SESSION ON OCT. 22, 2022

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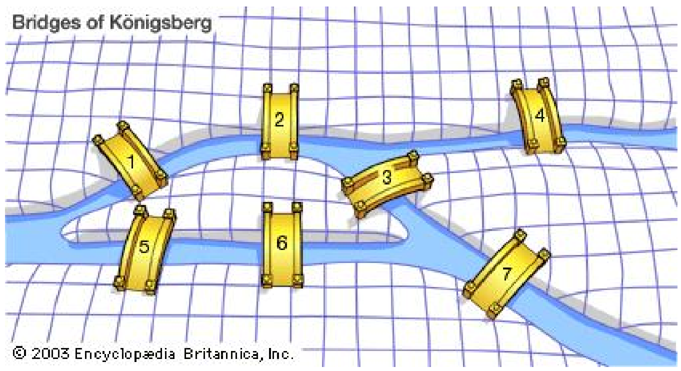
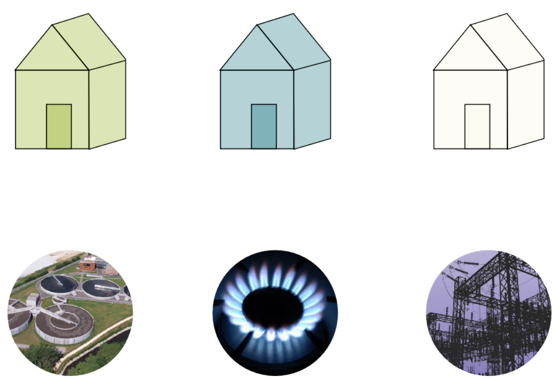
1. Graphs

1.1. **The Seven Bridges of Königsburg.** The city of Königsburg in Prussia (now Kaliningrad, Russia) was set on both sides of the Pregel River, and included two large islands which were connected to each other, or to the two mainland portions of the city, by seven bridges. In 1736, a mathematician by the name of Leonhard Euler visited the city and was fascinated by the bridges. Euler wondered whether one can devise a walk through the city that would cross each of those bridges once and only once. Take a few minutes to see if you can find a way on the map of Königsburg below.



1.2. **The Three Utilities Problem.** Suppose you have a neighbourhood with only three houses. Now, each house needs to be connected to a set of three utilities (gas, water, and electricity) in order for a family to live there. Your challenge is to connect all three houses to each of the utilities. Below is a diagram for you to draw out where each utility line/pipe should go. Without using a third dimension or sending any of the connections through another company or cottage, is there a way to make all nine connections without any of the lines crossing each other?

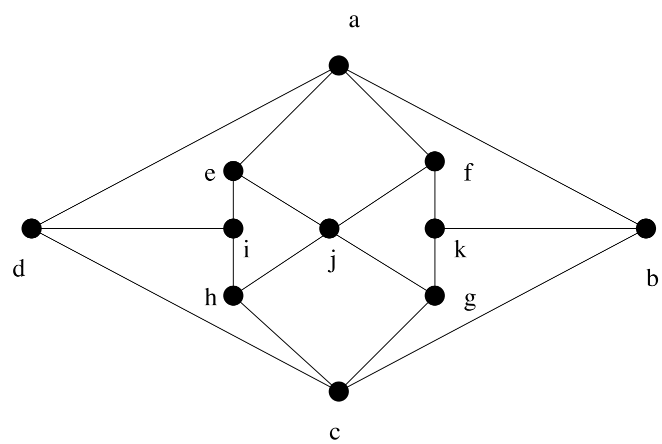


1.3. **Definitions.** Graph Theory is a relatively young branch of mathematics, and it was Euler's solution to the 7 Bridges problem in 1736 that represented the first formal piece of Graph Theory. But what is Graph Theory? In order to answer that question and to explore some of the applications of Graph Theory, we first need some definitions.

Definition 1.1.

(1) *A* **vertex** *(plural: vertices) is a point. In drawings, we usually represent these as circles.*
(2) *An* **edge** *e is an unordered pair of vertices. For example, $e = \langle a, b\rangle$. In drawings, edges are represented by lines between the two vertices.*
(3) *For an edge $e = \langle a, b\rangle$, we call a and b the* **endpoints** *of edge e.*
(4) *A vertex a is* **incident** *with an edge e if a is an endpoint of e.*
(5) *Two vertices a and b are* **adjacent** *if there is an edge e with a and b as its endpoints.*
(6) *The vertices adjacent to a vertex a are called the* **neighbors** *of a*
(7) *A* **graph** *$G = (V, E)$ is comprised of a set V of vertices, and a set of edges E .*
(8) *A* **walk** *is a sequence of vertices where each vertex is adjacent to the vertex before it and after it.*
(9) *A* **path** *is a walk that doesn't repeat vertices.*
(10) *A* **cycle** *is a path that begins and ends at the same vertex.*

These definitions can be difficult to understand as abstract concepts. To make these definitions more concrete, consider the graph G below:



Problem 1.

(1) List the vertices in G
(2) List the edges in G
(3) List the edges incident with vertex c
(4) List the vertices adjacent to g
(5) Find a path from j to b
(6) Find a cycle in G

Problem 2. Replace each land mass in the seven bridges problem with a vertex, and each bridge with an edge (this will allow us to record which pair of vertices (land masses) is connected by corresponding bridges). How the resulting graph looks like?

Problem 3. Euler observed that (except at the endpoints of the walk), whenever one enters a vertex by a bridge, one leaves the vertex by a bridge. In other words, during any walk in the graph, the number of times one enters a non-terminal vertex equals the number of times one leaves it. How this observation can be used to solve the seven bridges puzzle?

1.4. **Number of edges.** In this section we will learn how to count the number of the edges of a graph if we know how many edges are incident to each vertex. The **degree** of a vertex v, denoted $\deg(v)$ is the edges adjacent to v.

Problem 4. In a country there are 100 cities, each of which is connected by a road to exactly 4 others. How many roads are there in total ?

Theorem 1. *(a) In a graph, the number of vertices of odd degree is even.*

(b) Let $G = (V, E)$ be a graph with vertices $v_1, \ldots, v_n$ and let e be the number of edges. Then

$$2e = \deg(v_1) + \ldots + \deg(v_n).$$

Problem 5. There are 30 students in a class. Is it possible that 9 of them have 3 friends each, 11 have 4 friends each, and 10 have 5 friends each ?

Problem 6. At a party there are 15 people. Is it possible that each of them knows exactly 5 other people ?

1.5. **Connected graphs.** A graph is called **connected** if one can travel between any two vertices using edges of the graph.

Problem 7. Create a graph where each vertex represents a continent, and there is a edge if the continents are continents are joined by land or a bridge. Is the graph connected ?

Problem 8. In a small county from Transylvania there are 15 villages, each connected to at least 7 other villages. Prove that one can travel between any two villages (perhaps passing through some other villages in between).

Theorem 2. *Let G be a graph with n vertices such that each vertex has degree $\geq \frac{n-1}{2}$. Then G is connected.*

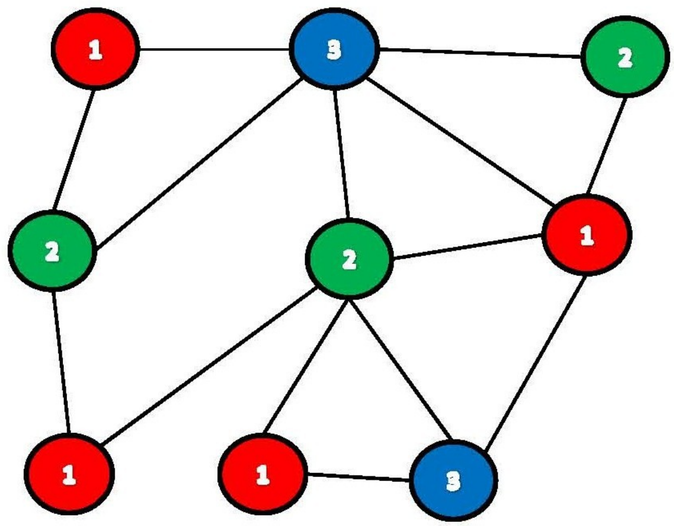
Problem 9.* In Mount Moria, the Dwarfs live in a number of cities, connected by tunnels. Their main city, called Khazad-dum is connected to exactly 7 other smaller cities. The farthest city, called Nogrod, is connected to another city by exactly one tunnel. All the remaining cities are connected to each other by exactly 4 tunnels. Show that it is possible for Gimli the Dwarf to travel from Khazad-dum to Nogrod.

Problem 10.* There used to be 26 football teams in the NFL, with two conferences divided 13 teams each. (Each conference was further divided into 2 divisions, but this is irrelevant here.) An NFL guideline said that each team's 14 game schedule should include exactly 11 games against teams in its own conference, and 3 games with teams in the other conference. By utilizing a graph model show that this guideline cannot be satisfied.

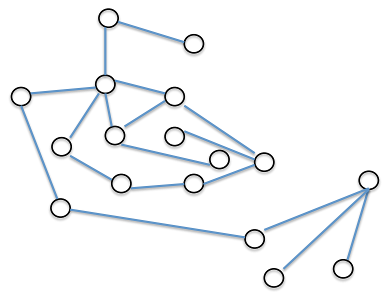
Note: Problems with $\star$ have a higher degree of difficulty.

1.6. **Colorings.** How many colors does it take to color a map of the United States? What about a map of Africa? While it may not be obvious, this problem (a more general form of it) has been one of the central problems in Graph Theory for a very long time. It turns out that we can color any planar graph in just four colors. In fact, this problem was initially posed in 1852, and a correct proof was not submitted until 1976, over 100 years later. But what does it mean to color a graph?

It's easy to understand how to color a map, we simply color each region a different color than it's neighbors. It turns out that coloring a graph is essentially the same thing. We assign each vertex a different color than the vertices adjacent to it. Even though we talk about colors like red or green, we usually just label the vertex with a number, and each different number represents a new color. Below is an example of a coloring. Notice that this graph can be colored using just three colors.



Problem 10. Try coloring the following graph with as few colors as possible. How many colors did you need?



Coloring Bipartite Graphs. A bipartite graph is a graph where we can split the vertices into two groups, A and B , and all of the edges have one endpoint in A and one in B. This means that for any vertex in A, all of the vertices it is adjacent to are in B. One example of a bipartite graph is $K_{3,3}$. We can see that if we call the top row of vertices A and the bottom row of vertices B, then all of the edges go from A to B, and none from A to A or B to B.