Lectures on quantum K theory of flag manifolds (1)

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Lecture notes, slides, and homework, available at
https://personal.math.vt.edu//lmihalce/slides.html

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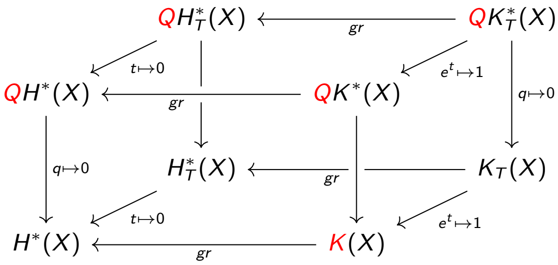
Where are we and where we are going ?



A way too early example in $\mathrm{QK}(Gr(3,6))$ (here $\deg q = 6$):

$$\begin{aligned}\mathcal{O}^{(3,2,1)} \circ \mathcal{O}^{(2,1)} = {} & q\mathcal{O}^{(3)} + 2q\mathcal{O}^{(2,1)} - 2q\mathcal{O}^{(2,2)} \\ & - 2q\mathcal{O}^{(3,1)} + q\mathcal{O}^{(3,2)} + q\mathcal{O}^{(1,1,1)} - 2q\mathcal{O}^{(2,1,1)} + q\mathcal{O}^{(2,2,1)} + q\mathcal{O}^{(3,1,1)} \\ & - q\mathcal{O}^{(3,2,1)} + \mathcal{O}^{(3,3,3)}.\end{aligned}$$

Use the A. Buch's `Equivariant Schubert Calculator` available at
`https://sites.math.rutgers.edu/ asbuch/equivcalc/`

K theory

X complex projective manifold. The K-theory

$$K(X) = \frac{\{[E] : E \to X \text{ vector bundle }\}}{[E] = [F] + [G]},$$

for any short exact sequence $0 \to F \to E \to G \to 0$. Addition and multiplication are given by

$$[E] + [F] := [E \oplus F]; \quad [E] \cdot [F] := [E \otimes F].$$

There is a pairing $\langle \cdot, \cdot \rangle : K(X) \times K(X) \to \mathbb{Z}$ defined by

$$\langle [E], [F] \rangle = \int_X E \otimes F = \sum (-1)^i \dim H^i(X; E \otimes F).$$

If $Y \subset X$ closed subvariety, $\mathcal{O}_Y$ has a **finite** resolution by vector bundles, thus $[\mathcal{O}_Y] \in K(X)$. More generally, any **coherent sheaf** $\mathcal{F}$ has such a resolution, and it gives $[\mathcal{F}] \in K(X)$. The K theory classes $[\mathcal{O}_Y], [\mathcal{F}]$ are called **Grothendieck classes**.

More generally, for arbitrary X:

- The (Grothendieck) ring of vector bundles $= K^\circ(X)$;
- The (Grothendieck) group of coherent sheaves $= K_\circ(X)$;
- If X is smooth $K^\circ(X) = K_\circ(X) = K(X)$;

$K(pt)$, $K(\mathbb{P}^1)$ (part I)

Intersections

Lemma (Fulton-Pragacz, Brion)

Let Y, Z be equidimensional Cohen-Macaulay subvarieties of a nonsingular variety X. Assume that the intersection $Y \cap Z$ is proper, i.e., it has the expected dimension $\dim Y + \dim Z - \dim X$. *Then each component of the scheme theoretic intersection $Y \cap Z$ has the expected dimension and $Y \cap Z$ is Cohen-Macaulay. Furthermore,*

$$[\mathcal{O}_Y] \cdot [\mathcal{O}_Z] = [\mathcal{O}_{Y \cap Z}] \quad \in K(X).$$

Example

- Any smooth variety is Cohen-Macaulay.
- Any Schubert variety is Cohen-Macaulay.
- More generally, we have a Kleiman's transversality statement: if $Y \subset X$, then for general $g_1, \dots, g_k \in G$, $Y \cap g_1 X^{w_1} \cap g_2 X^{w_2} \cap \dots \cap g_k X^{w_k}$ is either empty or purely-dimensional, of expected dimension, and Cohen-Macaulay.
- (To be defined later.) The moduli space of stable maps $\overline{\mathcal{M}}_{0,n}(G/P, d)$ is Cohen-Macaulay, because it is locally a smooth variety modulo a finite group.
- Smooth pull-backs preserve the Cohen-Macaulay property.

Example: $K(\mathbb{P}^n)$

Functoriality

- $K_\circ(X)$ a structure of $K^\circ(X)$-module. (Note the strong similarities to cohomology/homology versions!)
- If $f : X \to Y$ is a morphism, there is a **pull-back ring homomorphism**
$$f^* : K^\circ(Y) \to K^\circ(X), \quad [E] \mapsto [f^* E].$$
If f is flat and $Z \subset X$ is a subvariety, then $f^*[\mathcal{O}_Z] = [\mathcal{O}_{f^{-1}(Z)}]$.
- If $f : X \to Y$ is **proper**, there is a **push-forward**
$$f_* : K_\circ(X) \to K_\circ(Y), \quad f_*[\mathcal{F}] = \sum_{i \geq 0} (-1)^i [R^i f_* \mathcal{F}].$$
(This sum is finite, as the higher direct images vanish beyond the dimension of X.)
- The push-forward and pull-back satisfy the usual **projection formula**:
$$f_*(f^*[E] \otimes [\mathcal{F}]) = [E] \otimes f_*[\mathcal{F}] \quad \in K(Y).$$
- **Integration:** Let $p : X \to pt$ and assume X is proper. Then
$$p_*[\mathcal{F}] = \sum (-1)^i \dim H^i(X; \mathcal{F}) = \chi(X; \mathcal{F}).$$

The Chern character

As usual X is a manifold/smooth variety. The **Chern character**

$$ch : K(X) \to H^*(X)_{\mathbb{Q}}; \quad ch[L] = e^{c_1(L)} = 1 + c_1(L) + c_1(L)^2/2! + \dots$$

where $L \to X$ is a line bundle. For a general vector bundle $E \to X$, the **splitting principle** allows us to assume that $E = L_1 \oplus \dots \oplus L_r$ is a direct sum of line bundles with Chern roots $x_1, \dots, x_r$. Then

$$ch(E) = e^{x_1} + \dots + e^{x_r}.$$

If $Z \subset X$ is closed and irreducible, then

$$ch(Z) = [Z] + h.o.t.$$

where h.o.t. are terms in cohomological degree strictly larger than $\mathrm{codim} Z$. In other words $ch([\mathcal{O}_Z]) \in \oplus_{j \geq i} H^j(X)$, where subscripts denote dimension. The Chern character is always a **ring isomorphism**, if one works over $\mathbb{Q}$.

Example: $ch(\mathcal{O}_{\mathbb{P}^n}(1))$

The Hirzebruch λ_y class

Let $E \to X$ vector bundle of rank r. The **Hirzebruch λ_y class** of E is defined by

$$\lambda_y(E) = 1 + y[E] + y^2[\wedge^2 E] + \ldots + y^r[\wedge^r E] \quad \in K(X)[y].$$

This class is multiplicative: if $0 \to E_1 \to E_2 \to E_3 \to 0$ is a short exact sequence then

$$\lambda_y(E_1) \cdot \lambda_y(E_3) = \lambda_y(E_2).$$

The class $\lambda_{-1}(E^*)$ is sometimes called the K-theoretic Chern class of E, denoted by $cK(E)$. If L is a line bundle with first Chern class $c_1(L)$, then

$$ch(\lambda_{-1}(L^*)) = 1 - e^{-c_1(L)} = c_1(L) + h.o.t.$$

Furthermore, the identity

$$(1 - e^x)(1 - e^y) = (1 - e^x) + (1 - e^y) - (1 - e^{x+y})$$

implies that if L' is another line bundle, then

$$cK(L \oplus L') = cK(L) + cK(L') - cK(L \otimes L'),$$

recovering the **formal group law** for K theory.
Finally, note that the class $\lambda_{-1}(E)$ appears geometrically as an Euler class: if $E \to X$ is a vector bundle with a general section $s : X \to E$, then the zero locus of s has class

$$[\mathcal{O}_{Z(s)}] = \lambda_{-1}(E^*) \in K(X).$$

THEOREM. Let X be a projective manifold and $E \to X$ a vector bundle on E. Denote by T_X the tangent bundle. Then

$$\chi(X;E) = \int_X ch(E) \cdot Td(T_X),$$

where $Td(T_X)$ denotes the **Todd class** of T_X.

Flag manifolds

In these lectures we consider **partial flag manifolds**

$$\mathrm{Fl}(i_1, \ldots, i_k; n) = \{F_{i_1} \subset \ldots \subset F_{i_k} \subset \mathbb{C}^n\} = G/P,$$

with $G := GL_n$, $B :=$ Borel subgroup of UT matrices, B^- is the opposite subgroup.
Special cases:

$$\mathrm{Gr}(k; n) = \{V \subset \mathbb{C}^n : \dim V = k\} = G/P,$$

the **Grassmannian**, with P maximal parabolic, and

$$\mathrm{Fl}(n) = \{F_1 \subset \ldots \subset F_{n-1} \subset \mathbb{C}^n\} = G/B,$$

the **complete flag manifold**. The **Weyl group** $W = S_n$, and for $P \supset B$, define

$$W_P = \langle s_{i_1}, \ldots, s_{i_k} : s_{i_j} \in P\rangle \leq W$$

and let W^P the set of **minimal length representatives** in W/W_P.

- For $\mathrm{Fl}(n) = G/B$, $W_B = \langle 1 \rangle$ and $W^B = W$;
- For $\mathrm{Gr}(k, n) = G/P$, $W_P = \langle s_1, \ldots, \widehat{s_k}, \ldots, s_{n-1}\rangle$ and

  $$W^P \leftrightarrow \lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0) \text{ and } \lambda_1 \leq n - k,$$

  partitions included in the $k \times (n-k)$ rectangle.

Let $w_0 = \begin{pmatrix} 1 & 2 & \ldots & n \\ n & n-1 & \ldots & 1 \end{pmatrix}$ and $w^\vee =$ is the minimal length representative for $w_0 w$ in W^P.

Tautological sequences

Schubert varieties

Fix a flag $F_\bullet = \langle e_1 \rangle \subset \langle e_1, e_2 \rangle \subset \ldots \subset \mathbb{C}^n$. Schubert varieties:

- For $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_k \geq 0)$, and $\lambda_1 \leq n-k$,

$$X^\lambda = \{V \in \mathrm{Gr}(k,n) : \dim V \cap F_{n-k+i-\lambda_i} \geq i\}.$$

- There is also an opposite Schubert variety X_λ.
- $\dim X_\lambda = \mathrm{codim}\ X^\lambda = |\lambda| = \lambda_1 + \ldots + \lambda_k$.
- A similar description for Schubert varieties $X_u, X^u \subset G/P$, for permutations $u \in W^P$:

$$X^u = \overline{B^- uP/P}; \quad X_u = \overline{BuP/P}$$

  With these definitions

$$\dim_{\mathbb{C}} X_u = \mathrm{codim}_{\mathbb{C}}\ X^u = \ell(u).$$

- The **boundary** of X^w is defined by

$$\partial X^w = X^w \setminus X^{w,\circ} = \overline{B^- uP/P} \setminus B^- uP/P; \quad \partial X_w = \overline{BwP/P} \setminus BwP/P.$$

Denote

$$\mathcal{O}^w = [\mathcal{O}_{X^w}], \quad \mathcal{I}^w = [\mathcal{O}_{X^w}(-\partial X^w)] = [\mathcal{I}_{\partial X^w}] = [\mathcal{O}_{X^w}] - [\mathcal{O}_{\partial X^w}]$$

where $\mathcal{I}^w$ is the **ideal sheaf** of ∂X^w. The classes of the ideal sheaves of the boundaries $\partial X_w, \partial X^w$ are denoted by $\mathcal{I}_w, \mathcal{I}^w$. Note that

$$\mathcal{O}_w = \mathcal{O}^{w^\vee}, \quad \mathcal{O}^w = \mathcal{O}_{w^\vee} \text{ and } \mathcal{I}_w = \mathcal{I}^{w^\vee}, \quad \mathcal{I}^w = \mathcal{I}_{w^\vee}.$$

Example: the ideal sheaves in Grassmannians

Assume that $G/P = \mathrm{Gr}(k,n)$ is a Grassmann manifold and let

$$\iota : \mathrm{Gr}(k,n) \to \mathbb{P} = \mathbb{P}(\wedge^k \mathbb{C}^n)$$

be the **Plücker embedding**. Then the boundary of Schubert varieties are also Cartier divisors, corresponding to the restriction of the line bundle $\mathcal{O}_{\mathbb{P}}(1)$ to X_w. This implies that for any partition $\lambda \subset k \times (n-k)$,

$$\mathcal{I}^\lambda = \mathcal{O}^\lambda \cdot \mathcal{O}_{\mathrm{Gr}(k,n)}(-1).$$

Combinatorially,

$$\mathcal{I}^\lambda = \sum_{\lambda \subset \mu} (-1)^{|\mu/\lambda|} \mathcal{O}^\mu; \quad \mathcal{O}^\lambda = \sum_{\lambda \subset \mu} \mathcal{I}^\mu,$$

where the sums are over partitions $\mu \supset \lambda$ such that μ/λ is a **rook strip**, i.e. the skew shape does not have two boxes in the same row or column. This is a particular case of the **Chevalley formula**.

The Schubert package ($X = G/P$)

- **Schubert basis.** The (Grothendieck) classes of the structure/ideal sheaves give bases:
$$K(X) = \bigoplus_{w \in W^P} \mathbb{Z}\mathcal{O}^w = \bigoplus_{w \in W^P} \mathbb{Z}\mathcal{I}^w.$$
- **Duals.** The duals of the Schubert classes are the (opposite) boundary classes:
$$\langle \mathcal{O}_v, \mathcal{I}^w \rangle = \langle \mathcal{O}^v, \mathcal{I}_w \rangle = \delta_{v,w}.$$
- **Positivity.** (Buch, Brion) If $\mathcal{O}^u \cdot \mathcal{O}^v = \sum c_{u,v}^w \mathcal{O}^w$ then
$$(-1)^{\ell(u)+\ell(v)-\ell(w)} c_{u,v}^w \geq 0.$$
- **Presentation.** The Whitney relations give a complete ideal of relations. E.g., for $X = \mathrm{Gr}(k;n)$ equipped with the tautological sequence $0 \to \mathcal{S} \to \mathbb{C}^n \to \mathcal{Q} \to 0$,
$$\lambda_y(\mathcal{S}) \cdot \lambda_y(\mathcal{Q}) = \lambda_y(\mathbb{C}^n) = (1+y)^n.$$
- **Polynomial representatives.** The **Grothendieck polynomials** represent Schubert classes.
- **Functoriality.** Let $P \subset Q$ be two parabolic subgroups and $\pi : G/P \to G/Q$ the projection. Then for any $v \in W^P$ and $w \in W^Q$,
$$\pi_* \mathcal{O}_v = \mathcal{O}_{vW_Q}; \quad \pi^* \mathcal{O}^w = \mathcal{O}^w.$$

  To emphasize that we utilize a Poincaré dual class rather than its precise formula, we will use the notation $(\mathcal{O}_w)^\vee = \mathcal{I}^w$, $(\mathcal{O}^w)^\vee = \mathcal{I}_w$.

Topological filtration

Recall that if $Y \subset X$ is irreducible, then

$$ch([\mathcal{O}_Y]) = [Y] + h.o.t.$$

This gives a decreasing (codimension) filtration

$$K(X) = \mathcal{K}^0(X) \supset \mathcal{K}^1(X) \supset \dots$$

which gives $K(X)$ a structure of **filtered ring**, in the sense that $\mathcal{K}^i(X) \cdot \mathcal{K}^j(X) \subset \mathcal{K}^{i+j}(X)$.

Example

In $K(\mathrm{Gr}(2,4))$,

$$\mathcal{O}^{\ydiagram{1}} \cdot \mathcal{O}^{\ydiagram{1}} = \mathcal{O}^{\ydiagram{2}} + \mathcal{O}^{\ydiagram{1,1}} - \mathcal{O}^{\ydiagram{2,1}}$$

Why ? Because the short exact sequence

$$0 \to \mathcal{O}_{X^{\ydiagram{2}} \cup X^{\ydiagram{1,1}}} \to \mathcal{O}_{X^{\ydiagram{2}}} \oplus \mathcal{O}_{X^{\ydiagram{1,1}}} \to \mathcal{O}_{X^{\ydiagram{2}} \cap X^{\ydiagram{1,1}}} \to 0.$$

Ideal sheaves and Schubert classes

Theorem

(a) Let $X = \mathrm{Fl}(n)$*. Then*

$$\mathcal{I}_w = \sum_{v \leq w} (-1)^{\ell(w)-\ell(v)} \mathcal{O}_v; \quad \mathcal{O}_w = \sum_{v \leq w} \mathcal{I}_w.$$

(b) Let $X = \mathrm{Gr}(k,n)$*. Then for any partition* $\lambda \subset k \times (n-k)$*,*

$$\mathcal{I}^\lambda = \sum_{\lambda \subset \mu} (-1)^{|\mu/\lambda|} \mathcal{O}^\mu; \quad \mathcal{O}^\lambda = \sum_{\lambda \subset \mu} \mathcal{I}^\mu,$$

where the sums are over partitions $\mu \supset \lambda$ *such that* μ/λ *is a* **rook strip***, i.e. the skew shape does not have two boxes in the same row or column.*

Example

In $\mathrm{Gr}(3,7)$,

$$\mathcal{I}^{(4,3,1)} = \mathcal{O}^{(4,3,2)} + \mathcal{O}^{(4,4,1)} - \mathcal{O}^{(4,4,2)}$$

Note that for Grassmannians this is equivalent to a **Chevalley formula**:

$$\mathcal{I}^\lambda = \mathcal{O}(-1) \cdot \mathcal{O}^\lambda = (1 - \mathcal{O}^\square) \cdot \mathcal{O}^\lambda$$

Positivity

Recall Buch/Brion's positivity theorem in $K(G/P)$: if $\mathcal{O}^u \cdot \mathcal{O}^v = \sum c_{u,v}^w \mathcal{O}^w$, then $(-1)^{\ell(u)+\ell(v)-\ell(w)} c_{u,v}^w \geq 0$.
The proof relies on a more a general result proved by Brion. A variety X has **rational singularities** if it has a proper resolution of singularities $\pi : X' \to X$ such that (as sheaves)

$$\pi_* \mathcal{O}_{X'} = \mathcal{O}_X \text{ and } R^i \pi_* \mathcal{O}_{X'} = 0, \quad i > 0.$$

A variety with rational singularities must be normal and Cohen-Macaulay. Schubert varieties have rational singularities, and so have general intersections of them.

Theorem (Brion.)

Let $X = G/P$ and $Y \subset X$ be a subvariety with rational singularities. Consider the expansion

$$[\mathcal{O}_Y] = \sum a_w \mathcal{O}_w.$$

Then $(-1)^{\ell(w)-\dim Y} a_w \geq 0$.

To prove positivity, apply the theorem to Y equal to the **Richardson variety**

$$\mathcal{O}_{X^u \cap X_{v^\vee}} = \mathcal{O}^u \cdot \mathcal{O}_{v^\vee} = \mathcal{O}^u \cdot \mathcal{O}^v.$$

Idea of proof

Assume Y is smooth and $X = \mathbb{P}^n$. Then

$$a_w = \chi(Y \cdot (\mathcal{O}^w)^\vee)) = \chi([\mathcal{O}_Y] \cdot \mathcal{O}_{\mathbb{P}^i} \cdot \mathcal{O}(-1)).$$

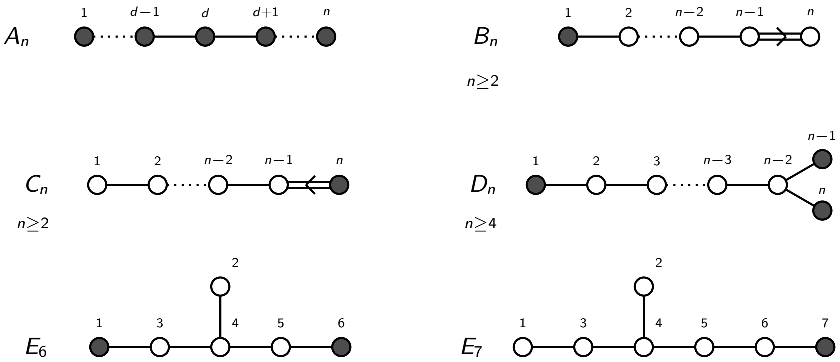
If nonempty, the general intersection is a (possibly disconnected) union of smooth varieties. The **Kodaira vanishing theorem** applied to each component of this intersection implies that

$$\chi([\mathcal{O}_Y] \cdot \mathcal{O}_{\mathbb{P}^i} \cdot \mathcal{O}(-1)) = (-1)^{\dim Y - n - i} H^{\dim Y - n - i}(Y \cap \mathbb{P}^i; \mathcal{O}(-1))$$

proving the claim.

THANK YOU !

Cominuscule spaces



The node k is cominuscule if the simple root α_k appears with multiplicity 1 in the highest root.