Math Circle 10/15/2076 Meeting: Isoperimetric Inequality, Part I

Queen Dido and the City of Carthage

Vergil's *Aeneid* describes the (apocryphal) foundation of the city of Carthage: Queen Dido from Phoenicia was obliged to flee from her bloodthirsty brother to North Africa. Once there, she made a deal with a local chieftain: In return for her fortune she would get as much land as she could isolate with the skin of a single ox. The deal was agreed upon and an ox was sacrificed.

Dido Purchases Land for the Foundation of Carthage. Engraving by Matthäus Merian the Elder, in *Historische Chronica*, Frankfurt a.M., 1630. Dido's people cut the hide of an ox into thin strips and try to enclose a maximal domain.

Queen Dido broke the skin of the ox down into very thin strips of leather, tied them together and constructed a huge semicircle, which, together with the natural boundary of the sea, turned out to be way bigger than anyone would have expected. Upon this land, Carthage was established. Apparently, the Queen knew the isoperimetric inequality and understood how to apply this knowledge to gain the best possible solution to her problem!

This city-state would become, through the Punic Wars, the only existential threat to the Roman Empire until the coming of the Vandals several centuries later. The Roman Republic destroyed Carthage in 146 BC during the Third Punic War, then re-developed it as Roman Carthage, which became the major city of the Roman Empire in the province of Africa.

Problem 1. Using graph paper, construct triangles with vertices at points

a. (0,0), (6,0), (0,3)
b. (0,0), (6,0), (1,3)
c. (0,0), (6,0), (3,3)

What can you say about their area? Which one has the smallest perimeter? What would be your conjecture about the shape of the triangle with the base 6 and the area 9 that has the smallest perimeter?

Problem 2. Using graph paper, construct triangles with vertices at points

a. (0,0), (4,0), (2,6)
b. (0,0), (6,0), (3,4)
c. (0,0), (12,0), (6,2)

What can you say about their area? Which one has the smallest perimeter? What would be your conjecture about the shape of the triangle with the area 12 that has the smallest perimeter?

Problem 3. Using graph paper, construct quadrilaterals with vertices at points

a. (0,0), (5,0), (8,5), (3,5)
b. (0,0), (5,0), (7,5), (2,5)
c. (0,0), (5,0), (5,5), (0,5)

What can you say about their area? Which one has the smallest perimeter? What would be your conjecture about the shape of the quadrilateral with the area 12 that has the smallest perimeter?

Problem 4. Suppose that $a + b = 8$, find the maximum value of ab.

Problem 5. Along all rectangles of perimeter 16, find the one with the largest area.

Problem 6. Show that $\text{Area}(\Delta ABC) = AB \times AC \times \sin(\angle BAC) \div 2$

Problem 7. Let Δ be the triangle that maximizes area among all triangles with two sides $a > 0$ and $b > 0$. What is the shape of Δ? What is its area?

Problem 8. Let Δ be the triangle that maximizes area among all triangles with a base $a = 6$ and perimeter $p = 16$. What is the shape of Δ? What is its area?

Hint: Fix side AB of length 6 and draw the set of all points C such that the perimeter is p. What is the shape of this set? At what point on this set the distance is furthest away from AB?

Problem 9. Let Δ be the triangle that maximizes area among all triangles of perimeter $p = 16$. What is the shape of Δ? What is the area of Δ?