# DETERMINATION OF AN NET MAP PRESENTATION FOR $f(z) = \frac{6z^3+1}{9z^2-2}$

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We begin by constructing a finite subdivision rule for $f(z) = \frac{6z^3+1}{9z^2-2}$.

**Dynamic portrait**

In accordance with Head's theorem (Prop 1.2 of [3]), we observe that

$$f(z) = z - \frac{p(z)}{p'(z)} = z - \frac{3z^3 - 2z - 1}{9z^2 - 2} = z - \frac{(z-1)(3z^2+3z+1)}{9z^2-2}.$$

Since $f'(z) = \frac{p(z)p''(z)}{(p'(z))^2}$, we observe that the superattracting fixed points of $f$ correspond to roots of the polynomial $p$.

Critical points of the map:

$$0, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{6}i$$

Fixed points of the map:

$$\infty, 1, -\frac{1}{2} \pm \frac{\sqrt{3}}{6}i$$

The unique non-fixed critical point has the following orbit:

$$0 \mapsto -\frac{1}{2} \mapsto 1$$

Note that $f$ has four postcritical points and that all critical points are simple, so $f$ is an NET map. Since $f$ has three superattracting fixed points, we conclude (by the holomorphic index formula, or by direct computation) that the fixed point $\infty$ is repelling. Thus $f$ is a Newton map in the sense of Definition 1.1 of [3] by Mikulich, Rückert and Schleicher.

**Finite subdivision rule for $f$**

We now construct the *channel diagram* $\Delta_0$ for $f$ following Hubbard, Schleicher, and Sutherland [1]. Denote by $U_R, U_G, U_B$ the immediate basins of $-\frac{1}{2} + \frac{\sqrt{3}}{6}i, -\frac{1}{2} - \frac{\sqrt{3}}{6}i, 1$ respectively. Each immediate basin has exactly one fixed internal ray connecting its fixed critical point to $\infty$ (see Prop 6 of [1]). The channel diagram $\Delta_0$ is taken to be the graph (embedded in the Riemann sphere) whose vertices are given by the four fixed points of $f$ and whose edges are these three internal rays. A portion of each of these edges is drawn as a black line segment in the Figure 1, which was drawn using Mandel (www.mndynamics.com).

Let $\Delta_1 := f^{-1}(\Delta_0)$. We argue that $\Delta_1$ consists of a single connected component (this is not always the case for general Newton maps). Since $f^{-1}(\Delta_0)$ must have valency 2 at each of the finite fixed points, $f^{-1}(\Delta_0)$ can have at most two components. If it had two components,

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it would mean that the closures of all three immediate basins intersect at a unique finite pole. But the cyclic ordering of the basins at the finite pole is reversed from the order at infinity. Thus $\Delta_1$ consists of a single component.

The closure of $\Delta_1 \setminus \Delta_0$ must consist of two components by valency considerations (both components appear in gray in Figure 1). By the real symmetry of $f$, one of these components must connect the two non-real finite fixed points of $f$.

Let $\Delta_2 := f^{-1}(\Delta_1)$. It is again seen that $\Delta_2$ is connected by counting preimages and valencies. Thus $0 \in \Delta_2$. We see that all postcritical points are in $\Delta_1$, and all critical and postcritical points are in $\Delta_2$. So $f$ is the subdivision map of a finite subdivision rule whose subdivision complex has 1-skeleton $\Delta_1$. The edges of $\Delta_2 \setminus \Delta_1$ are drawn as white line segments in Figure 1.

This finite subdivision rule is a bit inefficient. It can be simplified as follows. Let $\Delta_1'$ denote the union of $\Delta_0$ and the connected component of $\Delta_1 \setminus \Delta_0$ which connects the two non-real fixed points of $f$. This is a connected graph which contains all four postcritical points of $f$, and it is stable under $f$. So $f$ is the subdivision map of a finite subdivision rule whose subdivision complex has 1-skeleton $\Delta_1'$. This is the finite subdivision rule which we use to find an NET map presentation for $f$. It appears at the top of Figure 2, correct up to homeomorphism.

*Note:* the graph $\Delta_i$ is called the Newton graph of level $i$ for $f$, following Mikulich, Rückert and Schleicher. The Newton graph is one piece of the combinatorial invariant used to give a complete classification of postcritically *finite* Newton maps in [2].

## References


[1] John Hubbard, Dierk Schleicher, and Scott Sutherland, *How to find all roots of complex polynomials by Newton's method*, Invent. Math. **146** (2001), 1–33.

[2] Russell Lodge, Yauhen Mikulich and Dierk Schleicher, *A classification of postcritically finite Newton maps*, http://arxiv.org/abs/1510.02771.

[3] Yauhen Mikulich, Johannes Rückert, and Dierk Schleicher, *A combinatorial classification of postcritically fixed Newton maps*, http://arxiv.org/abs/1010.5280.

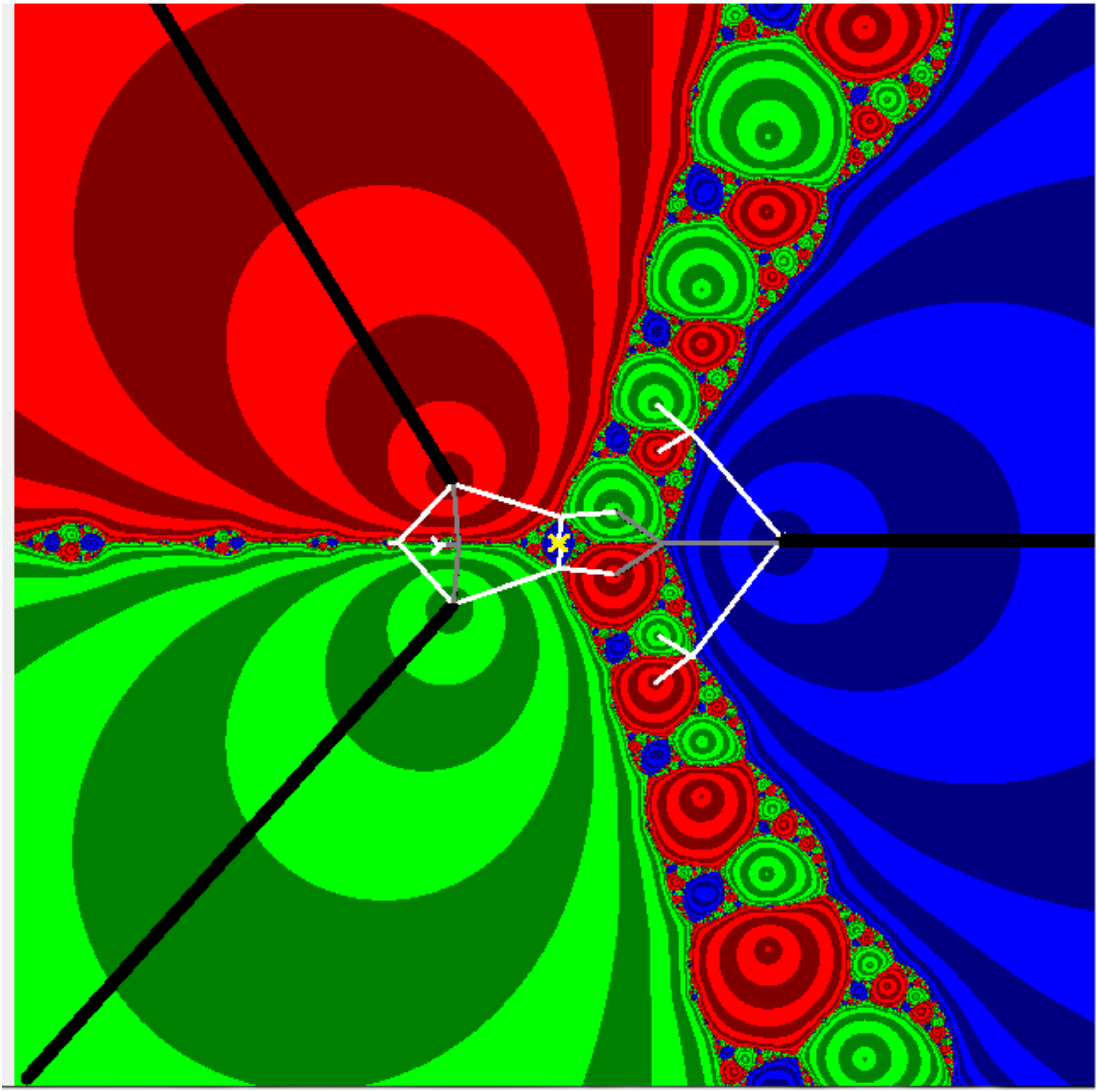

FIGURE 1. Constructing a finite subdivision rule for $f$

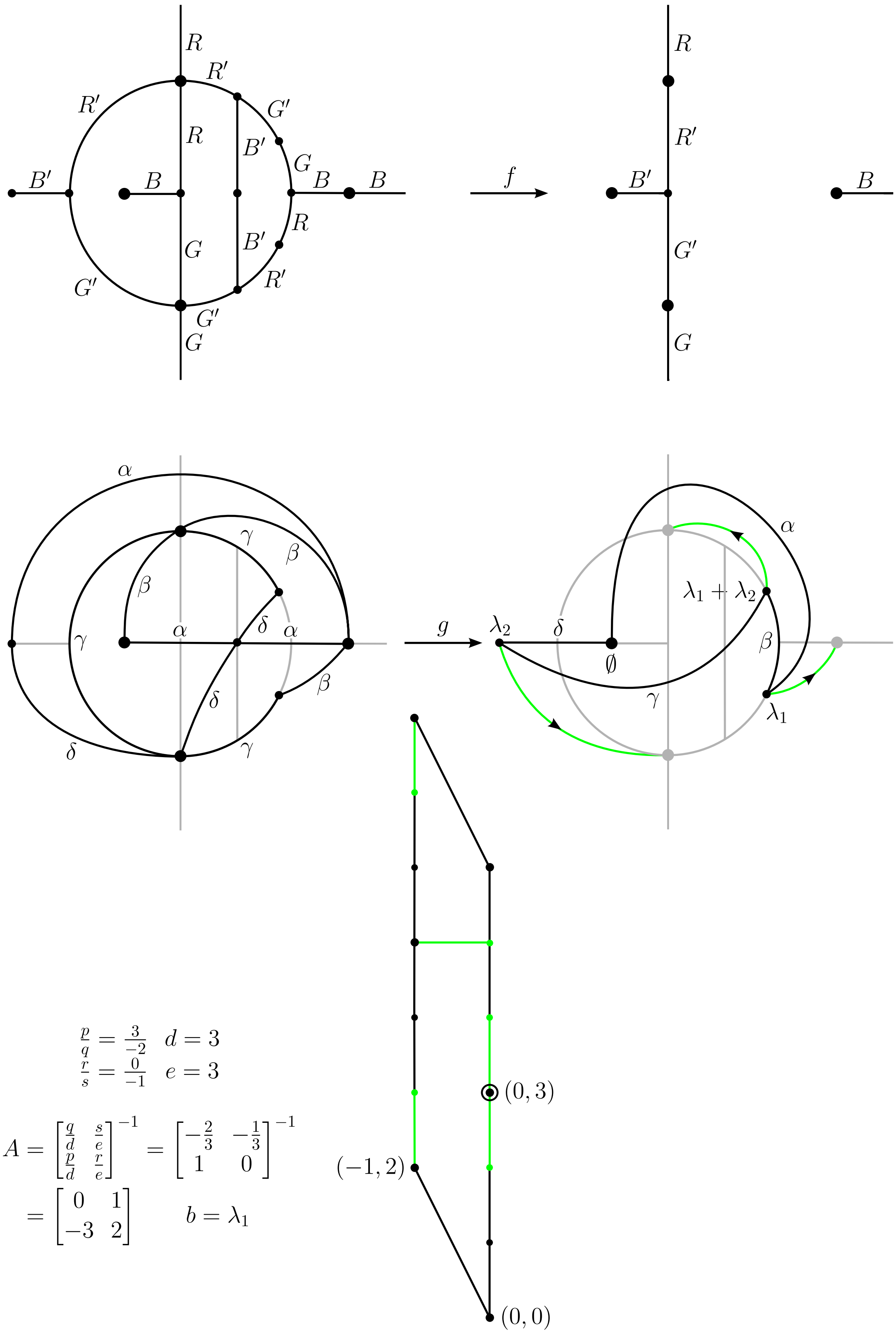


FIGURE 2. Constructing an NET map presentation for $f$