Modular Arithmetic, Grades 6 - 9

Blacksburg Math Circle

Warm-up Problem.

Let $a_1 = 7$ and $a_2 = 11$. We fix the divisor to be $d = 5$ in this problem.

(a) Compute the remainder r_1 of a_1. Compute the remainder r_2 of a_2 as well.
(b) Is the remainder of $a_1 + a_2$ equal to $r_1 + r_2$?
(c) For other choices of a_1 and a_2, do you think the remainder of $a_1 + a_2$ is always equal to $r_1 + r_2$? If not, can you find a relation between $a_1 + a_2$ and $r_1 + r_2$?
(d) Is the remainder of $a_1 a_2$ equal to $r_1 r_2$?
(e) For other choices of a_1 and a_2, do you think the remainder of $a_1 a_2$ is always equal to $r_1 r_2$? If not, can you find a relation between $a_1 a_2$ and $r_1 r_2$?

Definition: Let d be a positive integer. Two numbers a and b are called *congruent modulo* d if they have the same remainder when divided by d. i.e., $aq_1 d + r$ and $b = q_2 d + r$. We denote this by $a \equiv b \pmod{d}$.

Definition: The set consisting of the integers a modulo d is called the *congruence class* of the integer a, modulo d.

Problem 1.

Give the congruence classes of the following given modulus d.

(a) $d = 5$
(b) $d = 12$
(c) Where is the congruence class from part (b) used regularly?
(d) Since today is Saturday, find what day of the week it will be in 365 days?
(e) What day of the week were you born on? (Hint: Don't forget leap years!)

Problem 2. Fill in the blanks:

(1) $55 \equiv$ _____ $\pmod{7}$
(2) $2018 \equiv$ _____ $\pmod{3}$
(3) $406 \equiv$ _____ $\pmod{1056}$

Problem 3. What is the remainder of $2^{51} \pmod{5}$? How about $3^{2017} \pmod{7}$?

Problem 4.

Identify the last digit of $3^{3^{3^{3^{3}}}}$. (Hint: What would you choose for the modulus d?)

Problem 5.

Find a positive integer n such that $x \equiv 1 \pmod{3}$ and $x \equiv 3 \pmod{7}$. Can you list all such positive integers less than 50?

Problem 6.

Find a positive integer n such that $x \equiv 2 \pmod 4$, $x \equiv 7 \pmod 5$, and $x \equiv 4 \pmod 7$.

Problem 7.

Let n be an integer that is not divisible by 3.

(a) For $n = 5$, compute the remainder of $n^2 - 1 \pmod 3$.
(b) For $n = 10$, compute the remainder of $n^2 - 1 \pmod 3$.
(c) Try a few more examples. Based on your observations, formulate a conjecture.
(d) Prove your conjecture.

Problem 8.

Let n be an odd integer.

(a) For $n = 3$, compute the remainder of $n^2 \pmod 8$.
(b) For $n = 13$, compute the remainder of $n^2 \pmod 8$.
(c) Try a few more examples. Based on your observations, formulate a conjecture.
(d) Prove your conjecture.

Problem 9.

Let n be an odd integer.

(a) For $n = 3$, compute the remainder of $n(n+2) \pmod{16}$.
(b) For $n = 13$, compute the remainder of $n(n+2) \pmod{16}$.
(c) Try a few more examples. Based on your observations, formulate a conjecture.
(d) Prove your conjecture.

Problem 10.

Let n be an odd integer.

(a) For $n = 5$, compute the remainder of $1 + 2 + \cdots + n \pmod n$.
(b) For $n = 7$, compute the remainder of $1 + 2 + \cdots + n \pmod n$.
(c) What is your conjecture?
(d) Prove your conjecture.

Problem 11.

Let n be an even integer.

(a) For $n = 4$, compute the remainder of $1 + 2 + \cdots + n \pmod n$.
(b) For $n = 6$, compute the remainder of $1 + 2 + \cdots + n \pmod n$.
(c) What is your conjecture?
(d) Prove your conjecture.