Counting with Symmetry: Problems

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1. Consider a square cake with four conners labeled $1, 2, 3, 4$, each allowing a topping (denoted by letters a, b, c, d). A decoration is a way to assign a topping to each position, denoted by a string of letters. For example, $aabc$ means assigning topping a to 1 and 2, topping b to 3, and topping c to 4.

   Our goal is to count *distinct* decorated cakes. Note that the decorations $abcd$ and $bcda$ produce the *same* cake.

   There are four rotational symmetries, labeled g_0 (do nothing), g_1 (clockwise by 90°), g_2 (rotate by 180°), and g_3 (counterclockwise by 90°).

   (a) Write down $g_1(abcd)$ and $g_2(abcd)$.

   (b) For each of the following decorations, write down its orbit. Note that the orbit of a decoration consists of all rotated versions of it. Which symmetries fix this decoration? Verify that for each decoration, the size of its orbit multiplied by the number of symmetries fixing it equals 4, the total number of symmetries.

      i. $aabc$

      ii. $cdcd$

      iii. $aaaa$

   (c) Write down all decorations fixed by g_1.

   (d) Write down all decorations fixed by g_2. You may use x, y to denote unspecified toppings, but you must specify whether x and y are allowed to be the same or not.

   (e) Complete the table the following table:

	g_0	g_1	g_2	g_3
number of decorations fixed by the symmetry				

   (f) How many distinct decorated cakes can you make? You may use our theorem (Burnside's lemma), which in this case says that each symmetry contributes

   $$\frac{\text{number of decorations it fixes}}{4}.$$

2. We continue the above example, but now there are n types of toppings available.

(a) Find the number of decorations fixed by g_2.

(b) Complete the following table. Can you use it to count the number of decorations fixed by a symmetry systematically?

	g_0	g_1	g_2	g_3
Two-line notation				
Cycle notation				
number of cycles				

(c) How many distinct decorated cakes can you make?

3. Using Pólya's theorem, find the number of ways to make a necklace with 6 beads, where n types of beads are available. Two necklaces are considered the same if you can obtain one from the other via rotation and/or flipping. You will need to identify 12 symmetries for this task.

4. (Food of thought) In the example of Problem 1, try to use your own words to explain why Burnside's lemma

$$\text{Number of orbits} = \frac{1}{4}\left(\mathrm{Fix}(g_0) + \mathrm{Fix}(g_1) + \mathrm{Fix}(g_2) + \mathrm{Fix}(g_3)\right)$$

is true, where $\mathrm{Fix}(g_i)$ is the number of decorations fixed by g_i. The following table may be helpful.

	$xxxx$	$xyxy (x \neq y)$	others
Orbit			
Symmetries			
Fixed by g_0?			
Fixed by g_1?			
Fixed by g_2?			
Fixed by g_3?			

You might notice "counting row by row = counting column by column". This is the essence of our proof visualized.