NET MAPS WHICH ARE NEWTON MAPS

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We determine all NET maps which are Newton maps up to Thurston equivalence. NET maps which are Newton maps have degree either 3 or 4. There are two Thurston equivalence classes with degree 3 and one with degree 4. The two equivalence classes with degree 3 have different static portraits, and so they belong to different modular group Hurwitz classes, those represented by 31HClass3 and 31HClass7. The one with degree 4 belongs to the modular group Hurwitz class represented by 41HClass23. Here are three representative rational functions.

$$f(z) = \frac{6z^3+1}{9z^2-2} \qquad f(z) = \frac{2z^3+1}{3z^2}$$
$$f(z) = \frac{3z^4-2z^2+3}{4(z^3-3z)}$$

We begin with a definition and some fundamental results. In Definition 1.1 of [1] Mikulich, Rückert and Schleicher define a Newton map to be a rational function $f\colon \widehat{\mathbb{C}} \to \widehat{\mathbb{C}}$ of degree $d \geq 3$ for which ∞ is a repelling fixed point and for each fixed point $\xi \in \mathbb{C}$ there exists an integer $m \geq 1$ such that $f'(\xi) = \frac{m-1}{m}$. Head's Theorem (Proposition 1.2 of [1]) states that a rational map f of degree $d \geq 3$ is a Newton map if and only if there exists a polynomial $p\colon \mathbb{C} \to \mathbb{C}$ such that $f(z) = z - \frac{p(z)}{p'(z)}$. In this situation, the fixed points of f in $\mathbb{C}$ are the roots of p and $f'(z) = \frac{p(z)p''(z)}{p'(z)^2}$. Corollary 14.5 of Milnor's book [2] states that if a rational map f is postcritically finite, then every periodic orbit of f is either repelling or superattracting. Thus if $f(z) = z - \frac{p(z)}{p'(z)}$ is a postcritically finite Newton map for some polynomial p, then the above integers m are all 1 and the roots of p are distinct. The roots of p are distinct if and only if f and p have the same degree.

Now we determine all degrees of NET maps which are Newton maps. Let f be a NET map which is a Newton map. The previous paragraph shows that there exists a polynomial p with distinct roots whose degree equals the degree of f such that $f(z) = z - \frac{p(z)}{p'(z)}$. The roots of p are fixed critical points of f. So the roots of p are postcritical points of f. Since f has four postcritical points, the degree of f is at most 4. So the degree of f is either 3 or 4.

Our next goal is to determine all NET maps with degree 3 which are Newton maps. Let $f(z) = z - \frac{p(z)}{p'(z)}$ be a NET map with degree 3 which is a Newton map. So p is a polynomial with degree 3. Suppose that $p(z) = az^3 + bz^2 + cz + d$ with $a, b, c, d \in \mathbb{C}$ and $a \neq 0$. So

$$f(z) = z - \frac{az^3 + bz^2 + cz + d}{3az^2 + 2bz + c}.$$

We may divide the numerator and denominator of the fraction by a. In effect, we may assume that $a = 1$. Now with $a = 1$, we conjugate f to replace it with $f(z - b/3) + b/3$. In

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effect, we may assume that $b = 0$. Now we conjugate f by a dilation $z \mapsto \alpha z$. As a result we may assume that either $c = -1$ or $c = 0$ and $d = -1$.

This provides one example: the one with $a = 1$, $b = c = 0$ and $d = -1$. It is

$$f(z) = z - \frac{z^3 - 1}{3z^2} = \frac{2z^3 + 1}{3z^2}.$$

To check that it is a NET map, we note that the critical points of f are 0 and the third roots of 1. The local degree of f at each of these points is 2. Furthermore, the third roots of 1 are fixed points and $f(0) = \infty$, which is a fixed point. So the local degree of f at every critical point is 2, and f has four postcritical points. Thus f is a NET map.

Now we seek another example with degree 3 for which $p(z) = z^3 - z + d$ for some $d \in \mathbb{C}$. So

$$f(z) = z - \frac{z^3 - z + d}{3z^2 - 1} = \frac{2z^3 - d}{3z^2 - 1}.$$

In this case, the critical points of f consist of the roots of p and 0. We have that $f(0) = d$. This is not a root of p, for otherwise f would have only three postcritical points. So d is a postcritical point of f other than a root of p. In particular, $d \neq 0$. It can't be fixed by f, so $f(d)$ must be a root of p. So $\frac{2d^3 - d}{3d^2 - 1}$ is a root of p. We next evaluate p at this value, equate to 0 and solve for d. This can be done by hand or, more easily, by computer. We find that the only solutions are $d = 0$ and $d = \pm\sqrt{3/8}$. We have seen that $d \neq 0$. The values $d = \pm\sqrt{3/8}$ yield NET maps which are Newton maps. We note that conjugating $f(z)$ to $-f(-z)$ takes d to $-d$, and so these two rational maps belong to the same Thurston class. We use the value $d = -\sqrt{3/8}$ and conjugate f by $z \mapsto \sqrt{3/2}z$ to obtain

$$f(z) = \frac{6z^3 + 1}{9z^2 - 2}.$$

This is a NET map which is a Newton map.

Now we turn our attention to degree 4. We argue as for degree 3 that we may assume that p is monic. One verifies that because the local degree of f at every critical point is 2, the map f has six critical points and the polynomial p'' has two distinct roots. We conjugate f by a linear map so that we may assume that the roots of p'' are ± 1. So

$$p''(z) = 12z^2 - 12 \qquad p'(z) = 4z^3 - 12z + a \qquad p(z) = z^4 - 6z^2 + az + b$$

for some $a, b \in \mathbb{C}$. The postcritical set of f must consist of the four roots of p. So f maps each of its critical points to a root of p. The critical points of f consist of the roots of p together with the roots of p'', namely, ± 1. We have that

$$f(1) = 1 - \frac{-5 + a + b}{a - 8} = \frac{b + 3}{8 - a} \qquad f(-1) = -1 - \frac{-5 - a + b}{a + 8} = \frac{-b - 3}{8 + a}.$$

We evaluate p at these two values, equate to 0 and solve for a and b by computer. There are four solutions:

$$\begin{array}{cccc} a = 0 & a = 0 & a = 16 & a = -16 \\ b = -27 & b = 5 & b = 21 & b = 21. \end{array}$$

But either 1 or -1 is a root of p for all but the first solution, resulting in a critical point of f with local degree greater than 2. So only the first solution can yield a NET map which is a Newton map. One verifies that the first solution does indeed yield a NET map which is a Newton map. We obtain

$$f(z) = z - \frac{z^4 - 6z^2 - 27}{4z^3 - 12z} = \frac{3}{4}\frac{z^4 - 2z^2 + 9}{z^3 - 3z}.$$

Conjugating this by $z \mapsto \sqrt{3}z$, we obtain

$$f(z) = \frac{3z^4 - 2z^2 + 3}{4(z^3 - z)}.$$

References

[1] Yauhen Mikulich, Johannes Rückert and Dierk Schleicher, *A combinatorial classification of postcritically fixed Newton maps*, http://arxiv.org/abs/1010.5280.
[2] John Milnor, *Dynamics in One Complex Variable*, Princeton Univ. Press, Princeton, 2006.