Lectures on quantum K theory of flag manifolds (2)

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Lecture notes, slides, and homework, available at
https://personal.math.vt.edu//lmihalce/slides.html

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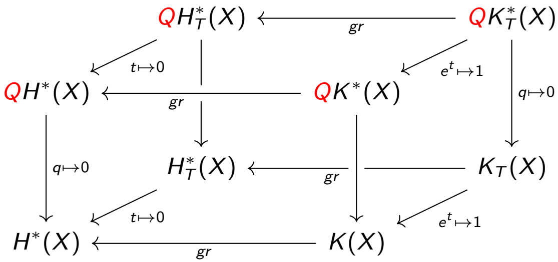
Where are we and where we are going ?



Example (Still too early)

In $\mathrm{QK}(\mathrm{Gr}(3,6))$ ($\deg q = 6$):

$$\begin{aligned}\mathcal{O}^{3,2,1} \circ \mathcal{O}^{3,2,1} &= q^2 - 2q^2\mathcal{O}^1 \\ &+ q^2\mathcal{O}^2 + q^2\mathcal{O}^{1,1} - q^2\mathcal{O}^{2,1} + q\mathcal{O}^{3,3} \\ &+ q\mathcal{O}^{2,2,2} + 2q\mathcal{O}^{3,2,1} - 2q\mathcal{O}^{3,2,2} - 2q\mathcal{O}^{3,3,1} + q\mathcal{O}^{3,3,2}\end{aligned}$$

Use the A. Buch's `Equivariant Schubert Calculator` available at `https://sites.math.rutgers.edu/ asbuch/equivcalc/`

Quantum K theory (Givental, Y.P. Lee)

For simplicity we take $X = G/P$ (a complex, projective, manifold) and $q = (q_\beta)$ a sequence of (**Novikov**) variables indexed by a basis $\{[C_i]\} \in H_2(X)$. Define

$$\deg q_i = c_1(T_X) \cap [C_i] \in \mathbb{Z}.$$

(For $X = \mathrm{Gr}(k, n)$, $H_2(X) \simeq \mathbb{Z}$ and $\deg q = n$.)

- As a module, $\mathrm{QK}(X) = K(X) \otimes_{\mathbb{Z}} \mathbb{Z}[[q]]$, i.e.,

$$\mathrm{QK}(G/P)) = \bigoplus_{w \in W^P} \mathbb{Z}[[q]]\mathcal{O}^w.$$

- The ring structure of $\mathrm{QK}(G/P)$ is determined by the **QK metric** and the 2-point and 3-point **K-theoretic Gromov-Witten (KGW) invariants**.

The moduli space of stable maps

For an effective degree $d \in H_2(X)$, denote by $\overline{\mathcal{M}}_{0,n}(X,d)$ the **Kontsevich moduli space of (genus** 0**,** n **pointed) stable maps of degree** d. This is a projective scheme, with points stable maps:

$$f : (C, p_1, \dots p_n) \to X; \quad f_*[C] = d.$$

Here C is a tree of $\mathbb{P}^1$'s, and f satisfies a stability condition. There are **evaluation maps**

$$\mathrm{ev}_i : \overline{\mathcal{M}}_{0,n}(X,d) \to X \quad f \mapsto f(p_i).$$

If $n = 3$ and $d = 0$, then $\overline{\mathcal{M}}_{0.n}(X, 0) = X$ and $\mathrm{ev}_i = id_X$.
We list some important properties of the Kontsevich moduli space.
Let $X = G/P$ be a flag manifold. Then:

- $\overline{\mathcal{M}}_{0,n}(X,d)$ has finite quotient singularities, hence rational singularities (this follows from construction);
- $\overline{\mathcal{M}}_{0,n}(G/P,d)$ is a connected, thus irreducible variety (Thomsen);
- $\overline{\mathcal{M}}_{0,n}(X,d)$ is a rational variety (Kim and Pandharipande).

The Gromov-Witten invariants and the QK product

Let $a_1, \dots, a_n \in K(X)$ and $d \in H_2(X)$. The **K-theoretic Gromov-Witten invariant** is

$$\langle a_1, \dots, a_n \rangle_d = \int_{\overline{\mathcal{M}}_{0,n}(X,d)} \mathrm{ev}_1^*(a_1) \cdot \ldots \cdot \mathrm{ev}_n^*(a_n) = \chi(\mathrm{ev}_1^*(a_1) \cdot \ldots \cdot \mathrm{ev}_n^*(a_n)).$$

In general the moduli space is not smooth, but since X is, one may write each of the classes a_i as a finite alternating sum of classes of vector bundles.
The (small) **QK pairing** is defined by

$$((a, b)) = \langle a, b \rangle + \sum_{d>0} \langle a, b \rangle_d q^d \quad \in \mathbb{Z}[[q]]$$

The quantum K product is the unique product $\circ$ which satisfies

$$((a \circ b, c)) = \sum_{d \geq 0} \langle a, b, c \rangle_d q^d.$$

Examples of QK pairing; $a \circ 1$

For $X = \mathrm{Gr}(k, n)$, the pairing is

$$((1,1)) = 1 + q + q^2 + \ldots = \frac{1}{1-q}.$$

More generally, the (simplest example of the) **string equation** implies that

$$\langle a, b, 1 \rangle_d = \langle a, b \rangle_d$$

which furthermore gives that $((a \circ 1, b)) = ((a, b))$ for any $a, b \in K(X)$. Therefore

$$a \circ 1 = a \quad \in QK(X).$$

(For more details see examples 4.2 and 4.4 in the lecture notes.)

The quantum K ring

Theorem (Givental, Lee)

The product $\circ$ *equips* $\mathrm{QK}(X)$ *with a structure of a commutative, associative ring with identity* $1 = [\mathcal{O}_X]$.

From definition it follows that:

- $K(X) \simeq QK(X)/\langle q \rangle$;
- Since $K(X)$ is filtered algebra, it induces a filtration on $QK(X)$, with $\deg q_i = \int_X c_1(T_X) \cap [C_i]$. The associated graded algebra is

$$\mathrm{Gr} QK(X) = QH^*(X),$$

  the **quantum cohomology** of X.

Unraveling the QK product

We discuss two equivalent formulations of the definition. Consider the product

$$\mathcal{O}^u \circ \mathcal{O}^v = \sum N_{u,v}^{w,d} q^d \mathcal{O}^w.$$

Then:

$$N_{u,v}^{w,d} = \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^w)^\vee \rangle_d - \sum_{d'>0,\kappa} N_{u,v}^{\kappa, d-d'} \langle \mathcal{O}^\kappa, (\mathcal{O}^w)^\vee \rangle_{d'}.$$

or, equivalently,

$$\begin{aligned} N_{u,v}^{w,d} =& \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^w)^\vee \rangle_d \\ & + \sum (-1)^s \langle \mathcal{O}^u, \mathcal{O}^v, (\mathcal{O}^{\kappa_0})^\vee \rangle_{d_0} \cdot \langle \mathcal{O}^{\kappa_0}, (\mathcal{O}^{\kappa_1})^\vee \rangle_{d_1} \cdot \ldots \cdot \langle \mathcal{O}^{\kappa_s}, (\mathcal{O}^k)^\vee \rangle_{d_s}; \end{aligned}$$

here the sum is over effective degrees $d_0, \ldots, d_s$ such that $d_0 + \ldots + d_s = d$ and $d_p > 0$ if $p > 0$.

Some theorems for $X = G/P$

1. (QK metric) Let $u, v \in W^P$. Then for each d there is an explicitly defined element $u(d) \in W^P$ such that
$$\langle \mathcal{O}_u, \mathcal{I}^v \rangle_d = \delta_{u(d),v}.$$
The Schubert variety $X_{u(d)}$ is the **curve neighborhood** of X_u. The QK metric may be calculated by
$$((\mathcal{O}^u, \mathcal{O}^v)) = \frac{q^{d_{min}(u,v)}}{\prod(1 - q_i)}$$
where $q^{d_{min}(u,v)}$ is the minimum power of q in the **quantum cohomology** product $[X^u] \star [X^v]$.
2. (Finiteness) The quantum K product is finite, i.e., for any $u, v \in W^P$, $\mathcal{O}^u \circ \mathcal{O}^v \in \mathrm{K}(X) \otimes \mathbb{Z}[q]$.
3. (Positivity) Let $X = \mathrm{Gr}(k, n)$ and consider
$$\mathcal{O}^\lambda \circ \mathcal{O}^\mu = \sum N_{\lambda,\mu}^{\nu,d} q^d \mathcal{O}^\nu.$$
Then $(-1)^{|\nu|+nd-|\lambda|-|\mu|} N_{\lambda,\mu}^{\nu,d} \geq 0$.

‘Quantum=classical’

Assume $X = \mathrm{Gr}(k,n)$ is a Grassmannian. Consider the incidence diagram

$$\begin{array}{ccc} Z_d := \mathrm{Fl}(k-d,k,k+d;n) & \xrightarrow{\;p_d\;} & X := \mathrm{Gr}(k,n) \\ \downarrow{\scriptstyle q_d} & & \\ Y_d := \mathrm{Fl}(k-d,k+d;n) & & \end{array}$$

Here, if $d \geq k$ then we set $Y_d := \mathrm{Fl}(k+d;n)$ and if $k+d \geq n$ then we set $Y_d := \mathrm{Gr}(k-d;n)$. If $d \geq \min\{k, n-k\}$, then Y_d is a single point. Then for any $a,b,c \in \mathrm{K}(X)$ and any effective degree d

$$\langle a,b,c\rangle_d = \int_{Y_d} (q_d)_* p_d^*(a) \cdot (q_d)_* p_d^*(b) \cdot (q_d)_* p_d^*(c).$$

The ‘quantum = classical’ theorem has many applications, including:

- explicit combinatorial **Pieri/Chevalley formulae** for any (co)minuscule Grassmannians X;
- **Presentations** of $\mathrm{QK}(\mathrm{Gr}(k,n))$ by generators and relations which quantize the Whitney presentation;
- Positivity;
- An extension of **Seidel representation** and combinatorics of quantum shapes, generalizing Postnikov’s cylinder.

Example

Recall that $\mathrm{QK}(\mathrm{Gr}(3,6))$ ($\deg q = 6$):

$$\begin{aligned}\mathcal{O}^{3,2,1} \circ \mathcal{O}^{3,2,1} &= q^2 - 2q^2\mathcal{O}^1 \\ &+ q^2\mathcal{O}^2 + q^2\mathcal{O}^{1,1} - q^2\mathcal{O}^{2,1} + q\mathcal{O}^{3,3} \\ &+ q\mathcal{O}^{2,2,2} + 2q\mathcal{O}^{3,2,1} - 2q\mathcal{O}^{3,2,2} - 2q\mathcal{O}^{3,3,1} + q\mathcal{O}^{3,3,2}\end{aligned}$$

Then

$$((\mathcal{O}^{(3,2,1)}, \mathcal{O}^{(3,2,1)})) = \frac{q}{1-q}.$$

Curve neighborhoods

Let $\Omega_1, \dots, \Omega_n \subset X$ be closed subvarieties and fix an effective degree $d \in H_2(X)$.

1. The (n-point) **Gromov-Witten variety** is the intersection

$$\mathrm{GW}_d(\Omega_1, \dots, \Omega_n) = \mathrm{ev}_1^{-1}(\Omega_1) \cap \dots \cap \mathrm{ev}_n^{-1}(\Omega_n) \subset \overline{\mathcal{M}}_{0,n+a}(X, d).$$

   If $\Omega_2 = \dots = \Omega_n = X$ we will simply use the notation $\mathrm{GW}_d(\Omega_1) = \mathrm{GW}_d(\Omega_1, X, \dots, X)$.

2. The (n-point) **curve neighborhood** of $\Omega_1, \dots, \Omega_n$ is defined as the image of the corresponding Gromov-Witten variety:

$$\Gamma_d(\Omega_1, \dots, \Omega_n) = \mathrm{ev}_{n+1}(\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)).$$

   As before, $\Gamma_d(\Omega) := \mathrm{ev}_{n+1}(\mathrm{GW}_d(\Omega))$.
   All these may be extended to the case when one has a sequence of degrees $d_1, \dots, d_k$, by replacing the moduli space with an appropriate stratum in the boundary.

Examples

Example

(a) If $d = 0$, then $\Gamma_0(\Omega_1, \Omega_2) = \Omega_1 \cap \Omega_2$.
(b) Take $X = \mathbb{P}^n$ and $d > 0$. Then $\Gamma_d(pt) = \mathbb{P}^n$ and

$$\Gamma_d(pt, pt) = \begin{cases} \text{line} & d = 1 \\ \mathbb{P}^n & d \geq 2. \end{cases}$$

Basic properties of curve neighborhoods

Theorem (Buch-Chaput-M.-Perrin)

Let $\Omega_1, \dots, \Omega_n$ be general translates of Schubert varieties in X. Then the following hold:
(a) The GW variety $\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)$ is either empty, or locally irreducible of expected dimension, and with rational singularities. Furthermore,

$$\langle [\mathcal{O}_{\Omega_1}], \dots, [\mathcal{O}_{\Omega_n}] \rangle_d = \chi([\mathcal{O}_{\mathrm{GW}_d(\Omega_1, \dots, \Omega_n)}]).$$

(b) The non-empty Gromov-Witten varieties $\mathrm{GW}_d(\Omega_1, \Omega_2)$ are irreducible and rationally connected. In particular, the 2-point curve neighborhood $\Gamma_d(\Omega_1, \Omega_2)$ is also irreducible and rationally connected.
(c) If Ω is any Schubert variety, then $\Gamma_d(\Omega)$ is again a Schubert variety and the evaluation map $\mathrm{ev}_i : \mathrm{GW}(\Omega) \to \Gamma_d(\Omega)$ is cohomologically trivial.

Two-point KGW invariants

For $u \in W^P$ and $d \geq 0$ an effective degree, define $u(d), u(-d) \in W^P$ by:

$$X_{u(d)} = \Gamma_d(X_u); \quad X^{u(-d)} = \Gamma_d(X^u).$$

Then for $u, v \in W^P$, the 2-point KGW invariants are given by:

$$\langle \mathcal{O}_u, \mathcal{O}^v \rangle_d = \langle \mathcal{O}_{\Gamma_d(X_u)}, \mathcal{O}^v \rangle_0 = \begin{cases} 1 & v \leq u(d) \\ 0 & \text{otherwise} \end{cases}.$$

In particular the minimum quantum degree in $[X_u] * [X^v]$ is

$$d_{min}(u, v) = \min\{d : v \leq u(d)\}.$$

From the duality between structure and ideal sheaves:

$$\langle \mathcal{O}^u, (\mathcal{O}^v)^\vee \rangle_d = \delta_{u(-d),v},$$

(the Kronecker delta symbol).

Curve neighborhoods: the moment graph method

The **moment graph** of G/P has:

- **vertices** corresponding to $u \in W^P$;
- **edges** $u \xrightarrow{d(i,j)} v$ if $\ell(v) > \ell(u)$ and $u \cdot (i,j) = v$ for $i < j$.

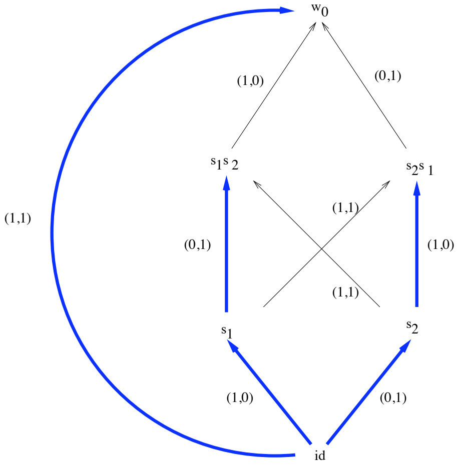
The edge has (multi)degree $d_{i,j} = \varepsilon_i - \varepsilon_j$ modulo Δ_P (the simple roots which are already in P).

Theorem (Buch-M.)

Then $\Gamma_d(X_u)$ is the (unique!) maximal element in the Bruhat order obtained from tracing a path from u of degree $\leq d$.

Example

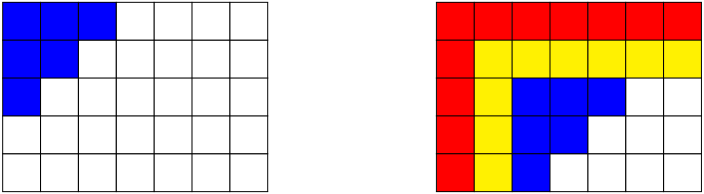
Below is the moment graph for $\mathrm{Fl}(3)$. With blue we drew the paths giving $\Gamma_{(1,0)}(pt) = X_{s_1}, \Gamma_{(0,1)}(pt) = X_{s_2}, \Gamma_{(1,1)}(pt) = X_{s_1 s_2 s_1}$.



Curve neighborhoods of Grassmannians

Let λ included in the $k \times (n-k)$ rectangle. The curve neighborhoods have nice combinatorial descriptions:

- $\lambda(d)$ is obtained from λ by adding d rim hooks of maximal length;
- $\lambda(-d)$ is obtained from λ by removing d rim hooks of maximal length.



The curve neighborhood $\lambda(2)$, for $\lambda = (3, 2, 1)$.

How to calculate curve neighborhoods for general flag manifolds

The **Demazure product** $\cdot$ of two Weyl group elements is defined as follows. If $u \in W$ and $s_i \in W$ is a simple reflection,

$$u \cdot s_i = \begin{cases} us_i & \ell(us_i) > \ell(u) \\ u & \ell(us_i) < \ell(u). \end{cases}$$

If $v = s_{i_1} \dots s_{i_k}$ is a reduced decomposition, then $u \cdot v = (((u \cdot s_{i_1}) \cdot s_{i_2}) \dots) \cdot s_{i_k}$. This equips $(W, \cdot)$ with a structure of an associative monoid. Let also $z_d \in W$ be the unique element defined by

$$X_{z_d} = \Gamma_d(pt) \subset \mathrm{Fl}(n).$$

We have the following algorithm to calculate $u(d)$:

Theorem (BCMP,Buch-M)

The following hold:
(a) In G/B, $\Gamma_d(X_u) = X_{u \cdot z_d}$.
(b) Take $\alpha > 0$ be the largest positive root such that $d - \alpha^\vee \geq 0$ in $H_2(\mathrm{Fl}(n))$. Then

$$z_d = z_{d-\alpha^\vee} \cdot s_\alpha = s_\alpha \cdot z_{d-\alpha^\vee}.$$

(c) Same procedure applies to any G/P: take $\alpha \in R^+ \setminus R_P^+$ maximal such that $d - \alpha^\vee \geq 0$ in $H_2(\mathrm{Fl}(\mathbf{i}))$. Then

$$z_d W_P = s_\alpha \cdot z_{d-\alpha^\vee} W_P.$$

A ring homomorphism

Theorem (Buch-Chung-M.-Li)

Assuming *that the* QK *product is finite, consider the specialization at* $q_i \mapsto 1$ *for all* i *of the usual pairing* $\chi : \mathrm{QK}(X) \to \mathrm{QK}(pt) = \mathbb{Z}[q]$. *Then this is a* **ring** *homomorphism, i.e.*

$$\chi(a \circ b) = \chi(a) \cdot \chi(b).$$

This is **false** in any other (quantum or classical) cohomology theory. (Just try $[pt] \cdot [pt] \in K(\mathbb{P}^1)$ or $[pt] \star [pt] \in QH^*(\mathbb{P}^1)$; see more about this in the lecture notes, e.g., Ex. 6.10.)

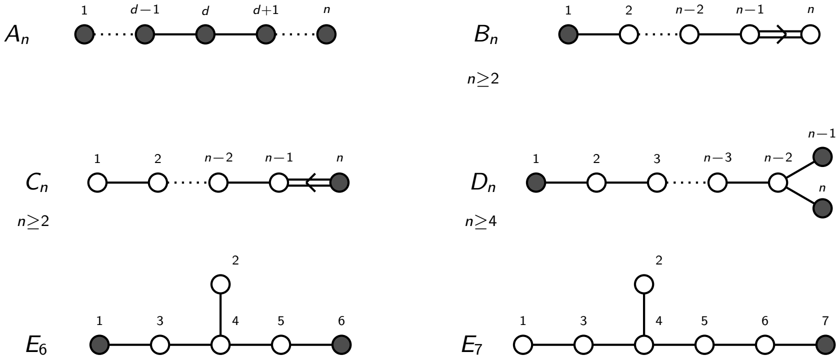
A more general ring homomorphism

Theorem (Kato)

Let $\pi : G/P \to G/Q$ be the natural projection for $P \subset Q$. Consider the $\mathbb{Z}[q]$-module projection $\pi_* : \mathrm{QK}(G/P) \to \mathrm{QK}(G/Q)$ defined by extending the usual projection $\pi_* : \mathrm{K}(G/P) \to \mathrm{K}(G/Q)$ and specializing $q_i \mapsto 1$ for all i such that $s_i \in W_Q \setminus W_P$. Then this is a ring homomorphism.

THANK YOU !

Cominuscule spaces



The node k is cominuscule if the simple root α_k appears with multiplicity 1 in the highest root.