QUANTUM SCHUBERT POLYNOMIALS FOR THE G_2 FLAG MANIFOLD

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Abstract. We study some combinatorial objects related to the flag manifold X of Lie type G_2. Using the moment graph of X we calculate all the curve neighborhoods for Schubert classes. We use this calculation to investigate the ordinary and quantum cohomology rings of X. As an application, we obtain positive Schubert polynomials for the cohomology ring of X and we find quantum Schubert polynomials which represent Schubert classes in the quantum cohomology ring of X.

1. Introduction

One of the major theorems in algebra is the classification of complex semisimple Lie algebras. There are four classical infinite series (of type A_n, B_n, C_n, D_n) and five exceptional finite series (of types E_6, E_7, E_8, F_4, G_2). To each algebra, one can associate a group and to each group a certain geometric object called a flag manifold. In type A_n, the points of this flag manifold are sequences $V_1 \subset V_2 \subset .... \subset \mathbb{C}^n$ of vector spaces V_i of dimension i. The algebra of type G_2 is considered the simplest among the exceptional series, and we denote by X the flag manifold for type G_2. The study of flag manifolds has a long and rich history starting in 1950's, and it lies at the intersection of Algebraic Geometry, Combinatorics, Topology and Representation Theory.

One can associate a ring to the flag manifold X called the cohomology ring $\mathrm{H}^*(X)$. This ring has a distinguished basis given by Schubert classes σ_w, indexed by the elements w in the Weyl group W of type G_2; see §4 below. We recall that W is actually isomorphic to the dihedral group with 12 elements, although we will use a different realization of it, which is more suitable to our purposes. This ring is generated by Schubert classes $\sigma_{s_1}, \sigma_{s_2}$ for the simple reflections s_1, s_2 in W. Therefore, at least in principle, the full multiplication table in the ring is determined by a formula to multiply one Schubert class by another for either s_1 or s_2. This is called a *Chevalley formula*. There has been substantial amount of work to find Chevalley formulas for this ring, starting with Chevalley [5] in 1950's. This formula can be expressed combinatorially in terms of the root system and the Weyl group for type G_2. Alternatively, the cohomology ring has a "Borel" presentation $\mathrm{H}^*(X) = \mathbb{Q}[x_1, x_2]/I$ where I is the ideal generated by $x_1^2 - x_1x_2 + x_2^2, x_1^6$. A natural question is to find out what is the relation between this "algebraic" presentation and the "geometric"

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one which involves the Schubert basis. In other words, one needs to find a polynomials in $\mathbb{Q}[x_1, x_2]$ which represents a Schubert class σ_w under the isomorphism $\mathrm{H}^*(X) = \mathbb{Q}[x_1, x_2]$. This is called a *Schubert polynomial*. Such polynomials are not unique, as their class in $\mathbb{Q}[x_1, x_2]/I$ is unchanged if one changes a polynomial by elements in I. In §5 we use the Chevalley rule to find Schubert polynomials for σ_w. Some of our polynomials coincide with similar Schubert polynomials found by D. Anderson [1], via different methods. The polynomials we found are homogeneous and have *positive* coefficients. Given that the positivity of Schubert polynomial coefficients has geometric interpretations in type A_n (see the paper of A. Knutson and E. Miller [10]), this is a desirable property.

The current paper also focuses on a deformation of the ring above called the *quantum cohomology ring* $\mathrm{QH}^*(X)$. It is a deformation of $\mathrm{H}^*(X)$ with the addition of quantum parameters $q^{\mathbf{d}} = q_1^{d_1} q_2^{d_2}$ for degrees $\mathbf{d} = (d_1, d_2)$. If $\mathbf{d} = (0, 0)$, or equivalently $q_1 = q_2 = 0$, the product reduces to the corresponding calculation in $\mathrm{H}^*(X)$. More detail will be given in section 4. See [6] for more information about the background/history of this ring. Similar to the ring $\mathrm{H}^*(X)$, the quantum cohomology ring has a $\mathbb{Z}[q]$-basis consisting of Schubert classes σ_w (where $q = (q_1, q_2)$ are the quantum parameters), and it is generated as a ring by the classes σ_{s_1} and σ_{s_2} for the simple reflections s_1 and s_2.

The *quantum Chevalley formula* is a formula for the quantum multiplication $\sigma_w \star \sigma_{s_i}$ $(i = 1, 2)$. An explicit form of this formula, which uses combinatorics of the root system of Lie type G_2 was obtained by Fulton and Woodward [7]. In this paper we use the "curve neighborhoods" method to write down the explicit Chevalley formula. This alternative method, obtained by Buch and Mihalcea [4], involves an interesting graph associated to the flag manifold, called the *moment graph*. Its definition and properties are found in section 3. It also has the advantage that it leads to a conjectural Chevalley formula in a further deformation of the quantum cohomology ring, called *quantum K theory*. This will be addressed in a follow-up paper.

Our main application is to obtain a quantum version of the Schubert polynomials. More precisely, it is known [6, Prop. 11] that $\mathrm{QH}^*(X) = \mathbb{Q}[x_1, x_2, q_1, q_2]/\tilde{I}$ where $\tilde{I}$ a certain ideal which deforms I. Then, as in the classical case, we would like to find the polynomials in $\mathbb{Q}[x_1, x_2, q_1, q_2]$ which represent each Schubert class σ_w via the isomorphism $\mathrm{QH}^*(X) = \mathbb{Q}[x_1, x_2, q_1, q_2]/\tilde{I}$. These are called *quantum Schubert polynomials*. As before, these polynomials are not unique, but we can impose some natural conditions that they satisfy, such as the fact that they deform the ordinary Schubert polynomials, and that they are homogeneous with respect to a certain grading. To our knowledge, such polynomials have not been explicitly calculated in the literature. As a byproduct, we also use the quantum Chevalley formula to recover the ideal $\tilde{I}$ of quantum relations. This ideal has been in principle calculated by Kim [9] using different techniques, but the explicit polynomials generating this ideal do not seem to appear in the literature. Our results are stated in Theorem 5.2 below. This work is part of an undergraduate research project conducted under the guidance of Prof. Leonardo C. Mihalcea

2. Preliminaries: the root system and the Weyl group of type G_2

2.1. **The G_2 Root System.** Denote R the root system of type G_2. It consists of 12 roots, which are non-zero vectors in the hyperplane in $\mathbb{R}^3$ given by the equation

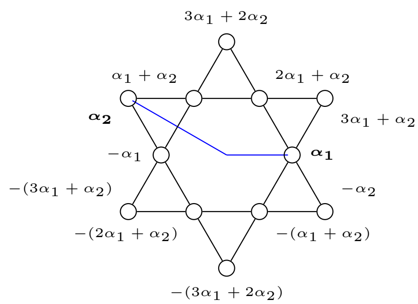

FIGURE 1. *The root system for G_2. Each node is a root. The blue lines represent the coordinate system using the Δ basis.*

$\xi_1 + \xi_2 + \xi_3 = 0$; our main reference is Bourbaki [3]. The roots are displayed in Table 1, in terms of the natural coordinates in $\mathbb{R}^3$. Each root α can be written uniquely as $\alpha = c_1\alpha_1 + c_2\alpha_2$ where α_1, α_2 are *simple roots* and $c_1 \cdot c_2 \geq 0$. A root is *positive* (*negative*) if both c_1, c_2 are non-negative (resp. non-positive). The set of simple roots is denoted $\Delta = \{\alpha_1, \alpha_2\}$, and they are $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = -2\epsilon_1 + \epsilon_2 + \epsilon_3$. For later purposes, we need to expand each root in terms of the simple roots. The full results for each root is shown in Table 1. The root vectors can be seen in Figure 1, in the Δ basis.

We also need the *dual* root system consisting of *coroots* $\alpha^\vee$. The coroot $\alpha^\vee$ of a root α is defined as $\alpha^\vee = \frac{2\alpha}{(\alpha,\alpha)}$ where (α, α) is the standard inner product in $\mathbb{R}^3$. Note that the coroots satisfy the properties $(\alpha^\vee)^\vee = \alpha$ and $(-\alpha)^\vee = -\alpha^\vee$. We denote the full set of coroots by $R^\vee$ and define the set $\Delta^\vee$ that holds the *simple coroots*: $\alpha_1^\vee$ and $\alpha_2^\vee$ for $R^\vee$. Table 1 shows the values for each of the coroots.

Natural Coordinates E Basis $(\epsilon_1, \epsilon_2, \epsilon_3)$	$\pm$	**Simple Roots Basis** (α_1, α_2)	**Coroot $\alpha^\vee$** $\alpha^\vee = \lambda\alpha_1 + \mu\alpha_2$
$\epsilon_1 - \epsilon_2$	$+$	α_1	$\alpha_1^\vee = \alpha_1$
$\epsilon_3 - \epsilon_1$	$+$	$\alpha_1 + \alpha_2$	$(\alpha_1 + \alpha_2)^\vee = \alpha_1 + \alpha_2$
$\epsilon_3 - \epsilon_2$	$+$	$2\alpha_1 + \alpha_2$	$(2\alpha_1 + \alpha_2)^\vee = 2\alpha_1 + \alpha_2$
$\epsilon_2 + \epsilon_3 - 2\epsilon_1$	$+$	α_2	$\alpha_2^\vee = \frac{1}{3}\alpha_2$
$\epsilon_1 + \epsilon_3 - 2\epsilon_2$	$+$	$3\alpha_1 + \alpha_2$	$(3\alpha_1 + \alpha_2)^\vee = \alpha_1 + \frac{1}{3}\alpha_2$
$-\epsilon_1 - \epsilon_2 + 2\epsilon_3$	$+$	$3\alpha_1 + 2\alpha_2$	$(3\alpha_1 + 2\alpha_2)^\vee = \alpha_1 + \frac{2}{3}\alpha_2$
$-(\epsilon_1 - \epsilon_2)$	$-$	$-\alpha_1$	$(-\alpha_1)^\vee = -\alpha_1$
$-(\epsilon_3 - \epsilon_1)$	$-$	$-(\alpha_1 + \alpha_2)$	$(-\alpha_1 - \alpha_2)^\vee = -\alpha_1 - \alpha_2$
$-(\epsilon_3 - \epsilon_2)$	$-$	$-(2\alpha_1 + \alpha_2)$	$(-2\alpha_1 - \alpha_2)^\vee = -2\alpha_1 - \alpha_2$
$-(\epsilon_2 + \epsilon_3 - 2\epsilon_1)$	$-$	$-\alpha_2$	$(-\alpha_2)^\vee = -\frac{1}{3}\alpha_2$
$-(\epsilon_1 + \epsilon_3 - 2\epsilon_2)$	$-$	$-(3\alpha_1 + \alpha_2)$	$(-3\alpha_1 - \alpha_2)^\vee = -\alpha_1 - \frac{1}{3}\alpha_2$
$-(-\epsilon_1 - \epsilon_2 + 2\epsilon_3)$	$-$	$-(3\alpha_1 + 2\alpha_2)$	$(-3\alpha_1 - 2\alpha_2)^\vee = -\alpha_1 - \frac{2}{3}\alpha_2$

TABLE 1. *Root System of type G_2. For each root, we give its sign, the root in terms of Δ basis, and the corresponding coroot.*

2.2. **The Weyl Group of G_2.**

The *Weyl group of G_2*, denoted W, is the group generated by reflections s_α, where $\alpha \in R$. Let $s_i := s_{\alpha_i}$. Geometrically, s_α is the reflection across the line perpendicular to the root α. For example, the reflection s_1 (corresponding to s_{α_1}) is the reflection across the line perpendicular to the α_1 axis (see Figure 2). As Figure 2 shows, for any root α, $s_\alpha = s_{-\alpha}$. Therefore only six unique reflections exist for the G_2 root system.

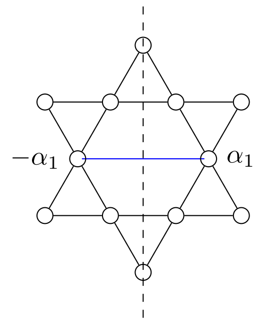

FIGURE 2. *The reflection s_{α_1} (dashed line) which is perpendicular to the α_1 axis (blue line).*

It is known (cf. e.g. [8]) that W has the presentation

$$W =< s_1, s_2 : {s_1}^2 = {s_2}^2 = 1, (s_1 s_2)^6 = 1 > .$$

From this it follows easily that W is isomorphic to the dihedral group with 12 elements. In order to determine the reflections in W, we need the following definitions.

Definition 2.1. *Consider $w \in W$. A **reduced expression** for w is an expression involving products of s_1 and s_2 in as short a way as possible (via the relations in the presentation). If $w \in W$ where w is a reduced expression, the **length** of w, $\ell(w)$, is the number of simple reflections (s_1 and s_2) that show up in the reduced expression.*

Example 2.2. *Consider $w = s_1 s_1 s_1 s_2 s_1 s_2$. From the presentation of W we know that ${s_1}^2 = s_1 s_1 = 1$ and so this expression is not reduced. However, $(s_1 s_1) s_1 s_2 s_1 s_2 = (1) s_1 s_2 s_1 s_2 = s_1 s_2 s_1 s_2$. The latter is a reduced expression and $\ell(w) = 4$.*

The 12 reduced expressions of the elements in W are:

$$W = \{1,\ s_1,\ s_2,\ s_1s_2,\ s_2s_1,\ s_1s_2s_1,\ s_2s_1s_2,\ s_1s_2s_1s_2,$$
$$s_1s_2s_1s_2,\ s_2s_1s_2s_1,\ s_1s_2s_1s_2s_1,\ s_2s_1s_2s_1s_2,\ s_1s_2s_1s_2s_1s_2\}$$

We denote by w_0 the longest element $s_1s_2s_1s_2s_1s_2$. Notice that among the twelve elements only six of them are the *root reflections* from the root system of G_2. Because any reflection has order 2, it is easy to check that the root reflections correspond to the reduced expressions of odd length.

Since the reflections s_1 and s_2 generate W, every reflection in the G_2 root system s_α can be expressed as a reduced expression product of s_1's and s_2's. Consider the action of W on the root system R given by the natural action of reflections on vectors in $\mathbb{R}^3$. Explicitly, this action is given by $s_\alpha \cdot \beta = s_\alpha(\beta) = \beta - (\beta, \alpha^\vee)\alpha$ (see [8, Pag. 43]). The following lemma in proved in *loc. cit.*

Lemma 2.3. *Let $w \in W$ and $\alpha \in R$. Then $w s_\alpha w^{-1} = s_{w\cdot\alpha}$.*

Example 2.4. *Consider $w = s_1 s_2 s_1$. We want to find reflection s_α that corresponds to w. By Lemma 2.3, $s_1 s_2 s_1 = s_{s_1(\alpha_2)}$ where s_1 is its own inverse and the action is*

$$s_1(\alpha_2) = \alpha_2 - (\alpha_2, {\alpha_1}^\vee)\alpha_1 = \alpha_2 - \left(\alpha_2, \frac{2\alpha_1}{(\alpha_1, \alpha_1)}\right)\alpha_1$$

We know $(\alpha_1, \alpha_1) = 2$ *(see Table 1) so*

$$\alpha_2 - \left(\alpha_2, \frac{2\alpha_1}{(\alpha_1,\alpha_1)}\right)\alpha_1 = \alpha_2 - \left(\alpha_2, \frac{2\alpha_1}{2}\right)\alpha_1 = \alpha_2 - (\alpha_2, \alpha_1)\,\alpha_1$$
$$= \alpha_2 - (-3)\alpha_1.$$

Thus $s_1(\alpha_2) = 3\alpha_1 + \alpha_2$. *The reflection* $s_1s_2s_1$ *is the reflection* $s_{3\alpha_1+\alpha_2}$.

Table 2 shows the reflection across the line perpendicular to each root. Notice that roots α and $-\alpha$ have the same reflection and all reflections listed have *odd* length.

Root (in Δ basis)	Reflection ($w \in W$)
$\pm\alpha_1$	s_1
$\pm\alpha_2$	s_2
$\pm(3\alpha_1 + \alpha_2)$	$s_1s_2s_1$
$\pm(\alpha_1 + \alpha_2)$	$s_2s_1s_2$
$\pm(2\alpha_1 + \alpha_2)$	$s_1s_2s_1s_2s_1$
$\pm(3\alpha_1 + 2\alpha_2)$	$s_2s_1s_2s_1s_2$

TABLE 2. *The root reflection corresponding to each root in* G_2.

3. The moment graph and curve neighborhoods

3.1. **Finding the Moment Graph.** Using the properties of the elements in the Weyl group for G_2, it is possible to define the following graph.

Definition 3.1. *The* ***moment graph*** *is an oriented graph that consists of a pair* (V, E) *where* V *is the set of vertices and* E *is the set of edges. To each Weyl group element* $v \in W$ *there corresponds a vertex* $v \in V$ *in this graph. For* $x, y \in V$, *an edge exists from* x *to* y, *denoted by*

$$x \xrightarrow{\alpha^\vee} y$$

if there exists a reflection s_α *such that* $y = xs_\alpha$ *and* $\ell(y) > \ell(x)$.

Definition 3.2. *A* ***degree d*** *is a non-negative combination* $d_1\alpha_1^\vee + d_2\alpha_2^\vee$ *of simple coroots. We will denote* $\boldsymbol{d} = (d_1, d_2)$.

Since any coroot $\alpha^\vee$ is a linear combination in terms of $\alpha_1^\vee$ and $\alpha_2^\vee$, it determines a degree. These degrees are given in Table 3 below.

Coroot $\alpha^\vee = d_1{\alpha_1}^\vee + d_2{\alpha_2}^\vee$	degree $\boldsymbol{d}$ (d_1, d_2)
${\alpha_1}^\vee$	$(1, 0)$
${\alpha_2}^\vee$	$(0, 1)$
$(3\alpha_1 + \alpha_2)^\vee$	$(1, 1)$
$(\alpha_1 + \alpha_2)^\vee$	$(1, 3)$
$(2\alpha_1 + \alpha_2)^\vee$	$(2, 3)$
$(3\alpha_1 + 2\alpha_2)^\vee$	$(1, 2)$

TABLE 3. *The degree for each coroot in the moment graph.*

Example 3.3. *An edge exists from s_1 to s_2s_1. This is so because*

$$\ell(s_2s_1) > \ell(s_1) \qquad \textit{and} \qquad s_2s_1 = s_1s_\alpha, \textit{ where } s_\alpha = s_1s_2s_1.$$

Example 2.4 shows $s_1s_2s_1 = s_{3\alpha_1+\alpha_2}$. The edge corresponding to these two edges has degree $(3\alpha_1+\alpha_2)^\vee$, i.e., $s_1 \xrightarrow{(3\alpha_1+\alpha_2)^\vee} s_2s_1$. Notice that $(3\alpha_1+\alpha_2)^\vee = 1\alpha_1^\vee + 1\alpha_2^\vee$ so $\boldsymbol{d} = (1,1)$. The edge from s_1 to s_2s_1 can be represented by the degree $(1,1)$.

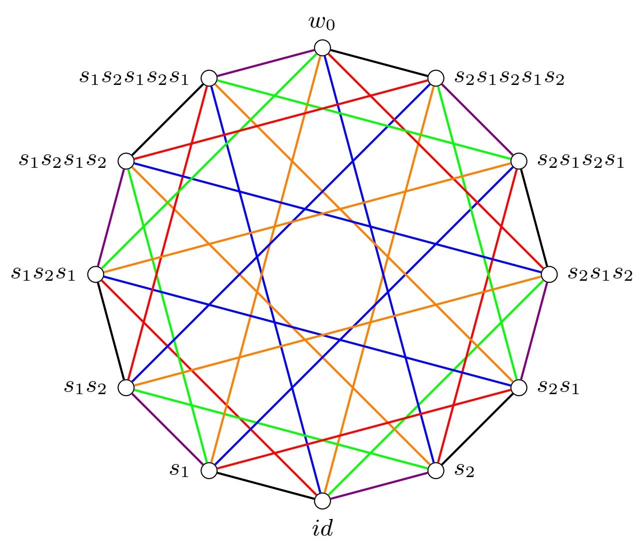

FIGURE 3. *The moment graph for G_2. The color code for the degrees is the following:*
Black-(1,0), Violet-(0,1), Red-(1,1), Green-(1,3), Blue-(2,3), Orange-(1,2).

We depict the moment graph as oriented upward, as in Figure 3 above. To help read the moment graph, a color code has been set up to represent the different edges. We review some of its relevant properties:

- The vertices correspond to the 12 Weyl group elements.
- The edges represent the root reflections associated to the G_2 root system. There are six different types of edges (different degree values) because there are exactly six reflections in the G_2 root system. Note that edges exist between Weyl group elements if the difference between lengths is odd.
- The bottom vertex is the element with the smallest length (id where $\ell(id) = 0$). The next "row" of vertices have length 1 (s_1 and s_2). The length of these elements increases by one as you travel up the graph. The top vertex is the element with the largest length, w_0, where $\ell(w_0) = 6$.
- For any vertex, there are six edges connected to it, corresponding to the 6 different coroots in $R^\vee$.
- For any $w_1, w_2 \in W$ where $\ell(w_1) = \ell(w_2)$, both w_1 and w_2 will have edges connecting to the same six vertices.

3.2. **Curve Neighborhoods.** In Section 3.1 we defined the degree $\mathbf{d}$ to help simplify the moment graph for use in future calculations. The importance of the moment graph can be realized with the following concept defined by A. Buch and L. Mihalcea [4]:

Definition 3.4. *Fix $\boldsymbol{d} = (d_1, d_2)$ a degree and $u \in W$ an element of the Weyl group, The* ***curve neighborhood****, $\Gamma_d(u)$ is a subset of W which consists of the maximal elements in the moment graph which can be reached from u with a path of total degree $\leq \boldsymbol{d}$.*

Example 3.5. *Consider $w = id$ and $\boldsymbol{d} = (1, 1)$. We want to determine the "highest" path (starting at the identity) where the total degree traveled is at most $(1, 1)$. By inspecting the moment graph, there are three initial paths starting from id:*

- *path $\mathbf{d} = (1, 0)$ which goes from id to s_1. Upon reaching s_1, one is not allowed to travel more than $\boldsymbol{d}' = (0, 1)$ upwards. Further inspection of the moment graph shows a path exists with degree $(0, 1)$ from s_1 to s_1s_2. We now have traveled a total degree of (1,1). Thus we are done and s_1s_2 is the largest element on this path.*
- *path $\boldsymbol{d} = (0, 1)$ which goes from id to s_2. Upon reaching s_2, one is not allowed to travel more than $\boldsymbol{d}' = (1, 0)$ upwards. Further inspection gives a path with degree $(1, 0)$ from s_2 to s_2s_1. We now have traveled a total degree of $(1, 1)$. Thus we are done and s_2s_1 is the largest element on this path.*
- *path $\boldsymbol{d} = (1, 1)$. which goes from id to $s_1s_2s_1$. Since we traveled a total degree of $(1, 1)$, we are done, and $s_1s_2s_1$ is the largest element on this path.*

We now take the maximal element that can be reached from id with degree $(1, 1)$. The largest of the three elements above is $s_1s_2s_1$, thus $\Gamma_{(1,1)}(id) = \{s_1s_2s_1\}$.

It is clear that for any $w \in W$ there exists some degree (a, b) where $\Gamma_{(a,b)}(w) = w_0$. Then for any larger degree (a', b') where $a' \geq a$ and $b' \geq b$, $\Gamma_{(a',b')}(w) = w_0$. Table 7 in A.1 shows the curve neighborhoods for every element of the Weyl group. For all the examples given, the curve neighborhood for some degree $\mathbf{d}$ at $u \in W$ is always unique, a fact which was initially proved in [4] for all Lie types.

4. Quantum Cohomology Ring for flag manifold X

Recall that X denotes the flag manifold of type G_2. The cohomology ring, denoted by $\mathrm{H}^*(X)$, consists of elements that can each be written uniquely as finite sums $\sum_{w \in W} a_w \sigma_w$ where $a_w \in \mathbb{Z}$ and σ_w is a (geometrically defined) *Schubert class*. Addition in this ring is given by:

$$\sum_{w \in W} a_w \sigma_w + \sum_{w \in W} b_w \sigma_w = \sum_{w \in W} (a_w + b_w) \sigma_w.$$

The quantum cohomology ring $\mathrm{QH}^*(X)$ is a deformation of $\mathrm{H}^*(X)$ by adding quantum parameters, $q^{\mathbf{d}} = q_1^{d_1} q_2^{d_2}$ for degrees $\mathbf{d} = (d_1, d_2)$. If $\mathbf{d} = (0, 0)$ for any calculation in $\mathrm{QH}^*(X)$, we reduce down to the corresponding calculation in $\mathrm{H}^*(X)$. Similarly to $\mathrm{H}^*(X)$, the elements of $\mathrm{QH}^*(X)$ can each be written uniquely as finite sums $\sum_{w \in W} a_w(\mathbf{d}) q^{\mathbf{d}} \sigma_w$ where $a_w(\mathbf{d}) \in \mathbb{Z}$. The addition in this ring is also straightforward:

$$\sum_{w \in W} a_w(\mathbf{d}) q^{\mathbf{d}} \sigma_w + \sum_{w \in W} b_w(\mathbf{d}) q^{\mathbf{d}} \sigma_w = \sum_{w \in W} (a_w(\mathbf{d}) + b_w(\mathbf{d})) q^{\mathbf{d}} \sigma_w.$$

The multiplication in this ring is given by certain integers $c_{u,v}^{w,\mathbf{d}}$ called the *Gromov-Witten invariants*:

$$\sigma_u \star \sigma_v = \sum_{w,\mathbf{d}} c_{u,v}^{w,\mathbf{d}} q^{\mathbf{d}} \sigma_w$$

where the sum is over $w \in W$ and degrees $\mathbf{d}$ which have non-negative components. The (quantum) cohomology ring has two generators, namely σ_{s_1} and σ_{s_2}, corresponding to the simple reflections $s_1, s_2 \in W$. As a result every element is a sum of monomials in σ_{s_i}'s and the quantum multiplication $\sigma_u \star \sigma_{s_i}$ by generators σ_{s_i} determines the entire ring multiplication. The formula for $\sigma_w \star \sigma_{s_i}$, the *(quantum) Chevalley rule*, is illustrated in Section 4.1. We list below few properties that will help to understand this ring and we refer e.g. to [6] for full details.

(1) the multiplication of quantum parameters is given by: $q_i^{d_s} q_i^{d'_s} = q_i^{d_s + d'_s}$.
(2) the quantum multiplication $\star$ is associative, commutative and it has unit $1 = \sigma_{id}$.
(3) the quantum multiplication is graded by imposing $\deg(\sigma_w) = \ell(w)$ and for $\mathbf{d} = (d_1, d_2)$, $\deg q^{\mathbf{d}} = 2(d_1 + d_2)$. This implies that $\deg(\sigma_u \star \sigma_v) = \deg(\sigma_u) + \deg(\sigma_v)$ and that $c_{u,v}^{w,\mathbf{d}} = 0$ unless $\ell(u) + \ell(v) = \ell(w) + \deg q^{\mathbf{d}}$.
(4) if we impose the substitution $q_1 = q_2 = 0$ in $\sigma_u \star \sigma_v$ then we obtain the multiplication $\sigma_u \cdot \sigma_v$ in the ordinary cohomology ring $\mathrm{H}^*(X)$.

4.1. **Quantum Chevalley Rule Via Curve Neighborhoods.** Recall that each coroot $\alpha^\vee$ can be written as a linear combination $\alpha^\vee = d_1\alpha_1^\vee + d_2\alpha_2^\vee$ where $\alpha_1^\vee, \alpha_2^\vee$ are the simple coroots and $d_1, d_2 \in \mathbb{Z}$. It follows that each $\alpha^\vee$ can be identified with the unique degree $\mathbf{d} = (d_1, d_2)$. Let $\mathbf{d}[\mathrm{i}]$ denote the i-th component of the degree $\mathbf{d}$ in the decomposition $\mathbf{d} = \mathbf{d}[1]\alpha_1^\vee + \mathbf{d}[2]\alpha_2^\vee$. In other words, $\mathbf{d}[\mathrm{i}] = d_i$. Note that $\alpha^\vee[i]$ means the same thing as $\mathbf{d}[\mathrm{i}]$.

The classical *Chevalley rule* (cf. [5], see also [7]) is a formula for the products $\sigma_u \cdot \sigma_{s_i} \in \mathrm{H}^*(X)$:

$$\sigma_u \cdot \sigma_{s_i} = \sum_{\alpha} (\alpha^\vee[i]) \sigma_{us_\alpha} \tag{1}$$

where the sum is over roots positive α such that $\ell(us_\alpha) = \ell(u) + 1$.

The *quantum Chevalley formula* for $\sigma_u \star \sigma_{s_i} = \sum_{w,\mathbf{d}} c_{u,s_i}^{w,\mathbf{d}}\, q^{\mathbf{d}}\, \sigma_w$ was first proved by Fulton and Woodward [7]. See Theorem 4.3 below. We follow here an approach based on curve neighborhoods, recently proved by Buch and Mihalcea [4]. If $\mathbf{d} = (0,0)$ then the coefficients $c_{u,s_i}^{w,\mathbf{d}}$ are those from identity (1) above. If $\mathbf{d} \neq (0,0)$ then the quantum coefficient $c_{u,s_i}^{w,\mathbf{d}}$ can be calculated as follows. First, let $w[\mathbf{d}] \in W$ be the curve neighborhood $\Gamma_{\mathbf{d}}(w)$. Then

$$c_{u,s_i}^{w,\mathbf{d}} = \mathbf{d}[i] \cdot \delta_{u,w[\mathbf{d}]} \tag{2}$$

where δ_{v_1,v_2} is the Kronecker symbol and w satisfies $\ell(w) + \deg q^{\mathbf{d}} = \ell(u) + 1$.

Remark 4.1. *Although it is not clear from definition, it turns out that if $\boldsymbol{d}[i] \neq 0$ then $u = w[\boldsymbol{d}]$ only if $\boldsymbol{d} = \alpha^\vee$ for some α such that $\ell(s_\alpha) = \deg q^{\alpha^\vee} - 1$. This recovers the original quantum Chevalley rule from [7].*

Example 4.2. *Consider $\sigma_{s_1} \star \sigma_{s_1}$.*

- *Assume $\boldsymbol{d} = (0,0)$. We need to determine roots α such that $\ell(s_1 s_\alpha) = \ell(s_1) + 1 = 2$. The only possible Weyl group elements to represent s_α are s_2 and $s_1 s_2 s_1$. If $s_\alpha = s_2$ then $\alpha = \alpha_2$. This implies $\alpha^\vee = (0,1)$ so $\alpha^\vee[1] = 0$.*

If $s_\alpha = s_1s_2s_1$ then $\alpha = 3\alpha_1+\alpha_2$. This implies $\alpha^\vee = (1,1)$ so $\alpha^\vee[1] = 1$. Thus

$$\sum_\alpha (\alpha^\vee[i])\sigma_{us_\alpha} = 0\cdot\sigma_{s_1s_2} + 1\cdot\sigma_{s_1s_1s_2s_1} = \sigma_{s_2s_1}$$

- *Assume $\boldsymbol{d} \neq (0,0)$. We need to determine $w \in W$ such that $w[\boldsymbol{d}] = \Gamma_{\boldsymbol{d}}(w) = s_1$. According to the curve neighborhood results table in A.1, the only possible $w \in W$ are id and s_1. For both elements, the possible nondegrees are $(N,0)$ where $N \in \mathbb{N}$. Note that we also need to choose w and $\boldsymbol{d}$ such that $\ell(w) + deg\, q^{\boldsymbol{d}} = \ell(s_1)+1 = 2$. Since $\deg q^{\boldsymbol{d}}$ is never odd, $\ell(w)$ must be even. This eliminates s_1. As for id, $\ell(id) = 0$ so then $\deg q^{\boldsymbol{d}} = 2$ where $\boldsymbol{d} = (N,0)$. This implies $N=1$. Therefore $c_{s_1,s_1}^{id,(1,0)} = \boldsymbol{d}[1]\cdot\delta_{s_1,s_1} = 1\cdot 1 = 1$ and this represents the only nonzero quantum term. Thus for $\boldsymbol{d} \neq (0,0)$,*

$$\sum_{w\in W,\mathbf{d}} c_{u,s_i}^{w,\boldsymbol{d}}\, q^{\boldsymbol{d}}\sigma_w = 1\cdot q^{(1,0)}\cdot\sigma_{id} = 1\cdot q_1\cdot 1 = q_1.$$

Combining the classical (i.e. from $\mathrm{H}^*(X)$) and pure quantum terms gives us $\sigma_{s_1}\star\sigma_{s_1} = \sigma_{s_2s_1}+q_1$.

Table 4 shows the results of our quantum Chevalley computations.

$\boldsymbol{w}$	$\boldsymbol{\sigma_w \star \sigma_{s_1}}$	$\boldsymbol{\sigma_w \star \sigma_{s_2}}$
$\boldsymbol{s_1}$	$\sigma_{s_2s_1}+q_1$	$\sigma_{s_1s_2}+\sigma_{s_2s_1}$
$\boldsymbol{s_2}$	$\sigma_{s_1s_2}+\sigma_{s_2s_1}$	$3\sigma_{s_1s_2}+q_2$
$\boldsymbol{s_1s_2}$	$\sigma_{s_1s_2s_1}+\sigma_{s_2s_1s_2}$	$2\sigma_{s_2s_1s_2}+q_2\sigma_{s_1}$
$\boldsymbol{s_2s_1}$	$2\sigma_{s_1s_2s_1}+q_1\sigma_{s_2}$	$3\sigma_{s_1s_2s_1}+\sigma_{s_2s_1s_2}$
$\boldsymbol{s_1s_2s_1}$	$\sigma_{s_2s_1s_2s_1}+q_1\sigma_{s_1s_2}+q_1q_2$	$\sigma_{s_1s_2s_1s_2}+2\sigma_{s_2s_1s_2s_1}+q_1q_2$
$\boldsymbol{s_2s_1s_2}$	$\sigma_{s_2s_1s_2s_1}+2\sigma_{s_1s_2s_1s_2}$	$3\sigma_{s_1s_2s_1s_2}+q_2\sigma_{s_2s_1}$
$\boldsymbol{s_1s_2s_1s_2}$	$\sigma_{s_1s_2s_1s_2s_1}+\sigma_{s_2s_1s_2s_1s_2}$	$\sigma_{s_2s_1s_2s_1s_2}+q_2\sigma_{s_1s_2s_1}$
$\boldsymbol{s_2s_1s_2s_1}$	$\sigma_{s_1s_2s_1s_2s_1}+q_1\sigma_{s_2s_1s_2}+q_1q_2\sigma_{s_2}$	$\sigma_{s_2s_1s_2s_1s_2}+3\sigma_{s_1s_2s_1s_2s_1}+q_1q_2\sigma_{s_2}$
$\boldsymbol{s_1s_2s_1s_2s_1}$	$q_1\sigma_{s_1s_2s_1s_2}+q_1q_2\sigma_{s_1s_2}$	$\sigma_{w_0}+q_1q_2\sigma_{s_1s_2}$
$\boldsymbol{s_2s_1s_2s_1s_2}$	$\sigma_{w_0}+q_1{q_2}^2$	$q_2\sigma_{s_2s_1s_2s_1}+2q_1{q_2}^2$
$\boldsymbol{w_0=(s_1s_2)^3}$	$q_1\sigma_{s_2s_1s_2s_1s_2}+q_1q_2\sigma_{s_2s_1s_2}+q_1{q_2}^2\sigma_{s_1}$	$q_2\sigma_{s_1s_2s_1s_2s_1}+q_1q_2\sigma_{s_2s_1s_2}+2q_1{q_2}^2\sigma_{s_1}$

TABLE 4. *Quantum Chevalley Table.*

Theorem 4.3 (**The Quantum Chevalley Rule**, [4,7])**.** *The following holds in $\mathrm{QH}^*(X)$:*

$$\sigma_u\star\sigma_{s_i} = \sum_\alpha(\alpha^\vee[i])\sigma_{us_\alpha} + \sum_\beta(\beta^\vee[i])q^{\beta^\vee}\sigma_{us_\beta} \tag{3}$$

The first sum is over positive roots α such that $\ell(us_\alpha) = \ell(u)+1$ and the second sum is over positive roots β such that $\ell(us_\beta) = \ell(u)+1-deg(q^{\beta^\vee})$.

5. Quantum Schubert Polynomials

We know that $\mathrm{QH}^*(X)$ is generated as a $\mathbb{Q}[q] = \mathbb{Q}[q_1,q_2]$-algebra by the classes σ_{s_1} and σ_{s_2}. (This means that every element in $\mathrm{QH}^*(X)$ can be written as a sum of monomials in σ_{s_i}'s with coefficients in $\mathbb{Q}[q]$.) Then there exists a *surjective* homomorphism of $\mathbb{Q}[q]$-algebras $\Psi : \mathbb{Q}[x_1,x_2;q_1,q_2] \to \mathrm{QH}^*(X)$ sending

$$\Psi(q_i) = q_i;\quad \Psi(x_1) = \sigma_{s_1};\quad \Psi(x_1+x_2) = \sigma_{s_2}.$$

Note that for any $P, P' \in \mathbb{Q}[x_1, x_2, q_1, q_2]$ we have $\Psi(P \cdot P') = \Psi(P) \star \Psi(P')$. We call Ψ the *quantization map.* Let $\tilde{I}$ be the kernel of this homomorphism. By the first isomorphism theorem we have an isomorphism

$$\overline{\Psi} : \mathbb{Q}[x_1, x_2, q_1, q_2]/\tilde{I} \to \mathrm{QH}^*(X)$$

and this gives the presentation of the quantum cohomology ring. A *quantum Schubert polynomial* for the Schubert class σ_w is any polynomial $P_w \in \mathbb{Q}[x_1, x_2, q_1, q_2]$ such that the image of P_w under Ψ gives the class σ_w. Equivalently $\overline{\Psi}(P_w + \tilde{I}) = \sigma_w$.

To find a quantum Schubert polynomial P_w, we proceed by induction on $\ell(w)$, using the quantum Chevalley formula from Table 4, and starting from the "initial conditions" $P_{s_1} = x_1$ and $P_{s_2} = x_1 + x_2$. To obtain the corresponding classical Schubert polynomials for cohomology, set $q_1 = q_2 = 0$.

Example 5.1. *In order to calculate $P_{s_2 s_1}$ we use the identity $\sigma_{s_1} \star \sigma_{s_1} = \sigma_{s_2 s_1} + q_1$ (taken from Table 4). Using that Ψ is an algebra homomorphism, we know that $\Psi(x_1^2) = \Psi(x_1) \star \Psi(x_1) = \sigma_{s_1} \star \sigma_{s_1}$ and $\Psi(q_1) = q_1$. Since $\Psi(x_1^2 - q_1) = \Psi(x_1^2) - \Psi(q_1)$ then it follows that $\Psi(x_1^2 - q_1) = \sigma_{s_2 s_1}$. This shows that $x_1^2 - q_1$ is a quantum Schubert polynomial for $\sigma_{s_2 s_1}$. The corresponding ordinary Schubert polynomial is x_1^2, obtained by making $q_1 = 0$.*

Computations of ordinary Schubert polynomials were done for the ordinary cohomology ring $\mathrm{H}^*(X)$ of the G_2 flag manifold in a paper by Anderson [1]. A classical result of Borel [2] shows that $\mathrm{H}^*(X) = \mathbb{Q}[x_1, x_2]/I$, where $I = < x_1^2 - x_1 x_2 + x_2^2, x_1^6 >$. (This can also be deduced from the classical Chevalley formula). Anderson used this presentation and a different method to obtain different Schubert polynomials, but our answers and his must be equal modulo the ideal I. The classical Schubert polynomials we found are shown alongside Anderson's in Table 5. In order to check if our results are equal we verified that the difference between our resulting classical polynomials was a multiple of one of the elements of the ideal.

σ_{s_α}	Our Calculation	D. Anderson's Calculation [1]
w_0	$\frac{1}{2}(x_1^6 + x_1^5 x_2)$	$\frac{1}{2} x_1^5 x_2$
$s_1 s_2 s_1 s_2 s_1$	$\frac{1}{2} x_1^5$	$\frac{1}{2} x_1^5$
$s_2 s_1 s_2 s_1 s_2$	$\frac{1}{6}(x_1 + x_2)^3 x_1 x_2$	$\frac{1}{2}(x_1^3 + x_2 x_1^2 + x_2^2 x_1 + x_2^3) x_1 x_2$
$s_2 s_1 s_2 s_1$	$\frac{1}{2} x_1^4$	$\frac{1}{2}(4x_1^2 - 3x_1 x_2 + 3x_2^2) x_1^2$
$s_1 s_2 s_1 s_2$	$\frac{1}{6}(x_1 + x_2)^2 x_1 x_2$	$\frac{1}{2}(x_1^4 + x_1^3 x_2 + x_1^2 x_2^2 + x_1 x_2^3 + x_2^4)$
$s_1 s_2 s_1$	$\frac{1}{2} x_1^3$	$\frac{1}{2}(4x_1^2 - 3x_1 x_1 + 3x_2^2) x_1$
$s_2 s_1 s_2$	$\frac{1}{2}(x_1 + x_2) x_1 x_2$	$2x_1^3 + \frac{1}{2} x_1^2 x_2 + \frac{1}{2} x_1 x_2^2 + 2x_2^3$
$s_2 s_1$	x_1^2	$3x_1^2 - 2x_1 x_2 + 2x_2^2$
$s_1 s_2$	$x_1 x_2$	$2x_1^2 - x_1 x_2 + 2x_2^2$
s_2	$x_1 + x_2$	$x_1 + x_2$
s_1	x_1	x_1
id	1	1

TABLE 5. *Classical Schubert polynomials.*

We used our quantum Schubert polynomial results, found in Table 6 below to compute the ideal $\tilde{I}$ of the quantum cohomology ring $\mathrm{QH}^*(X)$. This ideal is a deformation of the ideal I of $\mathrm{H}^*(X)$. As an example, we will derive the degree 2 relation in $\tilde{I}$. From the quantum Chevalley table on page 9 we know the following identities:

- $\sigma_{s_1} \star \sigma_{s_1} = \sigma_{s_2 s_1} + q_1$,
- $\sigma_{s_1} \star \sigma_{s_2} = \sigma_{s_1 s_2} + \sigma_{s_2 s_1}$, and
- $\sigma_{s_2} \star \sigma_{s_2} = 3\sigma_{s_1 s_2} + q_2$.

These three equalities can be combined to obtain that:

$$3(\sigma_{s_1} \star \sigma_{s_1}) + (\sigma_{s_2} \star \sigma_{s_2}) = 3(\sigma_{s_1} \star \sigma_{s_2}) + 3q_1 + q_2.$$

Now apply the transformation under $\overline{\Psi}$ to get

$$3(x_1 \cdot x_1) + ((x_1 + x_2) \cdot (x_2 + x_2)) \equiv \big(3(x_1(x_1 + x_2)) + 3q_1 + q_2\big) \; + \; \tilde{I}$$

which is

$$3x_1^2 + x_1^2 + 2x_1x_2 + x_2^2 \equiv \big(3x_1^2 + 3x_1x_2 + 3q_1 + q_2\big) \; + \; \tilde{I}.$$

Their difference belongs to $\tilde{I} = \ker \Psi$ so (after simplification) we get $x_1^2 - x_1x_2 + x_2^2 - (3q_1 + q_2) \in \tilde{I}$. This is the degree 2 relation in $\tilde{I}$. Notice that this is clearly a deformation of the ideal term $x_1^2 - x_1x_2 + x^2$ in I. To get the degree 6 relation, one does a similar manipulation but using the higher degree terms in the quantum Chevalley table on page 9. The following is the main result of this paper.

Theorem 5.2. *The quantum cohomology ring of the flag manifold of type G_2 is*

$$\mathrm{QH}^*(X) = \mathbb{Q}[q_1, q_2, x_1, x_2]/\langle R_2, R_6\rangle$$

where $R_2 = x_1^2 - x_1x_2 + x_2^2 - (3q_1 + q_2)$ and

$$R_6 := x_1^6 + q_1\left(-2x_1^4 - \frac{13}{3}x_2^3x_2 - \frac{5}{3}x_1^2x_2^2 - \frac{1}{3}x_1x_2^3\right) + q_1^2\left(-\frac{10}{3}x_1^2 - \frac{5}{3}x_1x_2 - \frac{1}{3}x_2^2\right) + q_1q_2\left(-2x_1^2 - \frac{11}{3}x_1x_2\right) - \frac{8}{3}q_1^2q_2.$$

Under this presentation, the corresponding quantum Schubert polynomials are given in the following table:

$\boldsymbol{w_0 = (s_1s_2)^3}$	$\frac{1}{2}\left[x_1^6 + x_1^5x_2\right] + \frac{1}{2}\left[-2x_1^4 - 6x_1^3x_2 - 5x_1^2x_2^2 - x_1x_2^3\right]q_1 +$ $+\frac{1}{2}\left[-3x_1^2 - 7x_1x_2 - 2x_2^2\right]q_1q_2 + \frac{1}{2}\left[-3x_1^2 - 4x_1x_2 - x_2^2\right]q_1^2 - q_1^2q_2$
$\boldsymbol{s_2s_1s_2s_1s_2}$	$\frac{1}{6}\left[(x_1 + x_2)^3x_1x_2\right] + \frac{1}{6}\left[(x_1 + x_2)^3q_1 + (-6x_1^3 - 4{x_1}^2x_2 - x_1x_2^2)q_2 + (8x_1 + 5x_2)q_1q_2\right]$
$\boldsymbol{s_1s_2s_1s_2s_1}$	$\frac{1}{2}x_1^5 + \frac{1}{2}\left[(-2x_1^3 - 4x_1^2x_2 - x_1x_2^2)q_1 + (-3x_1 - 2x_2)q_1q_2 + (-3x_1 - x_2)q_1^2\right]$
$\boldsymbol{s_1s_2s_1s_2}$	$\frac{1}{6}\left[(x_1 + x_2)^2x_1x_2\right] + \frac{1}{6}\left[(x_1 + x_2)^2q_1 + (-3x_1^2 - x_1x_2)q_2 + 2q_1q_2\right]$
$\boldsymbol{s_2s_1s_2s_1}$	$\frac{1}{2}x_1^4 + \frac{1}{2}\left[(-2x_1^2 - 3x_1x_2)q_1 - 2q_1q_2 - 2q_1^2\right]$
$\boldsymbol{s_2s_1s_2}$	$\frac{1}{2}\left[(x_1 + x_2)x_1x_2\right] + \frac{1}{2}\left[(x_1 + x_2)\,q_1 - x_1q_2\right]$
$\boldsymbol{s_1s_2s_1}$	$\frac{1}{2}x_1^3 + \frac{1}{2}\left[(-2x_1 - x_2)q_1\right]$
$\boldsymbol{s_1s_2}$	$x_1x_2 + q_1$
$\boldsymbol{s_2s_1}$	$x_1^2 - q_1$
$\boldsymbol{s_2}$	$x_1 + x_2$
$\boldsymbol{s_1}$	x_1
$\boldsymbol{id}$	1

TABLE 6. *Quantum Schubert Polynomials.*

APPENDIX A. TABLES

A.1. **Curve Neighborhood Calculations.** This appendix contains the curve neighborhoods for all the Weyl group elements. In order to list them as short as possible, we need to define the following:

- $\ell, m = 0, 1, 2, 3, ...$
- $N, M = 1, 2, 3, ...$
- $N', M' = 2, 3, 4, ...$

If $w \in W$ then $\Gamma_{(0,0)}(w) = w$ so we won't include that condition in the tables.

$\boldsymbol{id}$	$\boldsymbol{s_1}$	$\boldsymbol{s_2}$
$\Gamma_{(N,0)}(id) = s_1$ $\Gamma_{(0,N)}(id) = s_2$ $\Gamma_{(N,1)}(id) = s_1s_2s_1$ $\Gamma_{(1,N')}(id) = s_2s_1s_2s_1s_2$ $\Gamma_{(N',M')}(id) = w_0$	$\Gamma_{(N,0)}(s_1) = s_1$ $\Gamma_{(0,N)}(s_1) = s_1s_2$ $\Gamma_{(N,1)}(s_1) = s_1s_2s_1$ $\Gamma_{(N,N')}(s_1) = w_0$	$\Gamma_{(N,0)}(s_2) = s_2s_1$ $\Gamma_{(0,N)}(s_2) = s_2$ $\Gamma_{(N,1)}(s_2) = s_2s_1s_2s_1$ $\Gamma_{(1,N')}(s_2) = s_2s_1s_2s_1s_2$ $\Gamma_{(N',M')}(s_2) = w_0$
$\boldsymbol{s_1s_2}$	$\boldsymbol{s_2s_1}$	$\boldsymbol{s_1s_2s_1}$
$\Gamma_{(N,0)}(s_1s_2) = s_1s_2s_1$ $\Gamma_{(0,N)}(s_1s_2) = s_1s_2$ $\Gamma_{(N,1)}(s_1s_2) = s_1s_2s_1s_2s_1$ $\Gamma_{(N,N')}(s_1s_2) = w_0$	$\Gamma_{(N,0)}(s_2s_1) = s_2s_1$ $\Gamma_{(0,N)}(s_2s_1) = s_2s_1s_2$ $\Gamma_{(N,1)}(s_2s_1) = s_2s_1s_2s_1$ $\Gamma_{(N,N')}(s_2s_1) = w_0$	$\Gamma_{(N,0)}(s_1s_2s_1) = s_1s_2s_1$ $\Gamma_{(0,N)}(s_1s_2s_1) = s_1s_2s_1s_2$ $\Gamma_{(N,1)}(s_1s_2s_1) = s_1s_2s_1s_2s_1$ $\Gamma_{(N,N')}(s_1s_2s_1) = w_0$
$\boldsymbol{s_2s_1s_2}$	$\boldsymbol{s_1s_2s_1s_2}$	$\boldsymbol{s_2s_1s_2s_1}$
$\Gamma_{(N,0)}(s_2s_1s_2) = s_2s_1s_2s_1$ $\Gamma_{(0,N)}(s_2s_1s_2) = s_2s_1s_2$ $\Gamma_{(N,M)}(s_2s_1s_2) = w_0$	$\Gamma_{(N,0)}(s_1s_2s_1s_2) = s_1s_2s_1s_2s_1$ $\Gamma_{(0,N)}(s_1s_2s_1s_2) = s_1s_2s_1s_2$ $\Gamma_{(N,M)}(s_1s_2s_1s_2) = w_0$	$\Gamma_{(N,0)}(s_2s_1s_2s_1) = s_2s_1s_2s_1$ $\Gamma_{(0,N)}(s_2s_1s_2s_1) = s_2s_1s_2s_1s_2$ $\Gamma_{(N,M)}(s_2s_1s_2s_1) = w_0$
$\boldsymbol{s_1s_2s_1s_2s_1}$	$\boldsymbol{s_2s_1s_2s_1s_2}$	$\boldsymbol{w_0}$
$\Gamma_{(N,0)}(s_1s_2s_1s_2s_1) = s_1s_2s_1s_2s_1$ $\Gamma_{(\ell,N)}(s_1s_2s_1s_2s_1) = w_0$	$\Gamma_{(0,N)}(s_2s_1s_2s_1s_2) = s_2s_1s_2s_1s_2$ $\Gamma_{(N,\ell)}(s_2s_1s_2s_1s_2) = w_0$	$\Gamma_{(\ell,m)}(w_0) = w_0$

TABLE 7. *The Curve Neighborhoods for every degree at every $w \in W$.*

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