(\varphi)(x) = \sum n_i \cdot \chi(f^{-1}(x) \cap W_i)$$

where χ is the topological Euler characteristic.

Theorem.(Grothendieck and Deligne, proved by MacPherson, also M. H. Schwartz)
There exists a unique natural transformation $c_{SM} : F(X) \longrightarrow A_*(X)$ such that:
1. if X is smooth, $c_{SM}(1_X) = c(TX) \cap [X]$.
2. c_{SM} commutes with proper morphisms.

CSM classes of Schubert cells

Let $\mathbb{S}_\lambda$ be the Schubert variety corresponding to the *homological* partition λ:



$$\{V \in Gr(p,m) : \dim(V \cap F_{\lambda_{p-i+1}+i}) \geqslant i\}$$

for a flag

$$F^m : (0) \subset F_1 \subset \cdots \subset F_m = \mathbb{C}^m.$$

Definition. Let $\mathbb{S}^o_\lambda$ be the Schubert cell corresponding to λ and

$$c_{SM}(\mathbb{S}^o_\lambda) := c_{SM}(1_{\mathbb{S}^o_\lambda}).$$

Schubert expansion

A $\mathbb{Z}$-basis for $A_*(\mathbb{S}_\lambda)$ is given by fundamental classes

$$A_*(\mathbb{S}_\lambda) = \oplus_{\mu\subset\lambda}\mathbb{Z}[\mathbb{S}_\mu].$$

Therefore, have an expansion,

$$c_{SM}(\mathbb{S}_\lambda^o) = \sum_{\mu\in\lambda} c(\lambda,\mu)[\mathbb{S}_\mu].$$

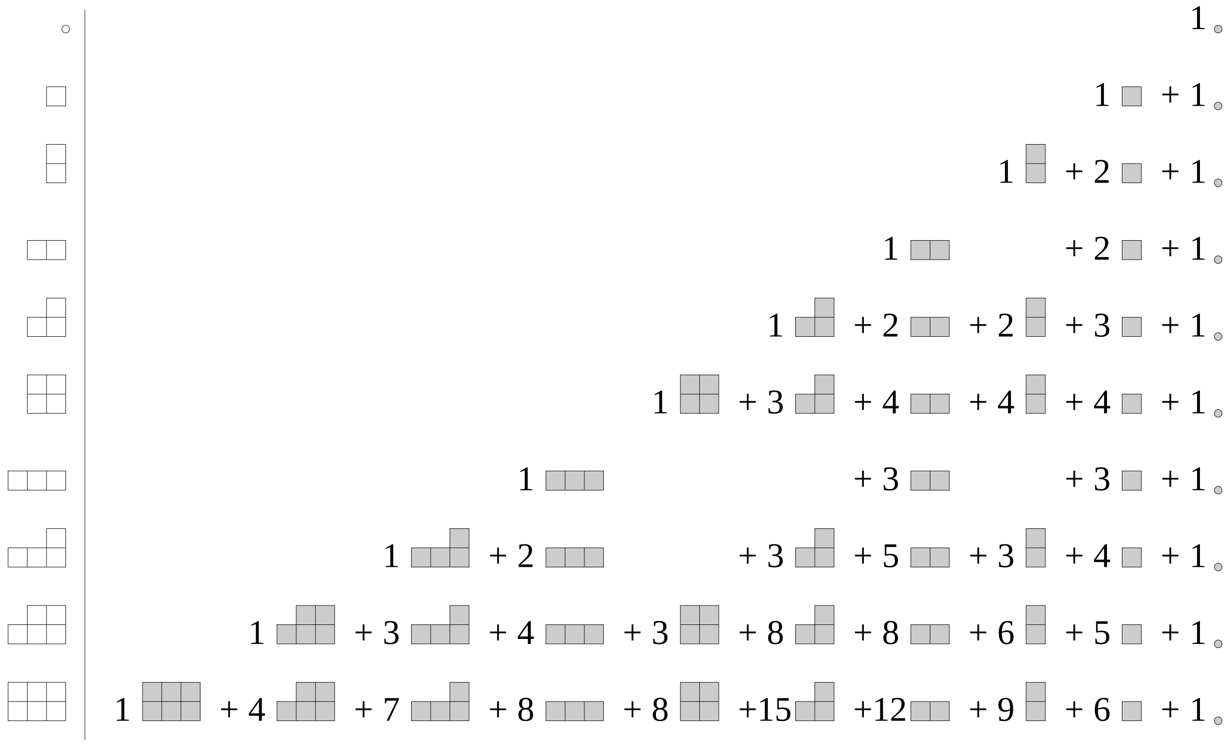
Since

$$\mathbb{S}_\lambda = \coprod_{\mu\subset\lambda} \mathbb{S}_\mu^o,$$

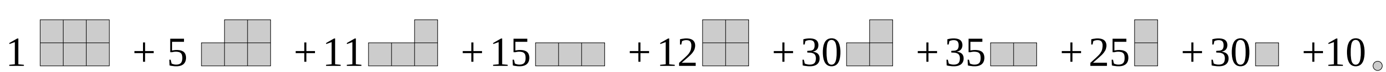
$$c_{SM}(\mathbb{S}_\lambda) = \sum_{\mu\subset\lambda} c_{SM}(\mathbb{S}_\mu^o).$$

Question: Find coefficients $c(\lambda,\mu)$!

An example



For example, $c_{SM}(Gr(2,5))$ may be obtained by adding up all these classes:



Note positivity of CSM classes of cells.

The Bott-Samelson resolution.

Let $\pi_\lambda : \mathbb{V}_\lambda \to \mathbb{S}_\lambda$ be the Bott-Samelson resolution of $\mathbb{S}_\lambda$.

Theorem.

1. π_λ is an isomorphism over $\mathbb{S}^o_\lambda$ and $\mathbb{V}_\lambda \backslash \mathbb{S}^o_\lambda$ is a simple normal crossing divisor $D = \bigcup_{i=1}^p D_i$. Let x_i be the class of D_i in $A^1(\mathbb{V}_\lambda)$.

2. $c_{SM}(\mathbb{S}^o_\lambda) = \pi_*(\frac{c(T\mathbb{V}_\lambda)}{\prod_{i=1}^p (1+D_i)} \cap [\mathbb{V}_\lambda]) \in A_*(\mathbb{S}_\lambda)$.

3. $c_{SM}(\mathbb{S}_{(\lambda_1,\lambda_2)}) = (\pi_{(\lambda_1,\lambda_2)})_*$ $((1+x_1)^{\lambda_1-\lambda_2} h_{\lambda_2}(1+x_1, x_2)(1+x_2)^{\lambda_2})$ (have a similar explicit formula in general).

Computing push-forwards

Let $N = m - p$, so have Grassmannian $Gr(p, N + p)$; (N^p) denotes the partition $(N, \cdots N)$, p components. Enough to compute push forwards

$$\pi_*(x_1^{r_1} \cdots x_p^{r_p} \cap [\mathbb{V}_{(N^d)}]).$$

These are

$$[\mathbb{S}_{(N-r_1,\cdots,N-r_p)}] \text{ if } r_1 \leqslant \cdots \leqslant r_p,$$

$$-\pi_*(x_1^{r_1} \cdots x_i^{r_{i+1}+1} x_{i+1}^{r_i-1} \cdots x_p^{r_p} \cap [\mathbb{V}_{(N^d)}]),$$

otherwise. E.g. in $Gr(2, 4)$,

$$\pi_*(x_2^2 \cap [\mathbb{V}_{(2,2)}]) = [\mathbb{S}_{(2,0)}] \quad \pi_*(x_1^2 x_2 \cap [\mathbb{V}_{(2,2)}]) = 0$$

$$\pi_*(x_1^2 \cap [\mathbb{V}_{(2,2)}])) = -[\mathbb{S}_{(1,1)}].$$

Binomial determinants; case $p = 3$.

Theorem.

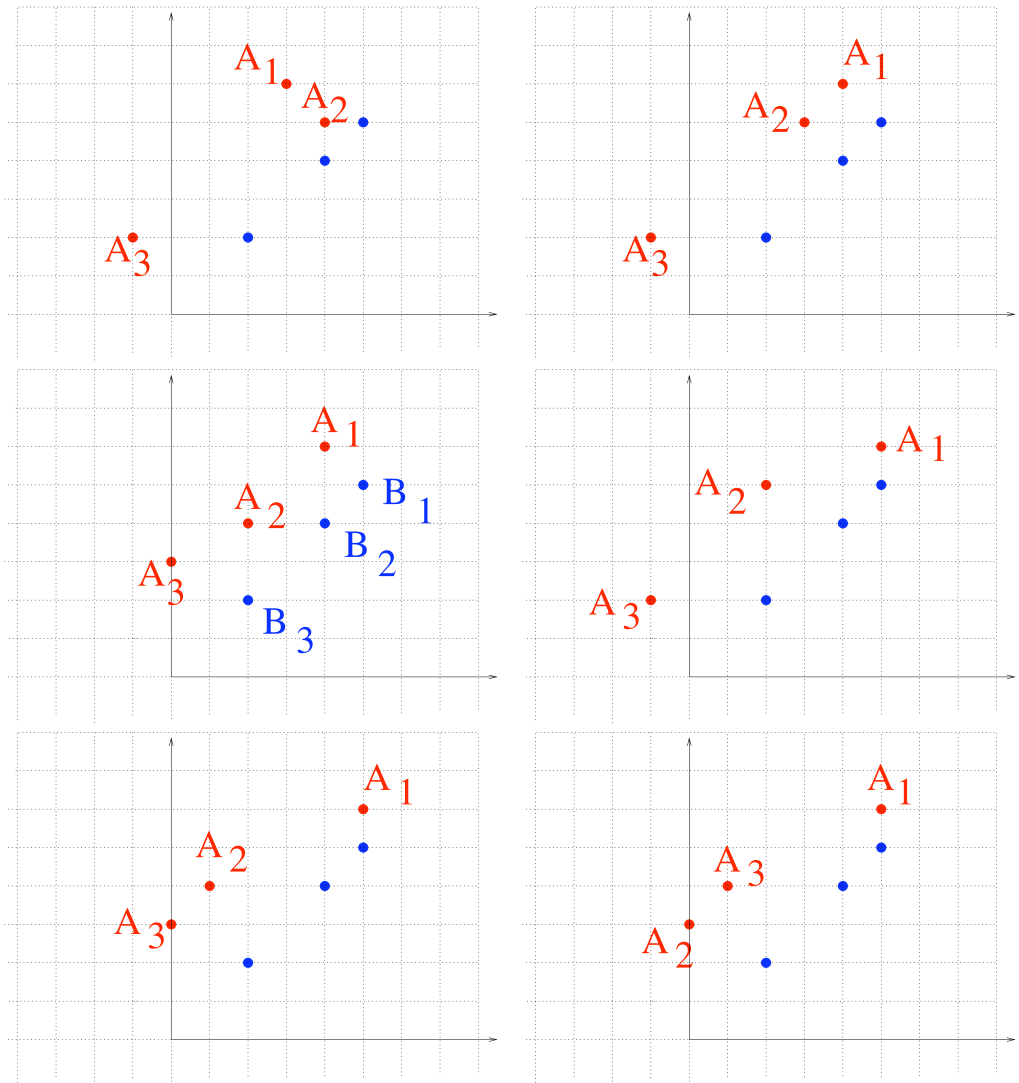
$$c(\lambda,\mu) = \sum_{k=0}^{\lambda_2} \sum_{i=0}^{k} \sum_{j=0}^{\lambda_3} \det M(i,j,k)$$

where $M(i,j,k)$ is the matrix

$$\begin{pmatrix} \binom{\lambda_1-k}{\mu_1-k} & \binom{\lambda_1-k}{\mu_2-1-k} & \binom{\lambda_1-k}{\mu_3-2-k} \\ \binom{\lambda_2-j}{\mu_1+1+i-j} & \binom{\lambda_2-j}{\mu_2+i-j} & \binom{\lambda_2-j}{\mu_3-1+i-j} \\ \binom{\lambda_3}{\mu_1+2+(k-i)+j} & \binom{\lambda_3}{\mu_2+1+(k-i)+j} & \binom{\lambda_3}{\mu_3+(k-i)+j} \end{pmatrix}.$$

Moreover, $\det M(i,j,k)$ counts 3−tuples of **signed non-intersecting lattice paths** from certain $A_1(i,j,k), A_2(i,j,k), A_3(i,j,k)$ to B_1, B_2, B_3, where B_i's depend only on λ.

An example using paths.



$\lambda = (3, 3, 3), \mu = (2, 2, 1),\ j + k = 2$; figures correspond, left-right, top-down to

$(0, 2, 0), (0, 1, 1), (1, 1, 1), (0, 0, 2), (1, 0, 2), (2, 0, 2).$

A positive formula for p=2,3.

Theorem. 1. If $p = 2$, all determinants are positive, and they count certain pairs of paths.

2. If $p = 3$, $c(\lambda, \mu)$ equals the number of certain *balanced* 3−tuples of non-intersecting paths. (Each balanced path counts with a plus sign.)

Conjecture. $c(\lambda, \mu) > 0$ for all p, λ, μ.

Computational evidence: true for $\lambda \subset (7^5), (5^6)$ etc.

If $\mu = (r)$ then $c(\lambda, (r))$ is the coefficient of t^r in

$$\prod_{i \geqslant 1} (1 + it)^{\lambda_i - \lambda_{i+1}}.$$

Further questions.

- Explain the positivity geometrically.

- State and prove a version of positivity of Gessel-Viennot determinants for families.

- Other homogeneous spaces ?

A quasi-generating function.

Theorem. $c(\lambda, \mu)$ equals the coefficient of

$$t_1^{\lambda_1} \cdots t_p^{\lambda_p} \cdot u_1^{\mu_1} \cdots u_p^{\mu_p}$$

in the expansion of the rational function

$$\Phi_p(\underline{t}, \underline{u}) = \frac{1}{(t_1^p \cdots t_p^1)(u_1^p \cdots u_p^1)} \cdot \prod_{1 \leqslant i < j \leqslant p} \frac{(t_i - t_j)(u_i - u_j)}{1 - 2t_j + t_i t_j}$$

as a Laurent polynomial in $\mathbb{Z}[[\underline{t}, \underline{u}]]$.